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The matching book embeddings of pseudo-Halin graphs

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Abstract

The *book embedding* of a graph G is to arrange the set of points of the graph on a line (spine) and embed the edges on the half-plane bounded by the spine so that the edges in the same page do not intersect with each other. If the maximum degree of vertices in each page is 1, the book embedding is *matching book embedding*. The *matching book thickness* of G is the minimum number n that G can be matching book embedded in n-page. In this paper, the matching book thickness of pseudo-Halin graphs is determined.

Keywords: book embedding, matching book embedding, matching book thickness, pseudo-Halin graph
Mathematics Subject Classification : 05C10; 68R10
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1. Introduction

The concept of book embedding was first proposed by Bernhart and Kainen [1]. A *book* consists of a spine and several pages. The spine can be seen as a line and the pages are half-planes with the spine as a common boundary. The *book embedding* of a graph consists of two parts, one is to arrange the vertices on the spine in order, and the other is to embed the edges into the pages so that the edges in the same page do not cross. The *book thickness* of a graph G is the least number n that G has an n-book embedding.

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A book embedding of a graph G is *matching* if the degree of vertices on each page is at most one. The *matching book thickness* of a graph G is the least number n that G can be matching book embedded in n pages, which we denoted by $mbt(G)$ [2]. A graph G is *dispersable* if $mbt(G) = \Delta(G)$, where $\Delta(G)$ is the maximum degree of G. A graph G is *nearly dispersable* if $mbt(G) = \Delta(G) + 1$, see [3].

Book embeddings have been extensively studied for various families of graphs [3-8]. Meanwhile, the question of determining a graph is dispersable or not has aroused widespread concern for its connection to edge coloring. Complete bipartite graphs $K_{n,n}$ $(n \geq 1)$, even cycles C_{2n} $(n \geq 2)$, binary n-cube $Q(n)$ $(n \geq 0)$ and trees are dispersable [1]. Kainen [2] shows that $mbt(C_p \Box C_q)$ is 4, when p and q are both even and $mbt(C_p \Box C_q)$ is 5, when p is even, q is odd. Kainen also gives some dispersability of circulant graphs [9] and bipartite cubic planar graphs are dispersable [10]. Joslin [11] shows that $mbt(C_m \Box C_n)$ is 5, when m is 3 or 5. In [3], it is proved that any regular dispersable graph G is bipartite and the matching book thickness of the complete graph K_n is n. Recently, the first author et al. have obtained the matching book thickness of generalized Petersen graphs [12],bipartite cubic planar graphs [13] and, cartesian product of complete graphs and cycles [14].

A 2-connected planar graph G with minimum degree at least 3 is a *pseudo-Halin graph* [15] if deleting the edges on the boundary of a single face f_0 yields a tree. It is a *Halin graph* if the vertices of f_0 all have degree 3 in G. The face f_0 is the *exterior face*; the others are *interior faces*. Vertices of f_0 are *exterior vertices*; the others are *interior vertices*. Vertices of f_0 with degree 3 in G are *regular vertices*; the others are *irregular vertices*. Let $R(f_0)$ and $IR(f_0)$ denote the sets of regular and irregular vertices in f_0, respectively. In particular, a pseudo-Halin graph with $\Delta(G) = 3$ is a cubic Halin graph, see Figure 1 for a Halin graph and a pseudo-Halin graph with $\Delta(G) = 4$, respectively.

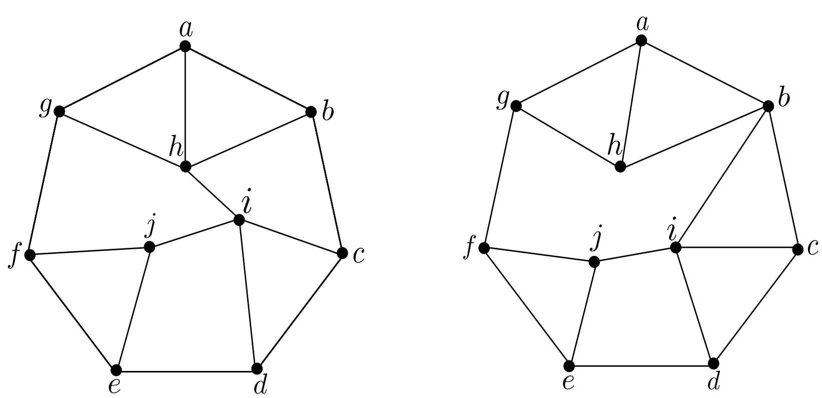

Figure 1 (Left) An example of a Halin graph. (Right) An example of a pseudo-Halin graph.

In this paper, we compute the matching book thickness of a pseudo-Halin graph G as follows.

Main theorem:

$$mbt(G) = \begin{cases} 4, & \Delta(G) = 3, \\ \Delta(G), & \Delta(G) \geq 4. \end{cases}$$

2. The proof of the main theorem

The proof will be completed by a sequence of results. We recall that the chromatic index of a graph G is denoted by $\chi'(G)$.

Remark 2.1 (3, pg. 87). For any simple graph G, $\Delta(G) \leq \chi'(G) \leq mbt(G)$.

Let G be a simple graph with p vertices. Suppose that G has an n-book embedding. All vertices of G occur in some specified order $v_1, \cdots, v_p$ from "top" to "bottom" along the spine and this sequence is called the *printing cycle* of the embedding [1]. Similarly, we can define a printing cycle for a matching book embedding of a graph. Sometimes it is convenient to understand the matching book embedding from another point of view as for book embedding of graphs in [16], that is, to embed the graph G so that its vertices lie on a circle and its edges are chords of the circle; to assign the chords to layers so that chords on the same layer do not cross and meet each other. By the rotations of the vertices on the circle, it is easy to get the following result which is a natural generalization of Remark 2.2 and Lemma 2.1 of [1].

Remark 2.2 (3, pg. 88). If a regular graph G is dispersable, then G is bipartite.

Lemma 2.1. [1] *Let G be a simple graph.*

(i) If G has an n-book matching embedding with printing cycle $v_1, v_2, ..., v_p$, then G also has an n-book matching embedding with printing cycle $v_2, ..., v_p, v_1$.

(ii) If G has an n-book matching embedding β with printing cycle $v_1, v_2, ..., v_p$, then G also has an n-book matching embedding β^- with printing cycle $v_p, ..., v_2, v_1$.

Lemma 2.2. *Let W_m be a wheel graph with m vertices* $(m \geq 4)$,

$$mbt(W_m) = \begin{cases} 4, & m = 4, \\ \Delta(W_m), & m \geq 5. \end{cases}$$

Proof. If $m = 4$, then W_4 is K_4. Hence $mbt(W_4) = 4$.

If $m \geq 5$, it follows by Remark 2.1 that $mbt(W_m) \geq \Delta(W_m)$. Assume u is the center vertex of W_m whose has neighbours on f_0, in counterclockwise order, denoted by $v_1, v_2, ..., v_{m-1}$. We embed the vertices of W_m along the spine according to the following ordering of $v_{\lfloor \frac{m-1}{2} \rfloor}, v_{\lfloor \frac{m-1}{2} \rfloor -1}, v_{\lfloor \frac{m-1}{2} \rfloor -2}, ..., v_1, u, v_{\lfloor \frac{m-1}{2} \rfloor +1}, v_{\lfloor \frac{m-1}{2} \rfloor +2}, ..., v_{m-1}$. We assign the edges of W_m to the $m-1$ pages as follows.

Page 1: the edges $\{(uv_1), (v_2v_3)\}$;

$\cdots\cdots$

Page i: the edges $\{(uv_i), (v_{i+1}v_{i+2})\}$;

$\cdots\cdots$

Page $m-3$: the edges $\{(uv_{m-3}), (v_{m-2}v_{m-1})\}$;

Page $m-2$: the edges $\{(uv_{m-2}), (v_{m-1}v_1)\}$;

Page $m-1$: the edges $\{(uv_{m-1}), (v_1v_2)\}$.

Therefore, $mbt(W_m) \leq m-1 = \Delta(W_m)$. The result is established, see Figure 2 for the case $m = 6$ and $m = 7$. □

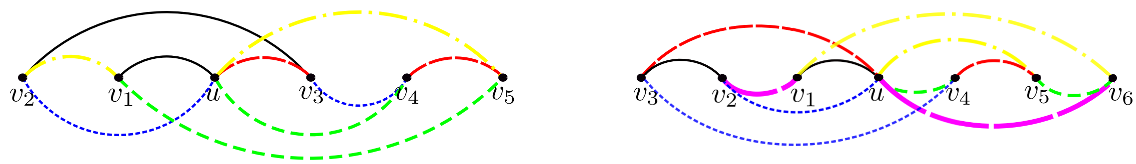

Figure 2 (Left) The matching book embedding of W_6. (Right) The matching book embedding of W_7.

Lemma 2.3. *If H is a cubic Halin graph, then $mbt(H) = 4$.*

Proof. It is easy to see that Halin graph H contains at least one 3-cycle. By Remark 2.2, we have H is not dispersable. Hence $mbt(H) \geq \Delta(H) + 1 = 4$.

Next, we show that $mbt(G) \leq 4$ by induction on the number n of interior vertices. If $n = 1$, then $H = K_4$. Thus, $mbt(H) = \Delta(K_4) = 4$. Assume that $mbt(H) \leq 4$ holds for $n = 1, 2, \cdots, m$. We need to show that $mbt(G) \leq 4$ for any cubic Halin graph H with $m + 1$ interior vertices. By the definition of Halin graph, H contains at least one interior vertex which is adjacent to at least two leaves. Assume w is an interior vertex of H whose has neighbours on f_0, in counterclockwise order, denoted by v_1, v_2, see Figure 3 (a). Because $\deg(w) = 3$, w has only one other neighbour which is denoted by u. Let x be the adjacent vertex on f_0 before v_1 and y be the adjacent vertex on f_0 behind v_2 ($x \neq y$).

Consider the graph H' obtained from H by contracting w, v_1, v_2 to a single vertex w', see Figure 3 (b). H' is still a cubic Halin graph with m interior vertices. By induction, $mbt(H') \leq 4$. In other words, the graph H' has a 4-page matching book embedding. We put the vertices of H' along the spine according to such a matching book embedding. By Lemma 2.1, it suffices to consider one vertex ordering of H' on the spine as ..., x, ..., w', ..., y', ..., see Figure 3 (d). Now we construct a matching book embedding of H from that of H'.

First, we replace the vertex w' with 3 vertices w, v_1, v_2 on the spine in the ordering of v_1, w, v_2. Without loss of generality, assume the edges $w'u, w'x, w'y$ are assigned to the page $1, 2, 3$, respectively, in the matching book embedding of H'. Now we assign the edges wu, v_1x, v_2y to the page $1, 2, 3$, respectively, and the edges wv_2, wv_1, v_1v_2 to the page $2, 3, 4$, respectively, see Figure 3 (c); Other edges of H are assigned to the same page as that of H'. It is easy to see that we get a matching book embedding of H in 4-book. Hence $mbt(H) \leq 4$.

The result is established. □

Lemma 2.4. *If G is a pseudo-Halin graph with $\Delta(G) = 3$, then $mbt(G) = 4$.*

Proof. Since a pseudo-Halin graph with $\Delta(G) = 3$ is a cubic Halin graph, it follows from Lemma 2.3 that $mbt(G) = 4$. □

Lemma 2.5. [15] *If G is a pseudo-Halin graph ($G \neq W_p$) with exterior face f_0 and $P = u_1u_2...u_k$ is the longest path in $G - E(f_0)$, $w \in \{u_2, u_{k-1}\}$, let $N(w)$ be the set of neighbours of the vertex w, then one of the following results holds:*

(1) *The vertex w is an interior vertex with $N(w) \subseteq V(f_0)$, $|N(w) \cap IR(f_0)| = 1$. Let $N(w) = \{y, v_1, v_2, ..., v_m\}$ $(m \geq 2)$; $xv_1, v_iv_{i+1}, v_my \in E(f_0)$, $y \in IR(f_0)$, $x \neq v_2$ and $y \neq v_{m-1}$, $i = 1, 2, ..., m-1$, then*

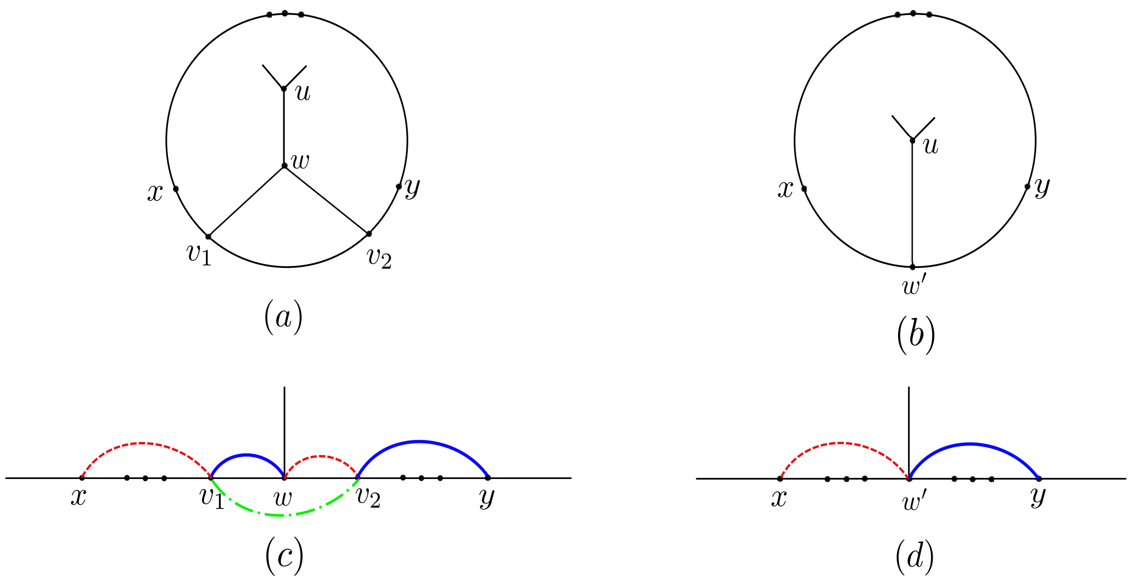

Figure 3 (a)The cubic Halin graph H. (b)The reduced graph $H^{'}$. (c)The matching book embedding of H. (d)The matching book embedding of H'.

$G_1^1 := G - \{w, v_i | i = 1, 2, ..., k\} + \{xy\}$ *and*
$G_1^2 := G - \{v_i, v_{i+1}, ..., v_j\} + \{v_{i-1}v_{j+1}\}$ $(2 \leq i < j \leq m-1)$ *are also pseudo-Halin graphs.*
(2) *The vertex* w *is an interior vertex with* $|N(w) \cap (V(G) \backslash V(f_0))| = 1$, $|N(w) \cap R(f_0)| = d(w) - 1$. *Let* $N(w) = \{u, v_1, ..., v_m\}$, u *being interior vertex,* $v_i \in R(f_0)$ $(i = 1, 2, ..., m)$, $xv_1, v_iv_{i+1}, v_my \in E(f_0)$ $(i = 1, 2, ..., m-1)$, $x \neq v_2$ *and* $y \neq v_{m-1}$, *then*
$G_2^1 := G - \{v_1, v_2, ..., v_m\} + \{xw, yw\}$ *and*
$G_2^2 := G - \{v_i, v_{i+1}, ..., v_j\} + \{v_{i-1}v_{j+1}\}$ $(2 \leq i < j \leq m,\ m \geq 3)$ *are also pseudo-Halin graphs.*

Lemma 2.6. *If* G *is a pseudo-Halin graph with* $\Delta(G) \geq 4$*, then* $mbt(G) = \Delta(G)$.

Proof. According to Remark 2.1, we have $mbt(G) \geq \Delta(G)$.

Next we show that $mbt(G) \leq \Delta(G)$ by induction on the number n of interior vertices. If $n = 1$, then G is a wheel graph. It follows from Lemma 2.2 that $mbt(G) \leq \Delta(G)$. Assume that $mbt(G) \leq \Delta(G)$ holds for $n = 1, 2, \cdots, m$. We need to show that $mbt(G) \leq \Delta(G)$ for any pseudo-Halin graph G with $m+1$ interior vertices. Let $P = u_1u_2...u_p$ is the longest path in $G - E(f_0)$, $w \in \{u_2, u_{p-1}\}$. In order to complete the proof, we need to consider the following two cases:

Case 1. The vertex w is an interior vertex with $N(w) \subseteq V(f_0)$, $|N(w) \cap IR(f_0)| = 1$.

Assume that the neighbours of w on f_0 are $v_1, v_2, ..., v_k, y$ $(k \geq 2)$ in counterclockwise order. Obviously, $xv_1, v_iv_{i+1}, v_ky \in E(f_0)$, $y \in IR(f_0)$, $x \neq v_2$ *and* $y \neq v_{k-1}$ $(i = 1, 2, ..., k-1)$. Let x be the adjacent vertex on f_0 before v_1, y' be the adjacent vertex on f_0 behind y and z_t $(t = 1, 2, ..., j)$ be the adjacent interior vertex of y, see Figure 4 (a) for the case $k = 4, j = 1$.

Consider the graph G' obtained from G by contracting $w, v_1, v_2, ..., v_k, y$ to a single vertex w', see Figure 4 (b) for the case $k = 4, j = 1$. By Lemma 2.6, G' is still a pseudo-Halin graph with m interior vertices. We need to consider two situations:

Situation 1: $4 \leq \Delta(G') \leq \Delta(G)$.

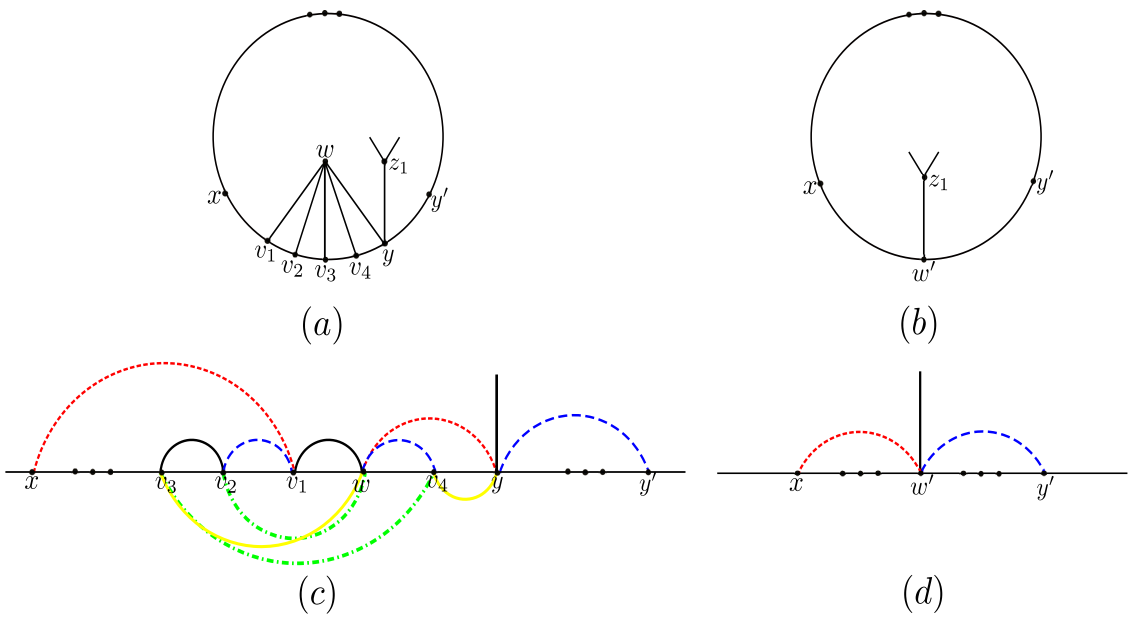

Figure 4 (a)The pseudo-Halin graph G. (b)The reduced graph $G^{'}$. (c)The matching book embedding of G. (d)The matching book embedding of G'.

By induction, $mbt(G') \leq \Delta(G')$. In other words, the graph G' has a $\Delta(G')$-page matching book embedding. We put the vertices of G' along the spine according to such a matching book embedding. By Lemma 2.1, it suffices to consider one vertex ordering of G' on the spine as $\cdots, x, ..., w', ..., y', ...$, see Figure 4 (d). Now we construct a matching book embedding of G from that of G'. First, we replace the vertex w' with $k+2$ vertices $w, v_1, v_2, ..., v_k, y$ on the spine in the ordering of $v_{\lceil \frac{k+1}{2} \rceil}, v_{\lceil \frac{k+1}{2} \rceil -1}, v_{\lceil \frac{k+1}{2} \rceil -2}, ..., v_1, w, v_{\lceil \frac{k+1}{2} \rceil +1}, ..., v_k, y$. Without loss of generality, assume the edges $w'x, w'y', w'z_1, ..., w'z_j$ are assigned to the page $1, 2, ..., j+2$, respectively, in the matching book embedding of G'. Next, we consider the matching book embedding of G from the following two cases.

(I) $deg(w) = k+1 = \Delta(G)$

We assign the edges of G to the $\Delta(G)$ pages as follows.

(a) $deg(w) = deg(y)$, i.e. $k = j+2$

Page 1: the edges $\{(wy), (xv_1)\}$;

Page 2: the edges $\{(yy'), (wv_k), (v_1v_2)\}$;

Page 3: the edges $\{(yz_1), (wv_1), (v_2v_3)\}$;

Page 4: the edges $\{(yz_2), (wv_2), (v_3v_4)\}$;

......

Page i: the edges $\{(yz_{i-2}), (wv_{i-2}), (v_{i-1}v_i)\}$;

......

Page $j+2$: the edges $\{(yz_j), (wv_j), (v_{j+1}v_{j+2})\}$;

Page $k+1$: the edges $\{(wv_{k-1}), (v_ky)\}$;

Other edges of G are assigned to the same pages as that of G'. It is immediate that we get a matching book embedding of G in $\Delta(G)$-book. Hence $mbt(G) \leq \Delta(G)$.

(b) $deg(w) > deg(y)$, i.e. $k > j + 2$

Page 1: the edges $\{(wy), (xv_1)\}$;

Page 2: the edges $\{(yy'), (wv_k), (v_1v_2)\}$;

Page 3: the edges $\{(yz_1), (wv_1), (v_2v_3)\}$;

Page 4: the edges $\{(yz_2), (wv_2), (v_3v_4)\}$;

......

Page i: the edges $\{(yz_{i-2}), (wv_{i-2}), (v_{i-1}v_i)\}$;

......

Page $j + 2$: the edges $\{(yz_j), (wv_j), (v_{j+1}v_{j+2})\}$;

Page $j + 3$: the edges $\{(wv_{j+1}), (v_{j+2}v_{j+3})\}$;

......

Page k: the edges $\{(wv_{k-2}), (v_{k-1}v_k)\}$;

Page $k + 1$: the edges $\{(wv_{k-1}), (v_ky)\}$;

Other edges of G are assigned to the same pages as that of G'. It is easily seen that we get a matching book embedding of G in $\Delta(G)$-book. Hence $mbt(G) \leq \Delta(G)$, see Figure 4 (c) for the case $k = 4, j = 1$.

(II) $deg(w) = k + 1 < \Delta(G)$

(a) $deg(w) = 3$

We assign the edges of G to the $j + 3$ pages as follows. Other edges of G are assigned to the same pages as that of G'. It is easy to see that we get a matching book embedding of G in $\Delta(G)$-book, see Figure 5 for the case $j = 1$.

Page 1: the edges $\{(wy), (xv_1)\}$;

Page 2: the edges $\{(yy'), (wv_2)\}$;

Page 3: the edges $\{(yz_1), (v_1v_2)\}$;

Page 4: the edges $\{(yz_2)\}$;

......

Page i: the edges $\{(yz_{i-2})\}$;

......

Page $j + 2$: the edges $\{(yz_j)\}$;

Page $j + 3$: the edges $\{(wv_1), (v_2y)\}$;

(b) $deg(w) \geq 4$ and $\Delta(G) = deg(y)$

If $\Delta(G) = deg(y) = j + 3$, we assign the edges of G to the $j + 3$ pages as follows. Other edges of G are assigned to the same pages as that of G'. It is obvious that we get a matching book embedding of G in $\Delta(G)$-book.

Page 1: the edges $\{(wy), (xv_1)\}$;

Page 2: the edges $\{(yy'), (wv_k), (v_1v_2)\}$;

Page 3: the edges $\{(yz_1), (wv_1), (v_2v_3)\}$;

Page 4: the edges $\{(yz_2), (wv_2), (v_3v_4)\}$;

......

Page i: the edges $\{(yz_{i-2}), (wv_{i-2}), (v_{i-1}v_i)\}$;

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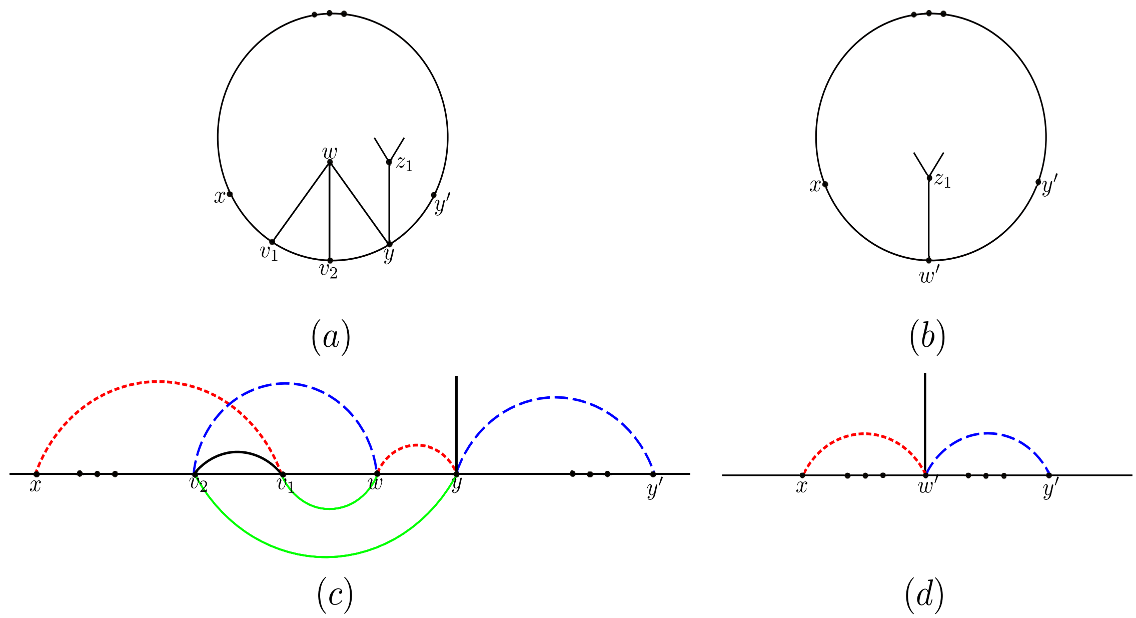

Figure 5 (a)The pseudo-Halin graph G. (b)The reduced graph G'. (c)The matching book embedding of G. (d)The matching book embedding of G'.

Page k: the edges $\{(yz_{k-2}), (wv_{k-2}), (v_{k-1}v_k)\}$;
Page $k+1$: the edges $\{(yz_{k-1})\}$;
......
Page $j+2$: the edges $\{(yz_j)\}$;
Page $j+3$: the edges $\{(v_k y), (wv_{k-1})\}$;

(c) $deg(w) \geq 4$ and $\Delta(G) \neq deg(y)$.

It is easy to verify that $\Delta(G) = \Delta(G')$. We assign the edges of G to the $max\{j+3, k+1\}$ pages. We can construct a $\Delta(G')$-page matching book embedding of G from that of G' in the same way as the case (I) or the case (II)(b). Other edges of G are assigned to the same pages as that of G'. Hence $mbt(G) \leq \Delta(G') = \Delta(G)$.

Situation 2: $\Delta(G') = 3$.

By Lemma 2.2, $mbt(G') = 4$. In a similar way as Situation 1 we can construct a matching book embedding of G in a $\Delta(G)$-page book. Hence $mbt(G) \leq \Delta(G)$.

Case 2. The vertex w is an interior vertex with $|N(w) \cap (V(G) \backslash V(f_0))| = 1$, $|N(w) \cap R(f_0)| = d(w) - 1$.

Assume that the neighbours of w on f_0 are $v_1, v_2, ..., v_k$ $(k \geq 2)$ in counterclockwise order. Obviously, $xv_1, v_i v_{i+1}, v_k y \in E(f_0)$, $x \neq v_2$ *and* $y \neq v_{k-1}$ $(i = 1, 2, ..., k-1)$. Let x be the adjacent vertex on f_0 before v_1, y be the adjacent vertex on f_0 behind v_k and u be the interior adjacent vertex of w, see Figure 6 (a).

Consider the graph G' obtained from G by contracting $w, v_1, v_2, ..., v_k$ to a single vertex w', see Figure 6 (b). By Lemma 2.5, G' is still a pseudo-Halin graph with m interior vertices. We need to consider two situations:

Situation 1: $4 \leq \Delta(G') \leq \Delta(G)$.

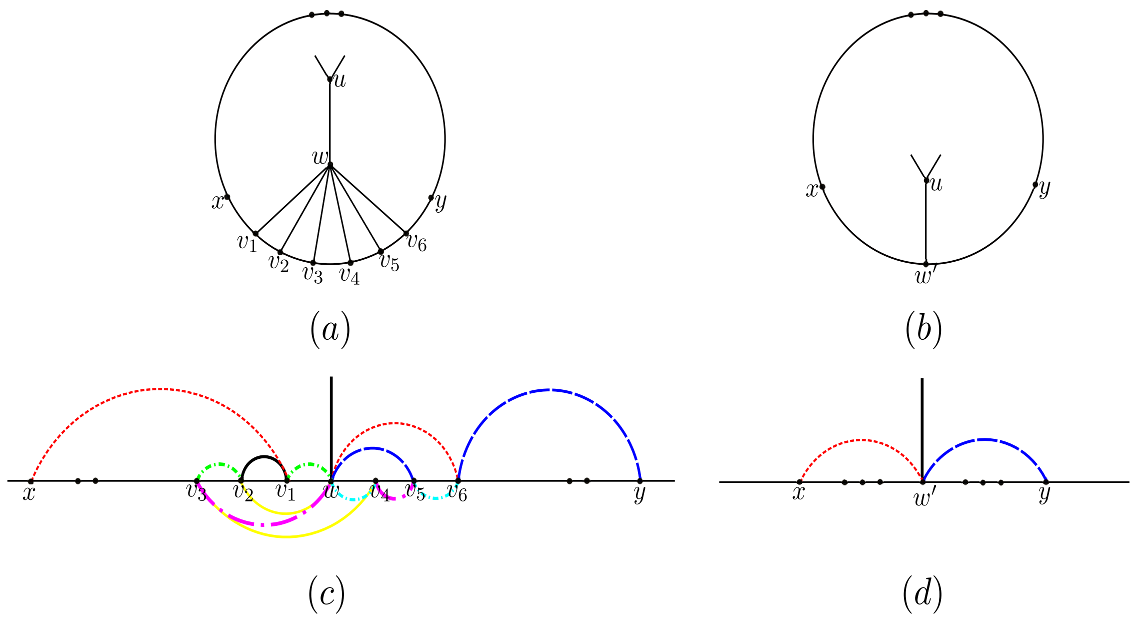

Figure 6 (a)The pseudo-Halin graph G. (b)The reduced graph $G^{'}$. (c)The matching book embedding of G. (d)The matching book embedding of G'.

By induction, $mbt(G') \leq \Delta(G')$. That is to say, the graph G' has a $\Delta(G')$-page matching book embedding. We put the vertices of G' along the spine according to such a matching book embedding. By Lemma 2.1, it is sufficient to consider one vertex ordering of G' on the spine as $..., x, ..., w', ..., y, ...$, see Figure 6 (d). Now we construct a matching book embedding of G from that of G'. First, we replace the vertex w' with $k+1$ vertices $w, v_1, v_2, ..., v_k$ on the spine in the ordering of $v_{\lceil \frac{k}{2} \rceil}, v_{\lceil \frac{k}{2} \rceil -1}, v_{\lceil \frac{k}{2} \rceil -2}, ..., v_1, w, v_{\lceil \frac{k+1}{2} \rceil +1}, ..., v_k$. Without loss of generality, assume the edges $w'u, w'x, w'y$ are assigned to the page $1, 2, 3$, respectively, in the matching book embedding of G'. Next we consider the matching book embedding of G from the following two cases.

(I) $deg(w) = k+1 = \Delta(G)$

We assign the edges of G to the $\Delta(G)$ pages as follows.

Page 1: the edges $\{(wu), (v_1v_2)\}$;

Page 2: the edges $\{(wv_k), (xv_1)\}$;

Page 3: the edges $\{(wv_{k-1})(v_ky)\}$;

Page 4: the edges $\{(wv_1), (v_2v_3)\}$;

Page 5: the edges $\{(wv_2), (v_3v_4)\}$;

. . . . . .

Page i: the edges $\{(yz_{i-3}), (v_{i-2}v_{i-1})\}$;

. . . . . .

Page k: the edges $\{(wv_{k-3}), (v_{k-2}v_{k-1})\}$;

Page $k+1$: the edges $\{(wv_{k-2}), (v_{k-1}v_k)\}$;

Other edges of G are assigned to the same pages as that of G'. It is easy to check that we get a matching book embedding of G in $\Delta(G)$-book. Hence $mbt(G) \leq \Delta(G)$, see Figure 6 (c) for the case $k = 6$.

(II) $deg(w) = k + 1 < \Delta(G)$

If $\Delta(G) > deg(w)$, then $\Delta(G) = \Delta(G')$, i.e. $deg(w) < \Delta(G')$. We can construct a $\Delta(G')$-page matching book embedding of G from that of G' in the same way as the case (I). Hence $mbt(G) \leq \Delta(G') = \Delta(G)$.

Situation 2: $\Delta(G') = 3$.

It is clear that $\Delta(G) = \Delta(w)$. In a similar way as Situation 1(I) we can construct a matching book embedding of G in a $\Delta(G)$-page book. Hence $mbt(G) \leq \Delta(G)$.

This completes the proof.

□

3. Conclusions

In this paper, we study the matching book embedding of a pseudo-Halin graph G and determine its matching book thickness. Specifically, we get the pseudo-Halin graph G is nearly dispersable when $\Delta(G) = 3$ and dispersable otherwise.

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