Electronic Journal of Graph Theory and Applications **12** (1) (2024), 147–156

Electronic Journal of Graph Theory and Applications

A note on lower bounds for boxicity of graphs

Akira Kamibeppu

Department of Creative Engineering, National Institute of Technology (KOSEN), Kushiro College, Kushiro, Hokkaido 084-0916, Japan.

kamibeppu@kushiro.kosen-ac.jp

Abstract

The *boxicity* of a graph G is the minimum non-negative integer k such that G is isomorphic to the intersection graph of a family of boxes in Euclidean k-space, where a *box* in Euclidean k-space is the Cartesian product of k closed intervals on the real line. In this short note, we define the *fractional boxicity* of a graph as the optimum value of the linear relaxation of a covering problem with respect to boxicity, which gives a lower bound for its boxicity. We show that the fractional boxicity of a graph is at least the lower bounds for boxicity given by Adiga et al. in 2014. We also present a natural lower bound for fractional boxicity of graphs. The aim of this note is to discuss and focus on "accuracy" rather than "simplicity" of these lower bounds for boxicity as the next step in the work by Adiga et al.

Keywords: hypergraph; boxicity; (co)interval graph; integer/linear program
Mathematics Subject Classification : 05C62, 05C65, 05C72
DOI: 10.5614/ejgta.2024.12.1.12

1. Introduction and Preliminaries

A *box* in Euclidean k-space is the Cartesian product of k closed intervals on the real line. The intersection graph of a family $\mathcal{F}$ of boxes in Euclidean k-space is the graph with $\mathcal{F}$ as the vertex set, where two boxes (vertices) in $\mathcal{F}$ are adjacent if and only if they have non-empty intersection in the space. The *boxicity* of a graph G, denoted by $\mathrm{box}(G)$, is the minimum non-negative integer k such that G is isomorphic to the intersection graph of a family of boxes in Euclidean k-space. For example, a complete graph K_n with n vertices, a path P_n with n vertices, and a cycle

Received: 15 November 2021, Revised: 12 March 2024, Accepted: 31 March 2024.

C_n with n vertices for $n \geq 4$ can be represented by the intersection graph of a family of boxes in 0-dimensional, 1-dimensional, and 2-dimensional space respectively (in fact, $\text{box}(K_n) = 0$, $\text{box}(P_n) = 1$, $\text{box}(C_n) = 2$).

The concept of boxicity of graphs was introduced by Roberts [15]. It has applications to measure the structural complexity of ecological and social networks (see [14, 16] for detail). So far many researchers have attempted to calculate or bound boxicity of graphs with specific structure. Roberts [15] found that the boxicity of a complete k-partite graph is equal to k, where the cardinality of each partite set is at least 2. Roberts also proved that the maximum boxicity of graphs with n vertices is $\lfloor \frac{n}{2} \rfloor$ (also see [7]), where $\lfloor x \rfloor$ denotes the largest integer at most x. Cozzens [6] found that the task of computing boxicity of graphs is NP-hard. Chandran and Sivadasan [5] presented upper bounds for chordal graphs, circular arc graphs, AT-free graphs, co-comparability graphs, and permutation graphs by relating boxicity to treewidth. Cozzens and Roberts [7] obtained an upper bound for boxicity of split graphs, which contributed to relating boxicity to the cardinality of minimum vertex cover and the chromatic number in [3]. Relationships between boxicity and (Euler) genus were found by Esperet and Joret in [9, 10, 11], which originated from researches of the boxicity of outerplanar graphs and planar graphs observed, respectively, by Scheinerman [17] and Thomassen [20]. In addition, boxicity has notable topics related to the following (graph) invariants: maximum degree [4, 8] and poset dimension [1, 12, 19].

In this short note we focus on lower bounds for boxicity of graphs. Adiga et al. [2] presented a lower bound for the boxicity of a graph as in Lemma 1.1 below, which also gives some lower bounds under various conditions on graphs. Those lower bounds for boxicity in addition to the lower bound in Lemma 1.2 are relatively easy to estimate by examination, but there is an example of a graph whose boxicity cannot be determined by those lower bounds (see Example 2.4 and Remark 2.5). In what follows, the symbol $\overline{G}$ denotes the complement of a graph G and the cardinality of a set X is denoted by $|X|$. The symbol $\lceil x \rceil$ denotes the smallest integer at least x. *Interval graphs* are graphs of boxicity at most 1.

Lemma 1.1 ([2], Lemma 3.1)**.** The inequality $\text{box}(G) \geq |E(\overline{G})|/|E(\overline{I_{\min}})|$ holds for a non-complete graph G, where $I_{\min}$ is an interval supergraph of G with $V(I_{\min}) = V(G)$ and with the minimum number of edges among all such interval supergraphs of G.

Lemma 1.2 ([7], Lemma 3)**.** Let G be a graph. Let $S_1 = \{u_1, u_2, \ldots, u_n\}$ and $S_2 = \{v_1, v_2, \ldots, v_n\}$ be disjoint subsets of $V(G)$ such that the only edges between S_1 and S_2 in $\overline{G}$ are the edges $u_i v_i$, where $i \in \{1, 2, \ldots, n\}$. Then $\text{box}(G) \geq \lceil n/2 \rceil$ holds.

The next step in the work by Adiga et al. is to discuss and focus on "accuracy" rather than "simplicity" of these lower bounds for boxicity. The purpose of this note is

- to review the lower bound in Lemma 1.1 for boxicity in the context of fractional graph theory,
- to introduce a fractional analogue of boxicity that will become a lower bound for boxicity, and
- to present a natural lower bound for our fractional analogue of boxicity, which works on calculation of boxicity of some graphs better than Lemmas 1.1 and 1.2.

In this note, all graphs are finite, simple and undirected. We use $V(G)$ for the vertex set of a graph G and $E(G)$ for the edge set of the graph G. These notations are also used for hypergraphs. A few concepts and results about (hyper)graphs are needed to present a fractional analogue of boxicity. A graph is said to be *cointerval* if its complement is an interval graph. A *cointerval edge covering* of a graph G is a family $\mathcal{C}$ of cointerval subgraphs of G such that each edge of G is in some graph in $\mathcal{C}$. The following is a basic result on boxicity.

Theorem 1.3 ([7], Theorem 3)**.** Let G be a graph. Then, $\text{box}(G) \leq k$ if and only if there exists a cointerval edge covering $\mathcal{C}$ of $\overline{G}$ with $|\mathcal{C}| = k$. Hence

$$\text{box}(G) = \min\{ |\mathcal{C}| : \mathcal{C} \text{ is a cointerval edge covering of } \overline{G}\,\}.$$

2. Main Results

In what follows, for n-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we write $\boldsymbol{u} \geq \boldsymbol{v}$ to mean that each coordinate of $\boldsymbol{u}$ is at least the corresponding coordinate of $\boldsymbol{v}$. Let $\mathcal{C}$ be a family of hyperedges of a hypergraph $\mathcal{H}$ and we write $\mathcal{C} = \{X_1, \ldots, X_k\}$. The family $\mathcal{C}$ is a covering of $\mathcal{H}$ if $V(\mathcal{H}) \subseteq X_1 \cup \cdots \cup X_k$ holds. Our key idea for the definition of a fractional analogue of boxicity is in the way to define a hypergraph associated with a graph. For a graph G, we define the hypergraph $\mathcal{H}_G$ as follows:

$$V(\mathcal{H}_G) = E(\overline{G}) \text{ and}$$
$$E(\mathcal{H}_G) = \{E \subset E(\overline{G}) : E \text{ corresponds to a cointerval subgraph of } \overline{G}\,\}.$$

Note that a covering of $\mathcal{H}_G$ corresponds to a cointerval edge covering of $\overline{G}$. Hence the covering number of the hypergraph $\mathcal{H}_G$, the minimum cardinality of a covering of $\mathcal{H}_G$, is equal to the boxicity of G by Theorem 1.3.

For a graph G, let e_i be an edge of $\overline{G}$ and E_j a hyperedge of $\mathcal{H}_G$. Moreover, let M_G be the incidence matrix of $\mathcal{H}_G$ whose rows are indexed by all edges of $\overline{G}$ and whose columns are indexed by all cointerval subgraphs of $\overline{G}$, that is, the i, j-entry of M_G is equal to 1 if $e_i \in E_j$, and otherwise 0. Write $E(\mathcal{H}_G) = \{E_1, \ldots, E_n\}$. Let $\mathcal{C}$ be a family of hyperedges in $E(\mathcal{H}_G)$ and $\boldsymbol{x} = {}^t(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ the indicator vector of hyperedges in $E(\mathcal{H}_G)$ that corresponds to the family $\mathcal{C}$, that is, x_i is equal to 1 if $E_i \in \mathcal{C}$, and otherwise 0. We see that $\mathcal{C}$ is a cointerval edge covering of $\overline{G}$ if and only if $M_G\boldsymbol{x} \geq \mathbf{1}$ holds, where $\mathbf{1}$ is a vector of all ones. We note that a subgraph of $\overline{G}$ with only one edge is a cointerval subgraph of $\overline{G}$. Hence the boxicity of a graph G can be defined as the optimum value of the integer program (that is feasible)

$$\text{(IP) minimize } {}^t\mathbf{1}\boldsymbol{x}$$
$$\text{subject to } M_G\boldsymbol{x} \geq \mathbf{1} \text{ and } \boldsymbol{x} \in \{0,1\}^n,$$

that is,

$$\text{box}(G) = \min\{{}^t\mathbf{1}\boldsymbol{x} : M_G\boldsymbol{x} \geq \mathbf{1}, \boldsymbol{x} \in \{0,1\}^n\}.$$

We relax the condition of the integer program (IP) and consider the linear program

$$\text{(LP) minimize } {}^t\mathbf{1}\boldsymbol{x}$$
$$\text{subject to } M_G\boldsymbol{x} \geq \mathbf{1} \text{ and } \boldsymbol{x} \geq \boldsymbol{o},$$

where $\boldsymbol{o}$ is a zero vector. We define the *fractional boxicity* of a graph G, denoted by $\mathrm{box}_f(G)$, to be the optimum value of (LP), that is,

$$\mathrm{box}_f(G) = \min\{{}^t\mathbf{1}\boldsymbol{x} : M_G\boldsymbol{x} \geq \mathbf{1}, \boldsymbol{x} \geq \boldsymbol{o}\}.$$

Hence $\mathrm{box}_f(G) \leq \mathrm{box}(G)$ holds for a graph G.

By the way, in the theory of linear programming, we usually consider the dual program of (LP):

$$\begin{aligned} \text{(D) maximize } & {}^t\mathbf{1}\boldsymbol{y} \\ \text{subject to } & {}^tM_G\boldsymbol{y} \leq \mathbf{1} \text{ and } \boldsymbol{y} \geq \boldsymbol{o}. \end{aligned}$$

The program (D) is clearly feasible. It is well-known in the theory of linear programming that a feasible linear program and its dual feasible program have the same optimum value. Hence we may consider the value of (D) instead of $\mathrm{box}_f(G)$. We notice that a vector $\boldsymbol{y}_*$ of all $1/p$'s is a feasible solution of (D), where $p = \max_{E_i \in E(\mathcal{H}_G)} |E_i|$. Hence, $\mathrm{box}_f(G) \geq {}^t\mathbf{1}\boldsymbol{y}_* = |E(\overline{G})|/p$. We note that this lower bound for fractional boxicity of graphs is identical to the lower bound for boxicity of graphs in Lemma 1.1.

An automorphism of a hypergraph $\mathcal{H}$ is a bijection π on $V(\mathcal{H})$ such that $X \in E(\mathcal{H})$ if and only if $\pi(X) \in E(\mathcal{H})$. A hypergraph $\mathcal{H}$ is *vertex-transitive* (*edge-transitive*) if for every pair (w_1, w_2) of vertices (hyperedges) there exists an automorphism π of $\mathcal{H}$ such that $\pi(w_1) = w_2$ holds. The following theorem is derived from Proposition 1.3.4 in [18].

Theorem 2.1. For a graph G, the inequalities

$$\mathrm{box}(G) \geq \mathrm{box}_f(G) \geq \frac{|E(\overline{G})|}{\max_{E_i \in E(\mathcal{H}_G)} |E_i|}$$

hold. In particular, if $\overline{G}$ is edge-transitive, we have the equality

$$\mathrm{box}_f(G) = \frac{|E(\overline{G})|}{\max_{E_i \in E(\mathcal{H}_G)} |E_i|}.$$

Proof. Note that the fractional boxicity of a graph G is the same concept with the fractional covering number of the hypergraph $\mathcal{H}_G$. In Lemma 2.2 below, we show the hypergraph $\mathcal{H}_G$ is vertex-transitive by the edge-transitivity of $\overline{G}$, so the above equality holds by Proposition 1.3.4 in [18]. The following Lemma 2.2 completes the proof of this theorem. □

Lemma 2.2. If $\overline{G}$ is edge-transitive for a graph G, the hypergraph $\mathcal{H}_G$ is vertex-transitive.

Proof. For every pair of vertices $e_1, e_2 \in V(\mathcal{H}_G) = E(\overline{G})$, there exists an automorphism $\pi : V(\overline{G}) \to V(\overline{G})$ such that $\pi(e_1) = e_2$ holds by our assumption. We can check that π induces a bijection $\overline{\pi}$ on $E(\overline{G})$ in a natural way: $\overline{\pi}(uv) = \pi(u)\pi(v)$ for an edge $uv \in E(\overline{G})$. Moreover E is in $E(\mathcal{H}_G)$ if and only if $\overline{\pi}(E)$ is in $E(\mathcal{H}_G)$ since π and its inverse π^{-1} map a subgraph H of $\overline{G}$ to a subgraph isomorphic to H. Hence $\overline{\pi}$ is the desired map. □

The fractional boxicity of a graph G is the same as the maximum value of ${}^t\mathbf{1}\boldsymbol{y}$ under the conditions ${}^tM_G\boldsymbol{y} \leq \mathbf{1}$ and $\boldsymbol{y} \geq \boldsymbol{o}$. We note that each entry of $\boldsymbol{y}$ is a weight of an edge of $\overline{G}$. The rows of ${}^tM_G$ are indexed by all cointerval subgraphs of $\overline{G}$, but we see that

- an inequality in ${}^tM_G\boldsymbol{y} \leq \mathbf{1}$ corresponding to a non-maximal cointerval subgraph (on their edge sets) is superfluous since $\boldsymbol{y} \geq \boldsymbol{o}$.

Hence we only have to focus on maximal cointerval subgraphs of $\overline{G}$ when we calculate $\mathrm{box}_f(G)$. In what follows, M_G always means the (reduced) incidence matrix of $\mathcal{H}_G$ whose columns are indexed by all maximal cointerval subgraphs of $\overline{G}$.

The boxicity and the fractional boxicity of a graph are different in general (also see Example 2.4 and Remark 2.5). As a simple example, let us consider the graph G in Figure 1 whose complement is isomorphic to K_3 with a pendant edge added at each vertex of K_3. It is easy to see that $\mathrm{box}(G) = 2 > 3/2 \geq \mathrm{box}_f(G)$ holds.

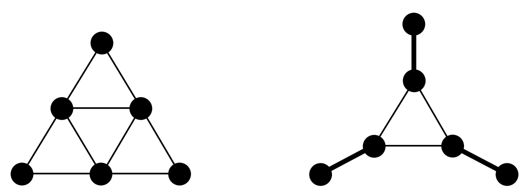

Figure 1. The graph G (left) and its complement $\overline{G}$ (right).

We can find three maximal cointerval subgraphs of $\overline{G}$ in total, each of which is isomorphic to the graph obtained from $\overline{G}$ by deleting one pendant edge. It is easy to check that $\boldsymbol{x} = {}^t(1/2, 1/2, 1/2)$ is a feasible solution for $M_G\boldsymbol{x} \geq \mathbf{1}$ and $\boldsymbol{x} \geq \boldsymbol{o}$. Hence $\mathrm{box}_f(G) \leq {}^t\mathbf{1}\boldsymbol{x} = 3/2$.

We will reduce unnecessary restrictions further within the same conditions $M_G\boldsymbol{x} \geq \mathbf{1}$ and $\boldsymbol{x} \geq \boldsymbol{o}$. Let $\mathcal{E}$ ($\subset E(\mathcal{H}_G)$) be the family of all maximal cointerval subgraphs of $\overline{G}$. Write $\mathcal{F}_e = \{E \in \mathcal{E} : e \in E\}$ for an edge $e \in E(\overline{G})$. An edge e of $\overline{G}$ is said to be *fundamental* if $\mathcal{F}_e$ is minimal as subfamily of $\mathcal{E}$ (see heavy edges in Figure 1 for definition). Let E^* be the set of all fundamental edges of $\overline{G}$. We define two edges e and e' in E^* to be equivalent, denoted by $e \sim e'$, if $\mathcal{F}_e = \mathcal{F}_{e'}$. We remark that

- an inequality in $M_G\boldsymbol{x} \geq \mathbf{1}$ corresponding to a non-fundamental edge of $\overline{G}$ is superfluous since $\boldsymbol{x} \geq \boldsymbol{o}$, and
- if $e \sim e'$ for e, $e' \in E^*$, the two inequalities in $M_G\boldsymbol{x} \geq \mathbf{1}$ which correspond to e and e' are the same inequalities.

The inequality corresponding to an equivalence class $[e]$ means an inequality in $M_G\boldsymbol{x} \geq \mathbf{1}$ corresponding to a representative of $[e]$. It does not depend on the choice of representatives of $[e]$. Let M_G^* be the reduced incidence matrix of $\mathcal{H}_G$ whose rows are indexed by all equivalence classes in $E^*/\!\sim$ and whose columns are indexed by all maximal cointerval subgraphs of $\overline{G}$. We see that

- $M_G\boldsymbol{x} \geq \mathbf{1}$ is equivalent to $M_G^*\boldsymbol{x} \geq \mathbf{1}$ under $\boldsymbol{x} \geq \boldsymbol{o}$.

Hence the fractional boxicity of a graph G is the same as the optimum value of the linear program

$$\text{(LP)}' \text{ minimize } {}^t\mathbf{1}\boldsymbol{x} \quad \text{subject to } M_G^*\boldsymbol{x} \geq \mathbf{1} \text{ and } \boldsymbol{x} \geq \boldsymbol{o}.$$

We consider a relaxation program of (LP)$'$ and get a natural lower bound for fractional boxicity of graphs.

Theorem 2.3. For a graph G, let $\{H_1, H_2, \dots, H_l\}$ be the family of all maximal cointerval subgraphs of $\overline{G}$ and let $E^*/\sim = \{[e_1], [e_2], \dots, [e_k]\}$. Let a_i be the number of fundamental edges of $\overline{G}$ in $\{e_1, e_2, \dots, e_k\}$ which are contained in H_i for $i \in \{1, 2, \dots, l\}$. Then

$$\text{box}_f(G) \geq \frac{k}{a^*}$$

holds, where $a^* = \max\{a_1, a_2, \dots, a_l\}$.

Proof. Note that $\text{box}_f(G) = \min\{{}^t\mathbf{1}\boldsymbol{x} : M_G^*\boldsymbol{x} \geq \mathbf{1}, \boldsymbol{x} \geq \boldsymbol{o}\}$. Sum up all k inequalities in $M_G^*\boldsymbol{x} \geq \mathbf{1}$, and then we obtain

$$a^*({}^t\mathbf{1}\boldsymbol{x}) = a^*(x_1 + x_2 + \cdots + x_l) \geq a_1x_1 + a_2x_2 + \cdots + a_lx_l \geq k,$$

where $\boldsymbol{x} = {}^t(x_1, x_2, \dots, x_l)$. Hence ${}^t\mathbf{1}\boldsymbol{x} \geq k/a^*$ holds, that is, $\text{box}_f(G) \geq k/a^*$. □

The fractional boxicity of a graph will measure its boxicity more accurately than the other lower bounds for boxicity given by Adiga et al. in 2014, although it is a difficult parameter to estimate by examination like the other fractional graph invariants.

Example 2.4. We consider the graph G_k whose complement is the graph in Figure 2 below (and is not edge-transitive), where $k \geq 4$. We will find all maximal cointerval subgraphs of $\overline{G_k}$ and prove $\text{box}_f(G_k) = k/2$.

Let H be a cointerval subgraph of $\overline{G_k}$. For example, we see that

(1) H cannot have edges $e_{11}, e_{12}, \dots, e_{5k-9}$ if H has the edge e_1, and

(2) H cannot have edges $e_{16}, e_{17}, \dots, e_{5k-9}$ if H has at least one of e_2, e_3, e_4 and e_5.

We will obtain similar statements to (1) or (2) if H has an edge e_i, where $i \in \{6, 7, \dots, 5k\}$.

Case 1. Assume that H contains the edge e_1. If it has at least one of e_7, e_8, e_9 and e_{10}, we can find maximal cointerval graphs containing H within the graph induced by $\{v_1, \dots, v_6, v_{2k-1}, v_{2k}\}$, and otherwise we can find them within the graph induced by $\{v_1, v_2, v_3, v_4, v_{2k-3}, v_{2k-2}, v_{2k-1}, v_{2k}\}$.

Case 2. Assume that H has at least one of e_2, e_3, e_4 and e_5. If it has at least one of e_{12}, e_{13}, e_{14} and e_{15}, we can find maximal cointerval graphs containing H within the graph induced by $\{v_1, \dots, v_8\}$, and otherwise we can find them within the graph induced by $\{v_1, \dots, v_6, v_{2k-3}, v_{2k-2}, v_{2k-1}, v_{2k}\}$.

As a result it is sufficient to find maximal cointerval subgraphs of the graph H_* in Figure 3. Clearly, H_* and $H_* - e$ (that is obtained from H_* by deleting e) are not cointerval for any $e \in E(H_*)$. We

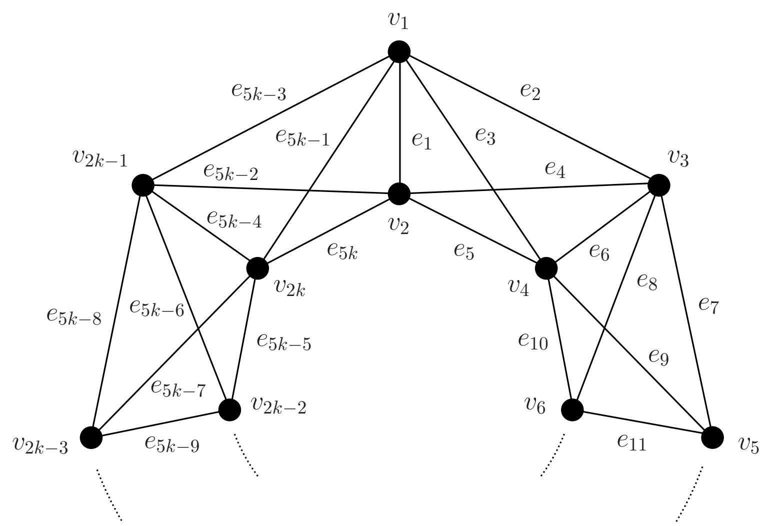

Figure 2. The complement of the graph G_k.

will find three maximal cointerval subgraphs of H_*, but two of them can be extended to the graph isomorphic to the graph with heavy edges in Figure 3 on the graph $\overline{G_k}$.

We have k maximal cointerval subgraphs of $\overline{G_k}$ in total, each of which is isomorphic to the graph with heavy edges in Figure 3. Hence the optimum value of the following linear program becomes the fractional boxicity $\mathrm{box}_f(G_k)$.

$$\begin{array}{lll}
\text{(D)} & \text{maximize} & y_1 + y_2 + \cdots + y_{5k} \\
& \text{subject to} & y_1 + y_2 + \cdots + y_{10} + y_{5k-3} + y_{5k-2} + \cdots + y_{5k} \leq 1 \\
& & y_{5i-13} + y_{5i-12} + \cdots + y_{5i} \leq 1 \quad (i \in \{3, 4, \ldots, k\}) \\
& & y_1 + y_2 + \cdots + y_5 + y_{5k-8} + y_{5k-7} + \cdots + y_{5k} \leq 1 \\
& & y_j \geq 0 \quad (j \in \{1, 2, \ldots, 5k\})
\end{array}$$

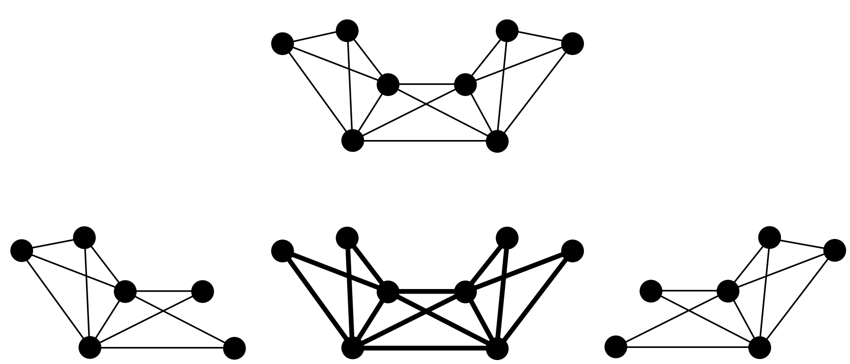

Figure 3. The graph H_* (top) and its maximal cointerval subgraphs (bottom).

We consider the dual program of (D) and reduce superfluous inequalities so that we can obtain the following linear program:

$$\begin{array}{lll} \text{(LP)}' & \text{minimize} & x_1 + x_2 + \cdots + x_k \\ & \text{subject to} & x_1 + x_k \geq 1 \\ & & x_i + x_{i+1} \geq 1 \quad (i \in \{1, 2, \ldots, k-1\}) \\ & & x_j \geq 0 \quad (j \in \{1, 2, \ldots, k\}). \end{array}$$

Then $(x_1, x_2, \ldots, x_k) = (1/2, 1/2, \ldots, 1/2)$ is a feasible solution of (LP)$'$, and hence $\text{box}_f(G_k) \leq k/2$.

Let E^* be the set of all fundamental edges of $\overline{G_k}$. It is easy to check that

- $E^* = \{e_1, e_6, \ldots, e_{5k-4}\}$,
- $E^*/\!\sim\, = \{[e_1], [e_6], \ldots, [e_{5k-4}]\}$ holds because $e \neq e'$ implies $e \not\sim e'$ for $e, e' \in E^*$, and
- $a^* = 2$ since every maximal cointerval subgraph of $\overline{G_k}$ contains two fundamental edges in $\{e_1, e_6, \ldots, e_{5k-4}\}$.

By Theorem 2.3, $\text{box}_f(G_k) \geq k/2$ holds, which implies our claim. □

Remark 2.5 (box$_f$ vs. the lower bounds in Lemmas 1.1 and 1.2)**.** It is easy to see that $\text{box}(G_k) \leq \lceil k/2 \rceil$ holds for any k by Theorem 1.3. Hence $\text{box}(G_k) = \lceil k/2 \rceil$ holds since $\text{box}_f(G_k) = k/2$. The lower bounds for boxicity in Lemmas 1.1 and 1.2 do not work on the graph G_k well for $k \geq 7$, that is, they cannot determine the boxicity of G_k.

We see that $\text{box}_f(G_k) > 5k/14 = |E(\overline{G_k})| / \max_{E_i \in E(\mathcal{H}_{G_k})} |E_i|$ holds. Let $m(G_k)$ be the maximum number of edges $a_i b_i$ of $\overline{G_k}$ with the condition in Lemma 1.2 and let M_k be a set of those edges of $\overline{G_k}$. For example, if $e_1 \in M_k$, any edge in $\{e_2, e_3, \ldots, e_{10}, e_{5k-8}, e_{5k-7}, \ldots, e_{5k}\}$ cannot be in M_k. If an edge $e \in \{e_2, e_3, e_4, e_5\}$ is in M_k, any edge in $\{e_1, e_2, \ldots, e_{11}, e_{5k-4}, e_{5k-3}, \ldots, e_{5k}\} \setminus \{e\}$ cannot be in M_k. It is not difficult to see that $m(G_k) \leq k/2$ holds in any case. Hence we have $\lceil m(G_k)/2 \rceil \leq \lceil k/4 \rceil < \text{box}_f(G_k)$. The difference between the fractional boxicity $\text{box}_f(G_k)$ and $|E(\overline{G_k})| / \max_{E_i \in E(\mathcal{H}_{G_k})} |E_i|$ (or $\lceil m(G_k)/2 \rceil$) can be arbitrary large. □

3. Further Observation

Finally we remark another way to calculate the fractional boxicity of graphs. Let s be a positive integer. The *s-fold boxicity* of a graph G, denoted by $\text{box}_s(G)$, is the minimum cardinality of a multiset $\{E_1, E_2, \ldots, E_k\}$ of cointerval subgraphs of $\overline{G}$ such that each edge of $\overline{G}$ is in at least s cointerval subgraphs in the multiset. Note that $\text{box}_1(G) = \text{box}(G)$. Since the subadditivity $\text{box}_{s+t}(G) \leq \text{box}_s(G) + \text{box}_t(G)$ holds for a graph G and $s, t \geq 1$, the following limit exists and we have the following equality by Fekete's subadditivity lemma [13]:

$$\lim_{s \to \infty} \frac{\text{box}_s(G)}{s} = \inf \left\{ \frac{\text{box}_s(G)}{s} : s \geq 1 \right\}.$$

Lemma 3.1 ([13])**.** Let $\mathbb{Z}^+$ and $\mathbb{R}^+$ be the set of all nonnegative integers and the set of all nonnegative real numbers, respectively. If $g : \mathbb{Z}^+ \to \mathbb{R}^+$ is subadditive, that is, $g(m+n) \leq g(m) + g(n)$ holds for any $m, n \in \mathbb{Z}^+$, the limit $\lim_{m\to\infty} g(m)/m$ exists and is equal to $\inf g(m)/m$. □

A basic result on the fractional covering numbers of hypergraphs guarantees $\text{box}_f(G) = \lim_{s\to\infty} \text{box}_s(G)/s$ (see Theorem 1.2.1 in [18]). Hence we may approach the study on the s-fold boxicity of graphs to calculate the fractional boxicity of graphs.

Acknowledgments

This work was initiated while the author was supported by JSPS KAKENHI Grant-in-Aid for Young Scientists (B), No.25800091 (2013-2015). This paper was significantly improved thanks to comments and suggestions from anonymous referees who were involved in the reviews. The author would like to show all anonymous referees my gratitude for his or her careful reading and suggestions.

References

[1] A. Adiga, D. Bhowmick, and L.S. Chandran, Boxicity and poset dimension, *SIAM J. Discrete Math.* **25** (2011), 1687–1698.

[2] A. Adiga, L.S. Chandran and N. Sivadasan, Lower bounds for boxicity, *Combinatorica* **34** (2014), 631–655.

[3] L.S. Chandran, A. Das, and C.D. Shah, Cubicity, boxicity, and vertex cover, *Discrete Math.* **309** (2009), 2488–2496.

[4] L.S. Chandran, M.C. Francis, and N. Sivadasan, Boxicity and maximum degree, *J. Combin. Theory Ser. B* **98** (2008), 443–445.

[5] L.S. Chandran, and N. Sivadasan, Boxicity and treewidth, *J. Combin. Theory Ser. B* **97** (2007), 733–744.

[6] M.B. Cozzens, Higher and Multidimensional Analogues of Interval Graphs, Ph.D. Thesis, Rutgers University, New Brunswick, NJ, 1981.

[7] M.B. Cozzens and F.S. Roberts, Computing the boxicity of a graph by covering its complement by cointerval graphs, *Discrete Appl. Math.* **6** (1983), 217–228.

[8] L. Esperet, Boxicity of graphs with bounded degree, *European J. Combin.* **30** (2009), 1277–1280.

[9] L. Esperet and G. Joret, Boxicity of graphs on surfaces, *Graphs Combin.* **29** (2013), 417–427.

[10] L. Esperet, Boxicity and topological invariants, *European J. Combin.* **51** (2016), 495–499.

[11] L. Esperet, Box representations of embedded graphs, *Discrete Comput. Geom.* **57** (2017), 590–606.

[12] L. Esperet and V. Wiechert, Boxicity, poset dimension, and excluded minors, *Electron. J. Combin.* **25** (2018) #P4.51.

[13] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math. Zeit.* **17** (1923), 228–249.

[14] L. C. Freeman, Spheres, cubes and boxes: Graph dimensionality and network structure, *Soc. Netw.* **5** (1983), 139–156.

[15] F.S. Roberts, On the boxicity and cubicity of a graph, in: Recent Progress in Combinatorics, Academic Press, New York (1969) 301-310.

[16] F.S. Roberts, Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems, Prentice-Hall, New Jersey, 1976.

[17] E.R. Scheinerman, Intersection Classes and Multiple Intersection Parameters, Ph.D. Thesis, Princeton University, 1984.

[18] E. R. Scheinerman and D.H. Ullman, Fractional Graph Theory, A Rational Approach to the Theory of Graphs, Dover Publications, Inc. Mineola, New York, 2011.

[19] A. Scott and D.R. Wood, Better bounds for poset dimension and boxicity, *Trans. Amer. Math. Soc. 373* (2020), 2157–2172.

[20] C. Thomassen, Interval representations of planar graphs, *J. Combin. Theory Ser. B* **40** (1986), 9–20.