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The local metric dimension of amalgamation of graphs

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Abstract

For any two adjacent vertices u and v in graph G, a set of vertices W *locally resolves* a graph G if the distance of u and v to some elements of W are distinct. The *local metric dimension* of G is the minimum cardinality of local resolving sets of G. For $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a finite collection of simple connected graphs. The *subgraph-amalgamation* of $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$, denoted by $Subgraph - Amal\{\mathcal{H}; J\}$, is a graph obtained by identifying all elements of $\mathcal{H}$ in J. The subgraph J is called as a *terminal subgraph* of $\mathcal{H}$. In this paper, we determine general bounds of the local metric dimension of subgraph-amalgamation graphs for any connected terminal subgraphs. We also determine the local metric dimension of $Subgraph - Amal\{\mathcal{H}; J\}$ for J is either K_1 or P_2.

Keywords: local basis, local metric dimension, local resolving set, subgraph-amalgamation
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1. Introduction

Throughout this paper, all graphs are finite, simple and connected. Let G be a graph. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The *distance* between two vertices u and v of G, denoted by $d_G(u, v)$, is the length of a shortest path from u to v in G. Let

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$W = \{w_1, w_2, ..., w_k\}$ be a subset of $V(G)$. The *representation* of v with respect to W is defined as the k-tuple $r(v \mid W) = (d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_k))$. The set W is called as a *resolving set* of G if every two distinct vertices u and v of G has different representations. A resolving set with minimum cardinality is called a *basis* of G and its cardinality is called as the *metric dimension* of G, denoted by $dim(G)$.

The metric dimension was first studied by Slater [34] and independently by Harary and Melter [15]. Since then, this topic has been widely investigated. All graphs with certain metric dimension have been studied in [9, 16, 18]. The metric dimension of certain classes of graph is also determined, including trees [15, 19, 9], cycles [9], unicyclic graphs [24], wheels [6, 7, 32], fans [7], Cayley graphs [12], Jahangir graphs [36], honeycomb networks [22], regular graphs [29], Sierpiński graphs [21], amalgamation of graphs [33], and fullerene graphs [2]. This topic also has been arised in many applications, such as network discovery and verification [5], robotic navigation [10, 19], and mastermind [14]. Some other results on metric dimension can be seen in [8, 17, 19, 27, 30, 31, 35, 37]

In this paper, we consider a variance of metric dimension, namely local metric dimension. In this version, two different vertices may have the same representation with respect to an ordered subset W of $V(G)$. In case $r(u \mid W) \neq r(v \mid W)$ for every adjacent vertices u and v in G, then the set W is called a *local resolving set* of G. A local resolving set with minimum cardinality is called a *local basis* of G and its cardinality is called the *local metric dimension* of G, denoted by $lmd(G)$. Since a resolving set of G also a local resolving set of G, then trivially we have $1 \leq lmd(G) \leq dim(G)$.

The local metric dimension problem was introduced by Okamoto *et al.* [23]. They proved that bipartite graphs are the only graphs having local metric dimension one. Moreover, they also showed that $lmd(G) = n - 1$ if and only if G is a complete graph of order n. Furthermore, they also characterized all graphs of order n whose local metric dimension $n - 2$. The local metric dimension of some certain particular graphs also has been determined, including graphs with small clique number [1], torus networks [11], regular graphs [28], block graphs [26], bouquet graphs [26], and split graphs [13].

Determining a relation, in terms of local metric dimension, between the origin graph and the resulting graph under a graph operation is also interesting to be considered. The local metric dimension of Cartesian product graphs has been investigated in [23]. Meanwhile, Rodríguez-Velázquez *et al.* studied the parameter for corona product graphs [25] and rooted product graphs [26]. The lower and upper bounds on the local metric dimension of the generalized hierarchical product are proved in [20]. Barragán-Ramírez *et al.* determined the local metric dimension of lexicographic product graphs [3] and strong product graphs [4].

Now, for $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$, let us consider a simple connected graph H_i containing a connected subgraph J. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a finite collection of simple connected graphs. The *subgraph-amalgamation* of $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$, denoted by $Subgraph - Amal\{\mathcal{H}; J\}$, is a graph obtained by identifying all elements of $\mathcal{H}$ in J. The subgraph J is called as a *terminal subgraph* of $\mathcal{H}$. In this paper, we determine general bounds of the local metric dimension of subgraph-amalgamation graphs for any connected terminal subgraphs. We also determine the local metric dimension of $Subgraph - Amal\{\mathcal{H}; J\}$ for J is either K_1 or P_2.

2. Subgraph Amalgamation

For $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph of order $k_i \geq 2$ containing a connected subgraph J of order p where $1 \leq p < k_i$. Let $V(H_i) = \{h_1, h_2, \ldots, h_p, h_{p+1}, \ldots, h_{k_i}\}$ where $V(J) = \{h_1, h_2, \ldots, h_p\}$.

Now, let us consider $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. In this section, we denote $H \cong Subgraph - Amal\{\mathcal{H}; J\}$. We define $V(H) = V(J) \cup \{h_{(i,j)} | 1 \leq i \leq n,\ p+1 \leq j \leq k_i\}$ and $H_i^\star = V(J) \cup \{h_{(i,j)} | p+1 \leq j \leq k_i\}$.

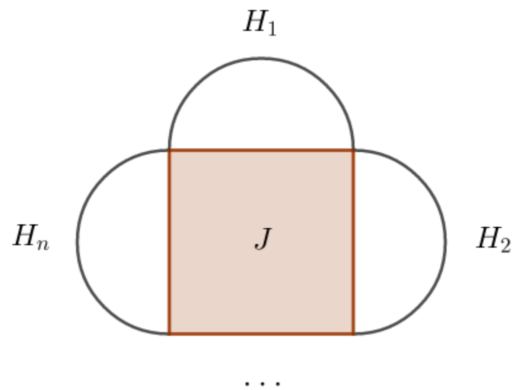

Figure 1. Subgraph amalgamation of $H_1, H_2, \ldots, H_n$

In Lemma 2.1 below, we prove that every graph $G \in \mathcal{H}$ contibutes at least $lmd(G) - p$ vertices in a local basis of H. Meanwhile in Lemma 2.2, we provide a local resolving set of subgraph-amalgamation H, which is union of local basis of every graph in $\mathcal{H}$.

Lemma 2.1. *Let W be a local basis of $H \cong Subgraph - Amal\{\mathcal{H}; J\}$. Then every graph $G \in \mathcal{H}$ contibutes at least $lmd(G) - p$ vertices in W.*

Proof. Suppose that there exists $i \in \{1, 2, \ldots, n\}$ such that H_i contributes at most $lmd(H_i) - p - 1$ vertices in W. Let $W_i = W \cap V(H_i)$ and $S_i = \{h_t \mid h(i,t) \in W_i\}$. Now, we define $A_i = S_i \cup V(J)$. Note that $|A_i| < lmd(H_i)$. Since all vertices of J are in A_i, so there exist two adjacent vertices h_k and h_l in $V(H_i) \setminus V(J)$ satisfying $r(h_k \mid A) = r(h_l \mid A)$, which implies $r(h_{(i,k)} \mid W_i) = r(h_{(i,l)} \mid W_i)$. Since every vertex $z \in V(H) \setminus V(H_i)$ and $u \in V(H_i) \setminus V(J)$ satisfies $d_H(z, u) = d_H(z, v) + d_H(v, u)$ for some $v \in V(J)$, it follows that $r(h_{(i,k)} \mid W) = r(h_{(i,l)} \mid W)$, a contradiction. □

Lemma 2.2. *Let B_i be a local basis of H_i. Then $W = \bigcup_{i=1}^{n} B_i$ be a local resolving set of $H \cong Subgraph - Amal\{\mathcal{H}; J\}$.*

Proof. Let us consider an edge $xy \in E(H)$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $x, y \in V(H_i)$. Since every two adjacent vertices in H_i are locally resolved by some vertices of B_i, we obtain that $W = \bigcup_{i=1}^{n} B_i$ is a local resolving set of H. □

By considering Lemmas 2.1 and 2.2, we obtain the lower and upper bounds for the local metric dimension of subgraph-amalgamation graphs, which can be seen in theorem below.

Theorem 2.1. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Let $|V(J)| = p$. Then*

$$\sum_{i=1}^{n} lmd(H_i) - pn \leq lmd(Subgraph - Amal\{\mathcal{H}; J\}) \leq \sum_{i=1}^{n} lmd(H_i).$$

In the theorem below, we provide a property of subgraph-amalgamation graph H such that its local metric dimension satisfies the upper bound in Theorem 2.1.

Theorem 2.2. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. If every local basis of H_i $(1 \leq i \leq n)$ does not contain all vertices of J and every vertex in J is adjacent to every vertex in $V(H_i) \setminus V(J)$, then $lmd(Subgraph - Amal\{\mathcal{H}; J\}) = \sum_{i=1}^{n} lmd(H_i)$.*

Proof. By Theorem 2.1, we only need to show that $lmd(H) \geq \sum_{i=1}^{n} lmd(H_i)$.

Suppose that $lmd(H) \leq (\sum_{i=1}^{n} lmd(H_i)) - 1$ and W be a local basis of H. Let B_i be a local basis of H_i. Note that B_i does not contain all vertices of J. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $h_{(i,l)} \notin W$ where $h_l \in B_i$. Let $B_i^{'} = B_i \setminus \{h_l\}$. Since $|B_i^{'}| < lmd(H_i)$, there exist two adjacent vertices h_s, h_t in H_i satisfying $r(h_s \mid B_i^{'}) = r(h_t \mid B_i^{'})$. Let $B(i)^{'} = \{h_{(i,k)} \mid h_k \in B_i^{'}\}$. Thus, $r(h_{(i,s)} \mid B(i)^{'}) = r(h_{(i,t)} \mid B(i)^{'})$. Since $d_H(h_{(i,s)}, h_j) = d_H(h_{(i,t)}, h_j)$ for each $h_{(i,s)}, h_{(i,t)} \in V(H) \setminus V(J)$ and $j \in \{1, 2, \ldots, p\}$, we obtain that $r(h_{(i,s)}|W) = r(h_{(i,t)}|W)$, a contradiction. □

In order to provide a property of subgraph-amalgamation graph H such that its local metric dimension satisfies the lower bound in Theorem 2.1, we need to prove Theorem 2.3 below. In this theorem, we give a property of subgraph-amalgamation graph whose local metric dimension is equal to $\sum_{i=1}^{n} lmd(H_i) - cn$ where $0 < c \leq p$. If $c = p$, then we have a subgraph-amalgamation graph H where $lmd(H)$ is equal to the lower bound in Theorem 2.1

Theorem 2.3. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Let $|V(J)| = p \geq 1$ and $V(J) = \{h_1, h_2, \ldots, h_p\}$. Let C be a non-empty subset of $V(J)$ where $C = \{h_1, h_2, \ldots, h_c\}$. Let $lmd(H_i) \geq 2c$ and every local basis B_i of H_i contains $x \in V(J)$ if and only if $x \in C$. If there exists a subset $\{h_{p+1}, h_{p+2}, \ldots, h_{p+c}\}$ of $B_i \setminus C$ such that $d_{H_i}(h_j, h_{p+j}) < d_{H_i}(h_k, h_{p+j})$ for every distinct $j, k \in \{1, 2, \ldots, c\}$, then $lmd(Subgraph - Amal\{\mathcal{H}; J\}) = \sum_{i=1}^{n} lmd(H_i) - cn$.*

Proof. Suppose that W is a local basis of H satisfying $|W| = lmd(H) \leq (\sum_{i=1}^{n} lmd(H_i) - cn) - 1$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that H_i contributes at most $lmd(H_i) - c - 1$ vertices in W. Let $S_i = W \cap V(H_i)$ and $W_i = \{h_j \in V(H_i) \mid h_{(i,j)} \in S_i\}$, then $|W_i| \leq lmd(H_i) - c - 1$. Let $A \subseteq C$ where every element of A, say h_b, locally resolves some two adjacent vertices in $H_i^{\star}$ and there exists $v \in W \setminus H_i^{\star}$ which satisfies $d_H(v, h_b) = min\{d_H(v, w) \mid w \in C\}$. Since every local basis B_i of H_i does not contain all vertices of $V(J) \setminus C$ and $|W_i \cup A| < |B_i|$, there exist two adjacent vertices h_k, h_l in H_i which are not locally resolved by vertices in $W_i \cup A$, which implies

$h_{(i,k)}, h_{(i,l)}$ are not locally resolved by vertices in $S_i \cup A$. It follows that $h_{(i,k)}, h_{(i,l)}$ are not locally resolved by W, a contradiction.

For $i \in \{1, 2, \ldots, n\}$, let $T_i = B_i \setminus C$. We define $U_i = \{h_{(i,l)} \mid h_l \in T_i\}$ and $W = \bigcup_{i=1}^{n} U_i$. We will show that W is a local resolving set of H.

We consider any two adjacent vertices $h_{(i,s)}, h_{(i,t)} \in V(H) \setminus W$ in two following cases.

- There exist h_s, h_t in H_i which are locally resolved by W_i.
  Let $h_l \in T_i$ locally resolves h_s and h_t. Then it is clear that $h_{(i,s)}, h_{(i,t)}$ in H will be locally resolved by $h_{(i,l)} \in W$.

- Every two adjacent vertices h_s, h_t in H_i are not locally resolved by W_i.
  Then there exists $h_j \in C$ such that $d_{H_i}(h_s, h_j) \neq d_{H_i}(h_t, h_j)$. Therefore, we have $d_H(h_{(i,s)}, h_j) \neq d_H(h_{(i,t)}, h_j)$. We consider $h_{(m,p+j)} \in W$ where $m \in \{1, 2, \ldots, n\} \setminus \{i\}$ and $d_{H_i}(h_j, h_{(p+j)}) < d_{H_i}(h_k, h_{(p+j)})$ for every distinct $j, k \in \{1, 2, \ldots, p\}$. We obtain

$$\begin{aligned} d_H(h_{(i,s)}, h_{(m,p+j)}) &= d_H(h_{(i,s)}, h_j) + d_H(h_j, h_{(m,p+j)}) \\ &= d_H(h_{(i,s)}, h_j) + d_{H_i}(h_j, h_{p+j}) \\ &\neq d_H(h_{(i,t)}, h_j) + d_{H_i}(h_j, h_{p+j}) \\ &= d_H(h_{(i,t)}, h_j) + d_H(h_{(j)}, h_{(m,p+j)}) \\ &= d_H(h_{(i,t)}, h_{(m,p+j)}). \end{aligned}$$

  Then $h_{(i,s)}, h_{(i,t)}$ in H are locally resolved by $h_{(m,p+j)} \in W$.

Therefore, W is a local resolving set of H. □

Note that, in Theorem 2.3 above, we consider a graph $Subgraph - Amal\{\mathcal{H}; J\}$ where every local basis of $H_i \in \mathcal{H}$ $(1 \leq i \leq n)$ contains the same exactly c vertices of J. In theorem below, we can prove that the upper bound of the local metric dimension of such subgraph-amalgamation graph is less than the upper bound in Theorem 2.1.

Theorem 2.4. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Let $|V(J)| = p \geq 1$ and C be a non-empty subset of $V(J)$ where $|C| = c$. Let every local basis B_i of H_i contains $x \in V(J)$ if and only if $x \in C$. Then*

$$lmd(Subgraph - Amal\{\mathcal{H}; J\}) \leq \sum_{i=1}^{n} lmd(H_i) - cn + c.$$

Proof. Let us consider any edges $xy \in E(H)$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $x, y \in V(H_i)$. For $i \in \{1, 2, \ldots, n\}$, let B_i be a local basis of H_i. Since every two adjacent vertices in H_i are locally resolved by some vertices in B_i, we obtain that $W = \bigcup_{i=1}^{n} B_i$ is a local resolving set of H. Since $|B_i \cap B_j| = c$ for distinct $i, j \in \{1, 2, \ldots, n\}$, we have $|W| = \sum_{i=1}^{n} |B_i| - (n-1)c$. Therefore, we obtain

$$lmd(H) \leq \sum_{i=1}^{n} lmd(H_i) - cn + c.$$

□

Next, we will show that the upper bound in Theorem 2.4 is sharp. In order to do so, first, we need to determine the local metric dimension of graph G_1, which can be seen in lemma below. The graph G_1 and its complement are shown in Figure 2.

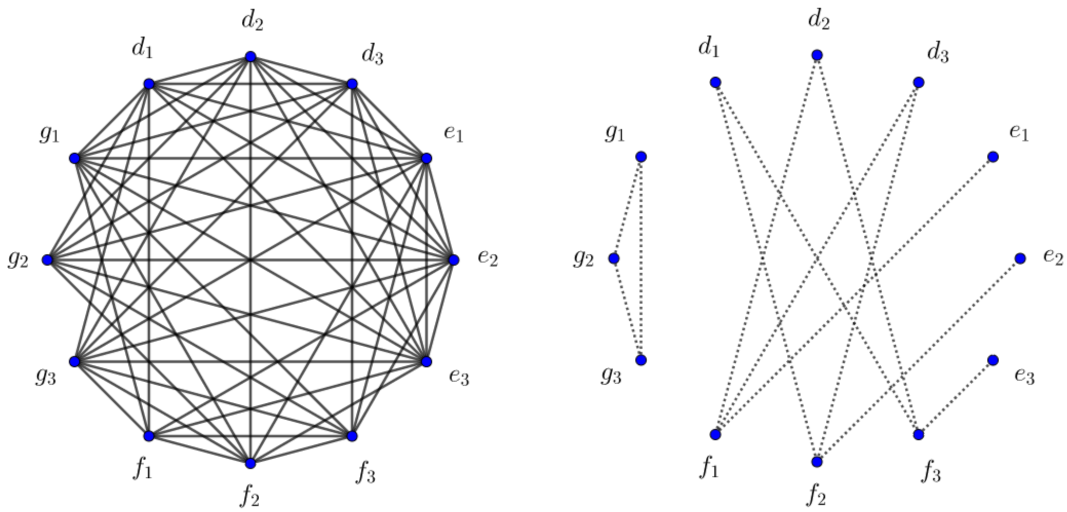

Figure 2. Graph G_1 (left) and its complement (right)

Lemma 2.3. *Let G_1 be a connected graph as stated in Figure 2. Then $lmd(G_1) = 3$. Moreover, the set $\{f_1, f_2, f_3\}$ is the only local basis of G_1.*

Proof. Let $D = \{d_i \mid i \in \{1, 2, 3\}\}$, $E = \{e_i \mid i \in \{1, 2, 3\}\}$, $F = \{f_i \mid i \in \{1, 2, 3\}\}$, and $G = \{g_i \mid i \in \{1, 2, 3\}\}$. Since G_1 contains an odd cycle, we have $lmd(G_1) > 1$. We will show that there is no local resolving set of G_1 whose cardinality is 2. Suppose that W be a local resolving set of G_1 with $|W| = 2$. We distinguish four cases.

Case 1. $W \subset D \cup F$
Since $|W| = 2$, there exists $i \in \{1, 2, 3\}$ such that $f_i \notin W$. So, we obtain that $r(e_i \mid W) = (1, 1) = r(g_i \mid W)$.

Case 2. $W \subset D \cup E \cup G$
Then $D \setminus W \neq \emptyset$ and $E \setminus W \neq \emptyset$. Let $d_i, e_j \notin W$. Note that d_i and e_j are adjacent. So, we obtain that $r(d_i \mid W) = (1, 1) = r(e_j \mid W)$.

Case 3. $W = \{e_i, f_j\}$ where $i, j \in \{1, 2, 3\}$.
Then two adjacent vertices f_k, g_l satisfy $r(f_k \mid W) = (1, 1) = r(g_l \mid W)$ with $k, l \in \{1, 2, 3\}$ and $k \neq i,\ l \neq j$.

Case 4. $W = \{f_i, g_j\}$ where $i, j \in \{1, 2, 3\}$.
Then two adjacent vertices e_k, e_l satisfy $r(e_k \mid W) = (1, 1) = r(e_l \mid W)$ with $k, l \in \{1, 2, 3\} \setminus \{i\}$

and $k \neq l$.

By all cases above, we conclude that W is not local resolving set of G_1. Since there is no local resolving set of G_1 with 2 elements, we obtain that $lmd(G_1) \geq 3$.

Now, we will construct a local resolving set of G_1 with 3 vertices. Define $B = \{f_1, f_2, f_3\}$. The representation of all vertices in G_1 with respect to B are as follows.

$$\begin{array}{lll} r(d_1|B) = (1,2,2) & r(e_2|B) = (1,2,1) & r(f_3|B) = (1,1,0) \\ r(d_2|B) = (2,1,2) & r(e_3|B) = (1,1,2) & r(g_1|B) = (1,1,1) \\ r(d_3|B) = (2,2,1) & r(f_1|B) = (0,1,1) & r(g_2|B) = (1,1,1) \\ r(e_1|B) = (2,1,1) & r(f_2|B) = (1,0,1) & r(g_3|B) = (1,1,1) \end{array}$$

Since there are no two adjacent vertices of G_1 having the same representation, we obtain that B is a local resolving set of G_1, which implies $lmd(G_1) \leq 3$.

Next, we will show that there is no local resolving set of G_1 with 3 vertices, except $\{f_1, f_2, f_3\}$. Let $W' \subset V(G_1)$ with $|W'| = 3$ and W' contains q vertices in F with $0 \leq q \leq 2$. We distinguish three cases.

Case 1. $q = 0$

- $W' \cap G = \emptyset$
  If $W' = E$, then two adjacent vertices $d \in D$ and $g \in G$ satisfy $r(d \mid W') = (1,1,1) = r(g \mid W')$. Otherwise, let $e_i \notin W$ $(i \in \{1,2,3\})$. Then we obtain $r(e_i \mid W') = (1,1,1) = r(g_i \mid W')$.

- $W' \cap G \neq \emptyset$
  Then there exist $d \in D$ and $e \in E$ which are not element of W'. Then we obtain $r(d \mid W') = (1,1,1) = r(e \mid W')$.

Case 2. $q = 1$

- $W' \cap G = \emptyset$
  If $W' \cap D = \emptyset$, then two adjacent vertices $d \in D$ and $g \in G$ satisfy $r(d \mid W') = (1,1,1) = r(g \mid W')$. Otherwise, let $e_i \notin W$ $(i \in \{1,2,3\})$. Then we obtain $r(e_i \mid W') = (1,1,1) = r(g_i \mid W')$.

- $W' \cap G \neq \emptyset$
  If $W' \cap E = \emptyset$, then there exist two distinct vertices of E which are adjacent to W'. Otherwise, all three vertices of D are adjacent to W'.

Case 3. $q = 2$
Let two distinct vertices $f_j, f_k \in W'$ where $j, k \in \{1,2,3\}$.

- $W' \cap D \neq \emptyset$
  Then for $l \in \{1,2,3\} \setminus \{j,k\}$, we have e_l is adjacent to W'. So, for $g \in G$, we obtain $r(e_l \mid W') = (1,1,1) = r(g \mid W')$.

- $W' \cap G \neq \emptyset$
  Then two adjacent vertices d_j, e_k satisfy $r(d_j \mid W') = r(e_k \mid W')$.
- $W' \cap E \neq \emptyset$
  Let $e_i \in W'$. If $i = j$, then two adjacent vertices d_j, e_k satisfy $r(d_j \mid W') = r(e_k \mid W')$. Otherwise, two adjacent vertices d_k, e_j satisfy $r(d_k \mid W') = r(e_j \mid W')$.

By all cases above, W' is not local resolving set of G_1. □

Now, we are ready to give an existence of $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ and a terminal subgraph J of H_i, such that the local metric dimension of $Subgraph - Amal\{\mathcal{H}; J\}$ is equal to the upper bound of Theorem 2.4.

Theorem 2.5. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let $H_i \cong G_1$, J be a terminal subgraph of H_i with $V(J) = \{f_1, f_2, e_3\}$, and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Then $lmd(Subgraph - Amal\{\mathcal{H}; J\}) = n + 2$.*

Proof. In this case, $C = \{f_1, f_2\}$ and $c = |C| = 2$. By Theorem 2.4, $lmd(H) \leq \sum_{i=1}^{n} lmd(H_i) - cn + c = n + 2$. Now, we will prove that $lmd(H) \geq \sum_{i=1}^{n} lmd(H_i) - cn + c = n + 2$.

Let $x_{(i,j)} \in V(H) \setminus V(J)$ whenever $x_j \in V(H_i) \setminus V(J)$ and W be a local resolving set of H. We consider two conditions below.

1. Two adjacent vertices d_1 and e_2 are not locally resolved by J. It follows that $d_{(i,1)}$ and $e_{(i,2)}$ in H are not locally resolved by $V(H) \setminus H_i^\star$. Since f_3 is an element of local basis of G_1 and f_3 locally resolves d_1 and e_2, we obtain that $f_{(i,3)}$ locally resolves $d_{(i,1)}$ and $e_{(i,2)}$. Thus, $f_{(i,3)} \in W$ for $1 \leq i \leq n$.
2. For $j \in \{1, 2\}$, two adjacent vertices e_j and g_j are not locally resolved by $J \setminus \{f_j\}$. According to Lemma 2.3, f_j is an element of a local basis of G_1. Note that if $f_j \notin W$, then $e_{(i,j)}$ and $g_{(i,j)}$ are not locally resolved by $V(H) \setminus H_i^\star$ since $d_H(v, e_{(i,j)}) < d_H(v, f_j) + d_H(f_j, e_{(i,j)})$ and $d_H(v, g_{(i,j)}) = d_H(v, f_j) + d_H(f_j, g_{(i,j)})$ with $v \in V(H) \setminus H_i^\star$. So, f_j must be in W for $j \in \{1, 2\}$.

By two conditions above, we obtain that $lmd(H) \geq n + 2$. □

In the next theorem, we will provide an existence of $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ and a terminal subgraph J of H_i, such that the local metric dimension of $Subgraph - Amal\{\mathcal{H}, J\}$ is not equal to both upper bound of Theorem 2.4 and lower bound of Theorem 2.1. In order to do so, first, we need to determine the local metric dimension of graph G_2, which can be seen in Lemma 2.4 below. The lemma is proved by using the similar argument of Lemma 2.3. Meanwhile, the graph G_2 and its complement are shown in Figure 2. Note that, the graph G_2 can be obtained from G_1 by adding an edge $f_1 f_3$.

Lemma 2.4. *Let G_2 be a connected graph as stated in Figure 3. Then $lmd(G_2) = 3$. Moreover, the set $\{f_1, f_2, f_3\}$ is the only local basis of G_2.*

Theorem 2.6. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let $H_i \cong G_2$, J be a terminal subgraph of H_i with $V(J) = \{f_1, f_2, e_3\}$, and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Then $lmd(Subgraph - Amal\{\mathcal{H}; J\}) = n + 1$.*

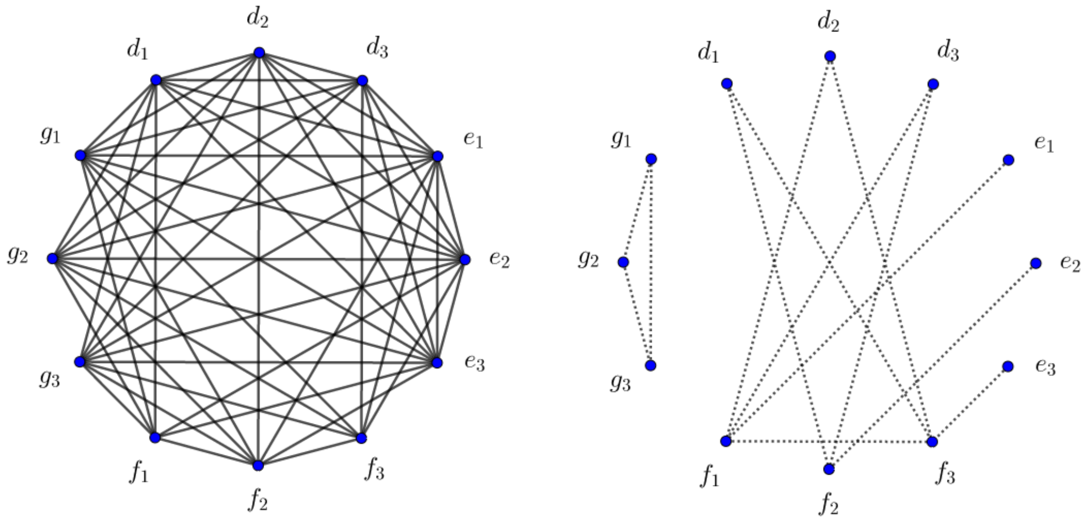

Figure 3. Graph G_2 (left) and its complement (right)

Proof. In this case, $C = \{f_1, f_2\}$ and $c = |C| = 2$. Note that, $\sum_{i=1}^{n} lmd(H_i) - cn = n < n+1 < \sum_{i=1}^{n} lmd(H_i) - cn + c = n + 2$.

Now, we will prove that $lmd(H) \geq n + 1$. Let $x_{(i,j)} \in V(H) \setminus V(J)$ whenever $x_j \in V(H_i) \setminus V(J)$ and W be a local resolving set of H. We consider two conditions below.

1. Two adjacent vertices d_1 and e_2 are not locally resolved by J. It follows that $d_{(i,1)}$ and $e_{(i,2)}$ in H are not locally resolved by $V(H) \setminus H_i^\star$. Since f_3 is an element of local basis of G_2 and f_3 locally resolves d_1 and e_2, we obtain that $f_{(i,3)}$ locally resolves $d_{(i,1)}$ and $e_{(i,2)}$. Thus, it must be $f_{(i,3)} \in W$ for $1 \leq i \leq n$.
2. Two adjacent vertices e_1 and g_1 are not locally resolved by $J \setminus \{f_1\}$. According to Lemma 2, f_1 is an element of a local basis of G_2. Note that, if $f_1 \notin W$, then $e_{(i,1)}$ and $g_{(i,1)}$ are not locally resolved by $V(H) \setminus H_i^\star$ since $d_H(v, e_{(i,1)}) < d_H(v, f_1) + d_H(f_1, e_{(i,1)})$ and $d_H(v, g_{(i,1)}) = d_H(v, f_1) + d_H(f_1, g_{(i,1)})$ with $v \in V(H) \setminus H_i^\star$. So, f_1 must be in W.

By two conditions above, we obtain that $lmd(H) \geq n + 1$.

Now, we will construct a local resolving set of H with $n+1$ vertices. Define $B = \{f_1\} \cup \{f_{(i,3)} \mid 1 \leq i \leq n\}$. The representation of all vertices in H with respect to B are as follows.

$$\begin{array}{ll}
r(f_1 \mid B) = (0,2,\ldots,2) & r(e_{(2,1)} \mid B) = (2,2,1,2,\ldots,2) \\
r(f_2 \mid B) = (1,1,\ldots,1) & r(e_{(2,2)} \mid B) = (1,3,1,3,\ldots,3) \\
r(e_3 \mid B) = (1,2,\ldots,2) & r(f_{(2,3)} \mid B) = (2,2,0,2,\ldots,2) \\
r(d_{(1,1)} \mid B) = (1,2,3,\ldots,3) & r(g_{(2,1)} \mid B) = (1,2,1,2,\ldots,2) \\
r(d_{(1,2)} \mid B) = (2,2,2,\ldots,2) & r(g_{(2,2)} \mid B) = (1,2,1,2,\ldots,2) \\
r(d_{(1,3)} \mid B) = (2,1,3,\ldots,3) & r(g_{(2,3)} \mid B) = (1,2,1,2,\ldots,2) \\
r(e_{(1,1)} \mid B) = (2,1,2,\ldots,2) & r(d_{(k,1)} \mid B) = (1,3,3,\ldots,3,2,3,\ldots,3) \\
r(e_{(1,2)} \mid B) = (1,1,3,\ldots,3) & r(d_{(k,2)} \mid B) = (2,2,2,\ldots,2,2,2,\ldots,2) \\
r(f_{(1,3)} \mid B) = (2,0,2,\ldots,2) & r(d_{(k,3)} \mid B) = (2,3,3,\ldots,3,1,3,\ldots,3) \\
r(g_{(1,1)} \mid B) = (1,1,2,\ldots,2) & r(e_{(k,1)} \mid B) = (2,2,2,\ldots,2,1,2,\ldots,2) \\
r(g_{(1,2)} \mid B) = (1,1,2,\ldots,2) & r(e_{(k,2)} \mid B) = (1,3,3,\ldots,3,1,3,\ldots,3) \\
r(g_{(1,3)} \mid B) = (1,1,2,\ldots,2) & r(f_{(k,3)} \mid B) = (2,2,2,\ldots,2,0,2,\ldots,2) \\
r(d_{(2,1)} \mid B) = (1,3,2,3,\ldots,3) & r(g_{(k,1)} \mid B) = (1,2,2,\ldots,2,1,2,\ldots,2) \\
r(d_{(2,2)} \mid B) = (2,2,2,2,\ldots,2) & r(g_{(k,2)} \mid B) = (1,2,2,\ldots,2,1,2,\ldots,2) \\
r(d_{(2,3)} \mid B) = (2,3,1,3,\ldots,3) & r(g_{(k,3)} \mid B) = (1,2,2,\ldots,2,1,2,\ldots,2)
\end{array}$$

where $k \in \{3,4,\ldots,n\}$ and $d_H(v_{(k,l)}, f_{(i,3)}) = d_{G_2}(v_l, f_3)$ for each $v_l \in D \cup E \cup F \cup G$ where $i = k$. Since there are no two adjacent vertices of H having the same representations, we obtain that B is a local resolving set of H, which implies $lmd(H) \leq n+1$. □

3. Vertex Amalgamation

For $n \in \mathbb{N}$ and $i \in \{1,2,\ldots,n\}$, let H_i be a simple connected graph of order $k_i \geq 2$ containing a connected subgraph J of order p where $1 \leq p < k_i$. In this section, let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ where J only consists of one vertex. The graph $Subgraph-Amal\{\mathcal{H}, J\}$ then is called as a *vertex amalgamation* graph.

Let $V(H_i) = \{h, h_2, \ldots, h_{k_i}\}$ where $V(J) = \{h\}$. In this section, we denote $H \cong Subgraph-Amal\{\mathcal{H}; J\}$. We define $V(H) = \{h\} \cup \{h_{(i,j)} | 1 \leq i \leq n,\ 2 \leq j \leq k_i\}$, $H(i) = \{h_{(i,j)} | 2 \leq j \leq k_i\}$, and $H_i^\star = \{h\} \cup H(i)$. We also define:

- $\mathcal{S} = \{A \in \mathcal{H} \mid A$ is not bipartite and there exists a local basis of A containing $h\}$
- $\mathcal{T} = \{A \in \mathcal{H} \mid A$ is not bipartite and every local basis of A does not contain $h\}$

From now on, let $s = |\mathcal{S}|$, $t = |\mathcal{T}|$, and $\mathcal{H} = \{H_1, H_2, \ldots, H_s, H_{s+1}, \ldots, H_{s+t}, H_{s+t+1}, \ldots, H_n\}$ where $\mathcal{S} = \{H_1, H_2, \ldots, H_s\}$ and $\mathcal{T} = \{H_{s+1}, H_{s+2}, \ldots, H_{s+t}\}$.

Proposition 3.1. *For $s + t \geq 1$, there exist two adjacent vertices $h_{(i,j)}$ and $h_{(i,k)}$ satisfying $d_H(h_{(i,j)}, h) = d_H(h_{(i,k)}, h)$ where $i \in \{1,2,\ldots,s+t\}$ and $j,k \in \{2,3,\ldots,k_i\}$.*

Proof. Since $s + t \geq 1$, for $i \in \{1,2,\ldots,s+t\}$, we have H_i is not bipartite and contains an odd cycle C. We distinguish two cases.

Case 1. $h \in V(C)$
Then there exist two adjacent vertices $h_j, h_k \in V(C) \setminus \{h\}$ satisfying $d_{H_i}(h_j, h) = d_{H_i}(h_k, h)$. We

obtain that $d_H(h_{(i,j)}, h) = d_H(h_{(i,k)}, h)$.

Case 2. $h \notin V(C)$
Let $h_l \in V(C)$ such that $d_{H_i}(h_l, h) = min\{d_{H_i}(v, h)|v \in V(C)\}$. Then there exist two adjacent vertices $h_j, h_k \in V(C) \setminus \{h_l\}$ satisfying $d_{H_i}(h_j, h_l) = d_{H_i}(h_k, h_l)$ and

$$\begin{aligned} d_{H_i}(h_j, h) &= d_{H_i}(h_j, h_l) + d_{H_i}(h_l, h) \\ &= d_{H_i}(h_k, h_l) + d_{H_i}(h_l, h) \\ &= d_{H_i}(h_k, h). \end{aligned}$$

Therefore, we obtain that $d_H(h_{(i,j)}, h) = d_H(h_{(i,k)}, h)$. □

In the next two lemmas, we provide some properties of local basis of the vertex amalgamation graph H.

Lemma 3.1. *Let W be a local basis of H. If $s + t \geq 1$, then*

(i) $W \cap H(i) \neq \emptyset$ for every $i \in \{1, 2, \ldots, s + t\}$; and
(ii) $W \cap H(j) = \emptyset$ for every $j \in \{s + t + 1, s + t + 2, \ldots, n\}$.

Proof. We distinguish two cases of proof.

(i) $W \cap H(i) \neq \emptyset$ for every $i \in \{1, 2, \ldots, s + t\}$
Suppose that $W \cap H(i) = \emptyset$ for some $i \in \{1, 2, \ldots, s + t\}$. Let $w \in W$. Note that $w \notin H(i)$. By Proposition 3.1, there exist two adjacent vertices $h_{(i,j)}$ and $h_{(i,k)}$ in H satisfying $d_H(h_{(i,j)}, h) = d_H(h_{(i,k)}, h)$. So, we have

$$\begin{aligned} d_H(h_{(i,j)}, w) &= d_H(h_{(i,j)}, h) + d_H(h, w) \\ &= d_H(h_{(i,k)}, h) + d_H(h, w) \\ &= d_H(h_{(i,k)}, w) \end{aligned}$$

Therefore, we obtain $r(h_{(i,j)} \mid W) = r(h_{(i,k)} \mid W)$, a contradiction.

(ii) $W \cap H(j) = \emptyset$ for every $j \in \{s + t + 1, s + t + 2, \ldots, n\}$
Suppose that there exists $j \in \{s + t + 1, s + t + 2, \ldots, n\}$ such that $W \cap H(j) \neq \emptyset$. Let $w \in H(j)$ be an element of W. We consider $W' = W \setminus \{w\}$. We will show that W' is still a local resolving set of H.
By considering (i), let $S = \{x \in W \mid x \in H(i),\ 1 \leq i \leq s + t\}$. Note that, by the definition of W', we also have that $S \subseteq W'$. Let x, y be two adjacent vertices in $V(H) \setminus W'$. So, there exists $i \in \{1, 2, \ldots, n\}$ such that $x, y \in H_i^\star$. We distinguish two cases.

Case 1. $i \in \{1, 2, \ldots, s + t\}$.
By (i), we have that x, y are locally resolved by vertices in S. It follows that both vertices are locally resolved by W'.

Case 2. $i \in \{s+t+1, s+t+2, \ldots, n\}$.
Since every H_i with $i \in \{s+t+1, s+t+2, \ldots, n\}$ is bipartite, we have $lmd(H_i) = 1$. Let B_i be a local basis of H_i. Then for every $v \in V(H_i)$, there exists B_i where $B_i = \{v\}$. Thus, $d_{H_i}(x', v) \neq d_{H_i}(y', v)$ for every two adjacent vertices $x', y' \in V(H_i) \setminus \{v\}$. Let $v = h$, we obtain that $d_{H_i}(x', h) \neq d_{H_i}(y', h)$. Consider two vertices $x, y \in H_i^\star$ which are corresponded to $x', y' \in V(H_i) \setminus \{v\}$, respectively. For any $z \in W'$, we obtain that

$$\begin{aligned} d_H(x, z) &= d_H(x, h) + d_H(h, z) \\ &= d_{H_i}(x', h) + d_H(h, z) \\ &\neq d_{H_i}(y', h) + d_H(h, z) \\ &= d_H(y, h) + d_H(h, z) \\ &= d_H(y, z). \end{aligned}$$

So, $r(x|W') \neq r(y|W')$. Then every two adjacent vertices $x, y \in V(H) \setminus W'$ are locally resolved by vertices in W'. In both cases, we obtain that W' is a local resolving set of H, a contradiction.

□

Lemma 3.2. *Let W be a local basis of H. If $s > 1$ or $t \geq 1$, then $h \notin W$.*

Proof. Let W be a local basis of H where $h \in W$. We consider $W' = W \setminus \{h\}$. We will show that W' is also a local resolving set of H.

Let x and y be two adjacent vertices in $V(H) \setminus W'$. By considering properties in Lemma 3.1, we have that x and y are locally resolved by vertices in W'. Thus, we have a contradiction. □

From proposition and lemmas above we have the following theorem.

Theorem 3.1. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph of order at least 2 containing a subgraph J and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Let $J = K_1$ where $V(J) = \{h\}$. Let $\mathcal{S} = \{A \in \mathcal{H} \mid A$ is not bipartite and there exists a local basis of A containing $h\}$ and $\mathcal{T} = \{A \in \mathcal{H} \mid A$ is not bipartite and every local basis of A does not contain $h\}$. If $s = |\mathcal{S}|$, $t = |\mathcal{T}|$ and the first $s + t$ elements in $\mathcal{H}$ are from $\mathcal{S} \cup \mathcal{T}$, then*

$$lmd(Subgraph - Amal\{\mathcal{H}; J\}) = \begin{cases} 1, & s = t = 0; \\ lmd(H_s), & s = 1 \text{ and } t = 0; \\ \sum_{i=1}^{s+t} lmd(H_i) - s, & s \geq 2 \text{ or } t \geq 1. \end{cases}$$

Proof. Let $H \cong Subgraph - Amal\{\mathcal{H}; J\}$. If $s = t = 0$, then every graph H_i is bipartite. It follows that H is also bipartite. Okamoto *et al.* [23] have been proved that the local metric dimension of any bipartite graphs is 1.

Now, assume $s \geq 1$ or $t \geq 1$. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_s, H_{s+1}, \ldots, H_{s+t}, H_{s+t+1}, \ldots, H_n\}$ where $\mathcal{S} = \{H_1, H_2, \ldots, H_s\}$ and $\mathcal{T} = \{H_{s+1}, H_{s+2}, \ldots, H_{s+t}\}$. We have two cases.

Case 1. $s = 1$ and $t = 0$.
First, we will show that $lmd(H) \leq lmd(H_s)$ by constructing a local resolving set with $lmd(H_s)$ vertices. Let B_s be a local basis of H_s containing h. We define $B_s^{'} = B_s \setminus \{h\}$ and $W = \{h_{(s,t)} \mid h_t \in B_s^{'}\} \cup \{h\}$. Let us consider any two adjacent vertices $h_{(i,j)}, h_{(i,k)} \in V(H) \setminus W$. If $i = s = 1$, then h_j and h_k are locally resolved by B_s, which implies $h_{(1,j)}, h_{(1,k)}$ are locally resolved by W. Now, we assume that $i \in \{2, 3, \ldots, n\}$. We obtain that H_i is a bipartite graph. Since there exists a local basis of H_i containing exactly one vertex h, then $h_{(i,j)}, h_{(i,k)}$ of H are locally resolved by $h \in W$. Thus, W is a local resolving set of H.

Next, we will show that $lmd(H) \geq lmd(H_s)$. Suppose that $lmd(H) \leq (lmd(H_s)) - 1$. Let S be a local basis of H. So, $|S| \leq lmd(H_s) - 1$. By Lemma 3.1 (ii), $S \cap H_i = \emptyset,\ i \in \{2, 3, \ldots, n\}$. We define a subset U of $V(H_s)$ as $\{h\} \cup \{h_j \mid h_{(s,j)} \in S\}$. Since $|U| = |S| \leq lmd(H_s) - 1$, there exist two adjacent vertices $h_k, h_l \in V(H_s)$ satisfying $r(h_k \mid U) = r(h_l \mid U)$. It follows that $r(h_{(s,k)} \mid S) = r(h_{(s,l)} \mid S)$, a contradiction.

Case 2. $s \geq 2$ or $t \geq 1$.
First, we will show that $lmd(H) \leq \sum_{i=1}^{s+t} lmd(H_i) - s$ by constructing a local resolving set with $\sum_{i=1}^{s+t} lmd(H_i) - s$ vertices. Since $s \geq 2$ or $t \geq 1$ for $i \in \{1, 2, \ldots, s+t\}$, H_i is not bipartite and $lmd(H_i) > 1$.

- $s \geq 1$ and $i \in \{1, 2, \ldots, s\}$
  Let B_i be a local basis of H_i containing h. Then we define $W_i = \{h_{(i,j)} | h_j \in B_i \setminus \{h\}\}$.
- $t \geq 1$ and $i \in \{s+1, s+2, \ldots, s+t\}$
  Let C_i be a local basis of H_i. We define $W_i = \{h_{(i,j)} | h_j \in C_i\}$.

Now, define $W = \bigcup_{i=1}^{s+t} W_i$. Note that W satisfies Lemmas 3.1 and 3.2. Let us consider any two adjacent vertices $x, y \in V(H) \setminus W$. Note that there exists $i \in \{1, 2, \ldots, n\}$ such that $x, y \in H_i^{\star}$.

- $s \neq 0$ and $i \in \{1, 2, \ldots, s\}$

  If $d_H(x, h) \neq d_H(y, h)$, then it is clear that x, y are locally resolved by vertices in $W \setminus W_i$. Otherwise, x, y are locally resolved by vertices in W_i.

- $t \neq 0$ and $i \in \{s+1, s+2, \ldots, s+t\}$

  Then x, y are locally resolved by vertices in W_i.

- $s + t < n$ and $i \in \{s+t+1, s+t+2, \ldots, n\}$

  Since x, y are locally resolved by the vertex h, it implies that they are locally resolved by all vertices in W.

Therefore, W is a local resolving set of H.

Next, suppose that $lmd(H) \leq (\sum_{i=1}^{s+t} lmd(H_i) - s) - 1$. Let W be a local basis of H. By Lemma 3.1 (ii) and Lemma 3.2, we have $W \cap H(j) = \emptyset$ for every $j \in \{s+t+1, s+t+2, \ldots, n\}$ and $h \notin W$. Since $|W| \leq \sum_{i=1}^{s+t} lmd(H_i) - s - 1$, we obtain two possibilities.

1. There exists $i \in \{1, 2, \dots, s\}$ such that $|W \cap H_i^\star| \leq lmd(H_i) - 2$
   Let $W_i = W \cap H_i^\star$ and $S_i = \{h_j \mid h_{(i,j)} \in W_i\} \cup \{h\}$. Since $|S_i| < lmd(H_i)$, then there exists two adjacent vertices h_k, h_l in H_i which are not locally resolved by S_i. It follows that $h_{(i,k)}, h_{(i,l)}$ are not locally resolved by W_i, which implies $r(h_{(i,k)} \mid W) = r(h_{(i,l)} \mid W)$.
2. There exists $i \in \{s+1, s+2, \dots, s+t\}$ such that $|W \cap H_i^\star| \leq lmd(H_j) - 1$
   Let $W_i = W \cap H_i^\star$ and $S_i = \{h_j \mid h_{(i,j)} \in W_i\} \cup \{h\}$. Note that, $|S_i| = lmd(H_i)$. However, every local basis of H_i does not contain h. It follows that S_i is not a local resolving set of H_i. Then there exists two adjacent vertices h_k, h_l in H_i which are not locally resolved by S_i. It follows that $h_{(i,k)}, h_{(i,l)}$ are not locally resolved by W_i, which implies $r(h_{(i,k)} \mid W) = r(h_{(i,l)} \mid W)$.

By both possibilities above, we have a contradiction. □

4. Edge Amalgamation

For $n \in \mathbb{N}$ and $i \in \{1, 2, \dots, n\}$, let H_i be a simple connected graph of order $k_i \geq 2$ containing a connected subgraph J of order p where $1 \leq p < k_i$. In this section, let $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ where $J \cong P_2$. Since P_2 has only one edge, the graph $Subgraph - Amal\{\mathcal{H}, J\}$ then is called as an *edge amalgamation* graph.

Let $V(H_i) = \{h_1, h_2, \dots, h_{k_i}\}$ where $V(J) = \{h_1, h_2\}$. In this section, we denote $H \cong Subgraph - Amal\{\mathcal{H}; J\}$. We define $V(H) = \{h_1, h_2\} \cup \{h_{(i,j)} | 1 \leq i \leq n,\ 3 \leq j \leq k_i\}$, $H(i) = \{h_{(i,j)} | 3 \leq j \leq k_i\}$, and $H_i^\star = \{h_1, h_2\} \cup H(i)$.

According to Theorem 2.1, we obtain the bounds for the local metric dimension of any edge amalgamation graphs, which can be seen in theorem below.

Theorem 4.1. *For $n \geq 2$ and $i \in \{1, 2, \dots, n\}$, let H_i be a simple connected graph of order at least 3 and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$. Then*

$$\sum_{i=1}^{n} lmd(H_i) - 2n \leq lmd(Subgraph - Amal\{\mathcal{H}; P_2\}) \leq \sum_{i=1}^{n} lmd(H_i).$$

In this section, we will show that all values between the lower and upper bound in Theorem 4.1 are achievable. In order to do so, we need to determine the local metric dimension of graphs G_3, G_4, and G_5. The graph G_3 and its complement are illustrated in Figures 4. Meanwhile the graphs G_4 and G_5 can be seen in Figures 5 and 6, respectively.

First, let us consider the graph G_3. Let $N_3 = \{a_1, a_2\}$, $O_3 = \{a_3, a_4\}$, $P_3 = \{a_5, a_6\}$, $R_3 = \{a_7, a_8\}$, and $S_3 = \{a_9, a_{10}, a_{11}, a_{12}\}$. Now, we are ready to determine the local metric dimension of G_3.

Lemma 4.1. *Let G_3 be a connected graph as stated in Figure 4. Then $lmd(G_3) = 4$. Moreover the set $\{a_1, a_2, a_7, a_8\}$ is a local basis of G_3.*

Proof. Since G_3 contains an odd cycle, we have $lmd(G_3) > 1$. Next, we will show that there is no local resolving set of G_3 with 2 elements. Suppose that W be a local resolving set of G_3 with

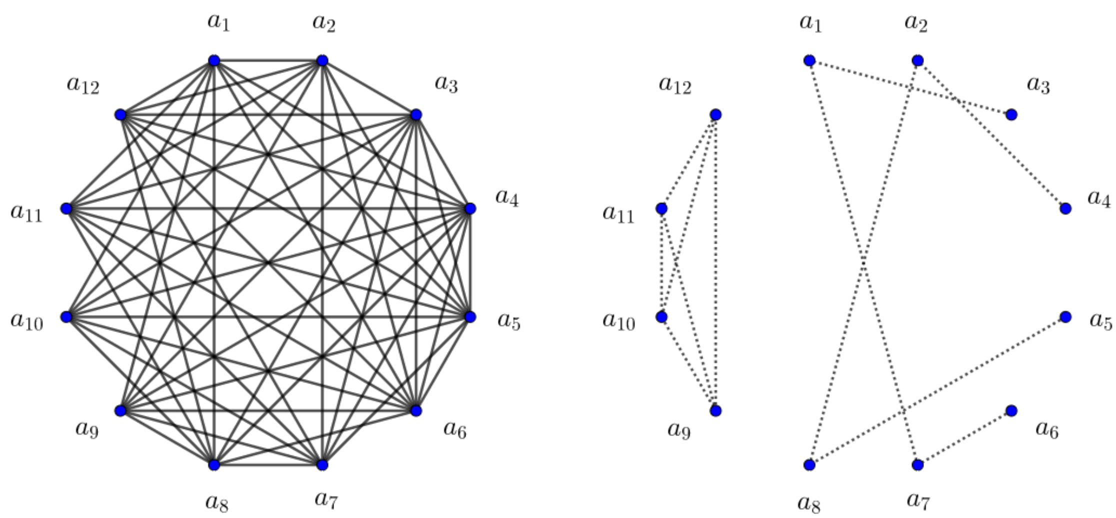

Figure 4. Graph G_3 (left) and its complement (right)

$|W| = 2$. We obtain two cases.

Case 1. $W \cap S_3 = \emptyset$.
Then there exist two distinct vertices $x, y \in O_3 \cup P_3$ which are adjacent to both vertices in W. Therefore, we have $r(x \mid W) = (1, 1) = r(y \mid W)$.

Case 2. $W \cap S_3 \neq \emptyset$.
Let $W = \{x, y\}$ where $x \in S_3$. If $y \in N_3 \cup O_3$, then we have $r(a_5 \mid W) = (1, 1) = r(a_6 \mid W)$. Otherwise, we have $r(a_3 \mid W) = (1, 1) = r(a_4 \mid W)$. By all cases above, we obtain that W is not local resolving set of G_3. Therefore, we can conclude that $lmd(G_3) \geq 3$. However, we will also show that there is no local resolving set of G_3 with cardinality 3. Suppose that W' be a local resolving set of G_3 with $|W'| = 3$. Note that, at least one vertices of S_3 are not in W'. Without loss of generality, let $a_9 \notin W'$. Let $|W' \cap S_3| = j$ where $j \in \{0, 1, 2, 3\}$. Then there exist $j + 1$ vertices $z_1, \ldots, z_{j+1} \in O_3 \cup P_3$ which are adjacent to every vertex in W'. If $j = 0$, we have $r(z_1 \mid W') = (1, 1) = r(a_9 \mid W')$. Otherwise, $r(z_1 \mid W') = (1, 1) = r(z_2 \mid W')$. Thus, W' is not local resolving set of G_3. Since there is no local resolving set of G_3 with cardinality 3, we obtain that $lmd(G_3) \geq 4$.

Next, we will show that $lmd(G_3) \leq 4$. Define $W" = \{a_1, a_2, a_7, a_8\}$. The representation of all vertices in G_3 with respect to $W"$ are as follows.

$$
\begin{array}{lll}
r(a_1|W") = (0,1,2,1) & r(a_3|W") = (2,1,1,1) & r(a_9|W") = (1,1,1,1) \\
r(a_2|W") = (1,0,1,2) & r(a_4|W") = (1,2,1,1) & r(a_{10}|W") = (1,1,1,1) \\
r(a_7|W") = (2,1,0,1) & r(a_5|W") = (1,1,1,2) & r(a_{11}|W") = (1,1,1,1) \\
r(a_8|W") = (1,2,1,0) & r(a_6|W") = (1,1,2,1) & r(a_{12}|W") = (1,1,1,1)
\end{array}
$$

Since there are no two adjacent vertices of G_3 having the same representation, we conclude that

W" is a local resolving set of G_3. □

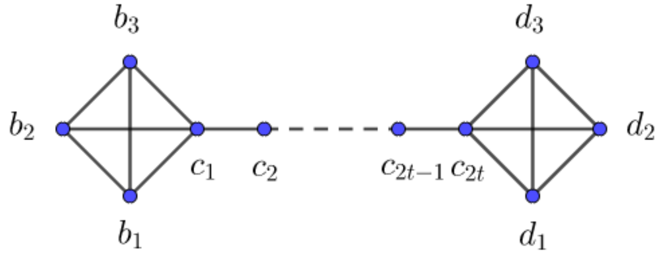

Figure 5. Graph G_4

Next, let us consider the graph G_4. Let $N_4 = \{b_1, b_2, b_3\}$, $O_4 = \{c_i \mid 1 \leq i \leq t,\ t \geq 1\}$, $P_4 = \{d_1, d_2, d_3\}$. Now, we are ready to determine the local metric dimension of G_4.

Lemma 4.2. *Let G_4 be a connected graph as stated in Figure 5. Then $lmd(G_4) = 4$. Moreover, every local resolving set of G_4 contains at least two vertices of N_4 and at least two vertices of P_4.*

Proof. Let W be a local resolving set of G_4. If $|W \cap N_4| \leq 1$ (or $|W \cap P_4| \leq 1$), then there exist two distinct vertices $x, y \in N_4$ (or $x, y \in P_4$) which are not element of W. Since x and y are adjacent, and for every $z \in V(G_4) \setminus \{x, y\}$, $d_{G_4}(x, z) = d_{G_4}(y, z)$, we have $r(x \mid W) = r(y \mid W)$, a contradiction. So, it must be $|W \cap N_4| \geq 2$ and $|W \cap P_4| \geq 2$, which implies $lmd(G_4) \geq 4$.

Next, we will show that $lmd(G_4) \leq 4$. Define $W' = \{b_1, b_2, d_1, d_2\}$. Let us consider two adjacent vertices $u, v \in V(G_4) \setminus W'$. If $d_{G_4}(u, b_1) \neq d_{G_4}(v, b_1)$, then we obtain $r(u \mid W') \neq r(v \mid W')$. Otherwise, we have $u = b_3$ and $v = c_1$. Since $d_{G_4}(u, d_1) = d_{G_4}(v, d_1) + 1$, we also obtain $r(u \mid W') \neq r(v \mid W')$. Thus, W' is a local resolving set of G_4. □

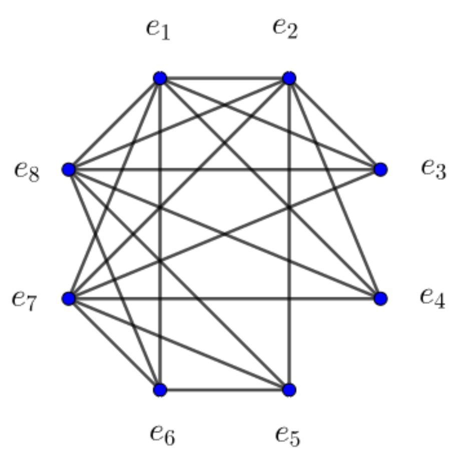

Figure 6. Graph G_5

Lemma 4.3. *Let G_5 be a connected graph as stated in Figure 6. Then $lmd(G_5) = 2$ where $\{e_5, e_6\}$ is a local basis of G_5. If a local resolving set B of G_5 contains e_1 or e_2, then $|B| \geq 3$. Moreover, the set $\{e_1, e_2, e_3\}$ is a local resolving set of G_5.*

Proof. Since G_5 contains an odd cylce, we have $lmd(G_5) > 1$. Next, we will construct a local resolving set of G_5 with 2 vertices. Define $W = \{e_5, e_6\}$. The representation of all vertices in G_5 with respect to W are as follows.

$$r(e_1|W) = (2,1) \quad r(e_3|W) = (2,2) \quad r(e_5|W) = (0,1) \quad r(e_7|W) = (1,1)$$
$$r(e_2|W) = (1,2) \quad r(e_4|W) = (2,2) \quad r(e_6|W) = (1,0) \quad r(e_8|W) = (1,1)$$

Since there are no two adjacent vertices having the same representation, we conclude that W is a local resolving set of G_5.

Next, we will show that every local basis B of G_5 satisfies $e_i \notin B$ for $i \in \{1, 2\}$. Suppose that $e_1 \in B$ or $e_2 \in B$. If $e_1, e_2 \in B$, then two adjacent vertices e_3 and e_8 are not locally resolved by B, a contradiction. Now, we assume that either $e_1 \in B$ or $e_2 \in B$. For $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, let $e_i \in B$ and $e_j \notin B$. Let $D = \{e_3, e_4, e_{4+i}\}$. If $B \cap D \neq \emptyset$, then two adjacent vertices e_j and e_8 are not locally resolved by B. Otherwise, two adjacent vertices e_j and e_3 are not locally resolved by B. Thus, we have that B is not local resolving set of G_5.

Now, let $S = \{e_1, e_2, e_3\}$. The representation of all vertices in G_5 with respect to S are as follows.

$$r(e_1|S) = (0,1,1) \quad r(e_3|S) = (1,1,0) \quad r(e_5|S) = (2,1,2) \quad r(e_7|S) = (1,1,1)$$
$$r(e_2|S) = (1,0,1) \quad r(e_4|S) = (1,1,2) \quad r(e_6|S) = (1,2,2) \quad r(e_8|S) = (1,1,1)$$

Since there are no two adjacent vertices having the same representation, we conclude that S is a local resolving set of G_5. □

Now, we are ready to show that all values between the lower and upper bound in Theorem 4.1 are achievable, which can be seen in theorem below.

Theorem 4.2. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, there exist simple connected graph H_i of order at least 3 and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ such that $lmd(Subgraph - Amal\{\mathcal{H}; P_2\}) = k$ for every integer k satisfying $\sum_{i=1}^{n} lmd(H_i) - 2n < k < \sum_{i=1}^{n} lmd(H_i)$.*

Proof. Let us consider a finite collection $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ where $H_i = G_3$. Let a_1a_2 be the terminal edge from every H_i and $H \cong Subgraph - Amal\{\mathcal{H}; P_2\}$. We will show that $lmd(H) = \sum_{i=1}^{n} lmd(H_i) - 2n$. By Theorem 2.1, we only need to show that $lmd(H) \leq \sum_{i=1}^{n} lmd(H_i) - 2n$.

Define $W = \bigcup_{i=1}^{n} \{a_{(i,7)}, a_{(i,8)}\}$. Let us consider any two adjacent vertices $x_1, x_2 \in V(H) \setminus W$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $x_1, x_2 \in H_i^{\star}$. Let $y_1, y_2 \in V(H_i)$ which are corresponded to $x_1, x_2 \in V(H)$, respectively. It is clear that y_1 is also adjacent to y_2 in H_i. If a_7 or a_8 is locally resolves y_1 and y_2, then it follows that x_1 and x_2 are locally resolves by W. Otherwise, y_1 and y_2 are locally resolved by a_1 or a_2.

- y_1 and y_2 are locally resolved by a_1
  Then for $l \in \{1, 2, \ldots, n\} \setminus \{i\}$, we have

$$\begin{aligned} d_H(x_1, a_{(l,8)}) &= d_H(x_1, a_1) + d_H(a_1, a_{(l,8)}) \\ &= d_{H_i}(y_1, a_1) + d_{H_i}(a_1, a_8) \\ &\neq d_{H_i}(y_2, a_1) + d_{H_i}(a_1, a_8) \\ &= d_H(x_2, a_1) + d_H(a_1, a_{(l,8)}) \\ &= d_H(x_2, a_{(l,8)}). \end{aligned}$$

- y_1 and y_2 are locally resolved by a_2
  Then for $l \in \{1, 2, \ldots, n\} \setminus \{i\}$, we have

$$\begin{aligned} d_H(x_1, a_{(l,7)}) &= d_H(x_1, a_2) + d_H(a_2, a_{(l,7)}) \\ &= d_{H_i}(y_1, a_2) + d_{H_i}(a_2, a_7) \\ &\neq d_{H_i}(y_2, a_2) + d_{H_i}(a_2, a_7) \\ &= d_H(x_2, a_2) + d_H(a_2, a_{(l,7)}) \\ &= d_H(x_2, a_{(l,7)}). \end{aligned}$$

Therefore, W is a local resolving set of H. It implies that $lmd(H) = \sum_{i=1}^{n} lmd(H_i) - 2n$.

To obtain an edge amalgamation graph whose local metric dimension is $k = \sum_{i=1}^{n} lmd(H_i) - 2n + q$ for $1 \leq q \leq 2n$, we replace some H_i in $\mathcal{H}$ by G_4 or G_5 according to the parity of q. For $l \in \{1, 2, \ldots, n\}$, we distinguish two cases.

Case 1. $q = 2l$
Let $\mathcal{H}' = \{H_1', H_2', \ldots, H_n'\}$ be a finite collection which is obtain from $\mathcal{H}$ by replacing l elements of $\mathcal{H}$ by G_4. Choose the terminal edge $c_t c_{t+1}$ for G_4 and $a_1 a_2$ for G_3.

Case 2. $q = 2l - 1$
Let $\mathcal{H}' = \{H_1', H_2', \ldots, H_n'\}$ be a finite collection which is obtain from $\mathcal{H}$ by replacing $l - 1$ elements of $\mathcal{H}$ by G_4 and an element of $\mathcal{H}$ by G_5. Choose the terminal edge $e_1 e_2$ for G_5, $c_t c_{t+1}$ for G_4, and $a_1 a_2$ for G_3.

By Lemma 4.1, the graph G_3 has a local basis S_3 containing a_1, a_2. However, by Lemmas 4.2 and 4.3, G_4 and G_5 do not have a local basis containing c_t, c_{t+1} and e_1, e_2, respectively. So, for G_4 and G_5, we consider a minimum resolving set S_4 and S_5, respectively, containing two vertices of their terminal edge. Note that, $|S_4| \geq lmd(G_4) + 2$ and $|S_5| \geq lmd(G_5) + 1$. Thus, by using the similar argument with the proof of Lemma 2.1, the graphs G_3, G_4, and G_5 must contribute at least $lmd(G_3) - 2$, $lmd(G_4)$, and $lmd(G_5) - 1$ vertices to a resolving set of $Subgraph - Amal\{\mathcal{H}', P_2\}$, respectively.

Now, we will construct a resolving set of $Subgraph - Amal\{\mathcal{H}', P_2\}$ with $\sum_{i=1}^{n} lmd(H_i) - 2n + q$ vertices. Let B_3, B_4, and B_5 be the set of vertices in $Subgraph - Amal\{\mathcal{H}', P_2\}$ which are corresponded to vertices a_7 and a_8 of G_3, vertices b_1, b_2, d_1, and d_2 of G_4, and vertex e_3 of

G_5, respectively. Then, define $B = B_3 \cup B_4 \cup B_5$. Note that, for q is even, we have $B_5 = \emptyset$. Let $H \cong Subgraph - Amal\{\mathcal{H}', P_2\}$. Let us consider any two adjacent vertices $x_1, x_2 \in V(H) \setminus B$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $x_1, x_2 \in H_i'^{\star}$. Let $H_i' \cong G_j$ where $j \in \{3, 4, 5\}$. If x_1, x_2 are resolved by B_j, then we are done. Otherwise, x_1, x_2 then are resolved by a vertex in terminal edge of H_i'. We distinguish two cases.

Case 1. $H_i' \cong G_3$
Let $y_1, y_2 \in V(H_i')$ which are corresponded to $x_1, x_2 \in V(H)$, respectively. Then it is clear that a_1 or a_2 is locally resolves y_1 and y_2. Note that, in H, the vertex a_1 of H_i' can be identified by c_t or c_{t+1} of H_r' where $H_r' \cong G_4$. So, x_1, x_2 are locally resolved by vertices in B_4.

Case 2. $H_i' \cong G_5$
Let $y_1, y_2 \in V(H_i')$ which are corresponded to $x_1, x_2 \in V(H)$, respectively. Then it is clear that e_1 or e_2 is locally resolves y_1 and y_2. Note that, in H, the vertex e_1 of H_i' can be identified by a_1 or a_2 of H_r' where $H_r' \cong G_3$, or by c_t or c_{t+1} of H_r' where $H_r' \cong G_4$. So, x_1, x_2 are locally resolved by vertices in B_3 or B_4. □

Conclusion

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a finite collection of simple connected graphs, where H_i is a graph containing a connected subgraph J. In this paper, we consider the subgraph-amalgamation graphs $Subgraph - Amal\{\mathcal{H}; J\}$. This graph is constructed by taking all elements of $\mathcal{H}$, then identifying all of them on J. We provide the lower and upper bounds of $lmd(Subgraph - Amal\{\mathcal{H}; J\})$ for any structures of J. These bounds are functions of $lmd(H_i)$ $(1 \leq i \leq n)$. We also provide some properties of $Subgraph - Amal\{\mathcal{H}; J\}$ whose local metric dimension is equal to some values, including the upper and lower bound values.

Furthermore, we consider $Subgraph - Amal\{\mathcal{H}; J\}$ for certain structure of J. For J is a vertex $(J = K_1)$, we determine an exact value of the local metric dimension of $Subgraph - Amal\{\mathcal{H}; J\}$. In case J is an edge $(J = P_2)$, we provide the lower and upper bounds of $lmd(Subgraph - Amal\{\mathcal{H}; J\})$. Moreover, we show that all values between those bounds are achievable.

For future work, we provide an interesting question that is whether all between the lower and upper bounds in Theorem 2.1 are achievalbe if the order of J is at least 3. The problem is stated as follows.

Problem 1. *Let J be a connected graph of order $p \geq 3$. Does there exist $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ where H_i is a connected graph containing J, such that for every integer t with $\sum_{i=1}^{n} lmd(H_i) - pn < t < \sum_{i=1}^{n} lmd(H_i)$, we have $lmd(Subgraph - Amal\{\mathcal{H}; J\}) = t$?*

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