On the Non-Commuting Graph of Dihedral Group

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Abstract

For a nonabelian group G, the non-commuting graph Γ_G of G is defined as the graph with vertex set $G - Z(G)$, where $Z(G)$ is the center of G, and two distinct vertices of Γ_G are adjacent if they do not commute in G. In this paper, we investigate the detour index, eccentric connectivity and total eccentricity polynomials of non-commuting graph on D_{2n}. We also find the mean distance of non-commuting graph on D_{2n}.

1 Introduction

The concept of non-commuting graph of a finite group has been introduced by Abdollahi *et al* in 2006 [1]. For a non-abelian group G, associate a graph Γ_G with it such that the vertex set of Γ_G is $G - Z(G)$, where $Z(G)$ is the center of G, and two distinct vertices x and y are adjacent if they don't commute in G, that is, $xy \neq yx$. Several works on assigning a graph to a group and investigation of algebraic properties of group using the associated graph have been done, for example, see [4, 5, 2].

All graphs are considered to be simple, which are undirected with no loops or multiple edges. Let Γ be any graph, the sets of vertices and edges of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The cardinality of the vertex set $V(\Gamma)$ is called the *order* of the graph Γ and is denoted by $|V(\Gamma)|$ and the number of edges of the graph Γ is called the *size* of Γ, and denoted by $|E(\Gamma)|$. The graph Γ is called *split* if $V(\Gamma) = S \cup K$, where S is an independent set and K is a complete set. For a vertex v in Γ, the number of edges incident to v is called the *degree* of v and is denoted by $deg_\Gamma(v)$. The *eccentricity* of a vertex v in Γ, denoted by $ecc(v)$, is the largest distance between v and any other vertex u in Γ. For vertices u and v in a graph Γ, a $u-v$ *path* in Γ is $u-v$ walk with no vertices repeated. The shortest (longest) $u-v$ path in a graph Γ, denoted by $d(u,v)$ $(D(u,v))$, is called the *distance(detour distance)* between vertices u and v in Γ. The *detour index*, *eccentric connectivity* and *total eccentricity polynomials* are defined as $D(\Gamma_\Omega, x) = \sum_{u,v\in V(\Gamma)} x^{D(u,v)}$ [9], $\Xi(\Gamma, x) = \sum_{u\in V(\Gamma)} deg_\Gamma(u)x^{ecc(u)}$ and $\Theta(\Gamma, x) = \sum_{u\in V(\Gamma)} x^{ecc(u)}$ [7], respectively. The *detour index* $dd(\Gamma)$, the *eccentric connectivity index* and the *total eccentricity* $\xi^c(\Gamma)$ of a graph Γ are the first derivatives of their corresponding polynomials at $x = 1$, respectively. A *transmission* of a vertex v in Γ is $\sigma(v,\Gamma) = \Sigma_{u\in V(\Gamma)} d(u,v)$. The transmission of a graph Γ is $\sigma(\Gamma) = \Sigma_{u\in V(\Gamma)}\sigma(u,\Gamma)$. The *mean(average) distance* of a graph Γ is $\mu(\Gamma) = \frac{\sigma(\Gamma)}{p(p-1)}$, where p is the order of Γ, see [8, 3, 6]. In this paper, we study some properties of non-commuting graph of dihedral groups. The dihedral group D_{2n} of order $2n$ is defined by

$$D_{2n} = \langle r, s : r^n = s^2 = 1, srs = r^{-1}\rangle$$

for any $n \geq 3$, and the center of D_{2n} is $Z(D_{2n}) = \begin{cases} \{1\}, & \text{if } n \text{ is odd} \\ \{1, r^{\frac{n}{2}}\}, & \text{if } n \text{ is even.} \end{cases}$
Throughout this article, we assume that $\Omega_1 = \{r^i : 1 \leq i \leq n\} - Z(D_{2n})$, and $\Omega_2 = \{sr^i : 1 \leq i \leq n\}$. This article is organized as follows: In the present section, we give some important definitions and notations. In Section 2, we study some basic properties of the non-commuting graph $\Gamma_{D_{2n}}$ of D_{2n}. We see that $\Gamma_{D_{2n}}$ is a split graph if n is an odd integer.
In Section 3, we find the detour index, eccentric connectivity and total eccentricity polynomials of the non-commuting graph $\Gamma_{D_{2n}}$. In Section 4, we find the mean distance of the graph $\Gamma_{D_{2n}}$.

2 Some properties of the non - commuting graph of D_{2n}

Recall that, for any $n \geq 3$, $D_{2n} = \langle r, s : r^n = s^2 = 1, srs = r^{-1}\rangle$, $\Omega_1 = \{r^i : 1 \leq i \leq n\} - Z(D_{2n})$, and $\Omega_2 = \{sr^i : 1 \leq i \leq n\}$.

We start with the following lemma, which has been proved in [1].

Lemma 2.1. *Let G be any non-abelian finite group and a be any vertex of Γ_G. Then $deg_{\Gamma_G}(a) = |G| - |C_G(a)|$, where $C_G(a)$ is the centralizer of the element a in the group G.*

According to the above lemma, we can state the following.

Theorem 2.2. *In the graph Γ_Ω, where $\Omega = \Omega_1 \cup \Omega_2$, we have*
1. $deg_{\Gamma_\Omega}(r^i) = n$ *for any* n,
2. $deg_{\Gamma_\Omega}(sr^i) = \begin{cases} 2n-2, & \text{if } n \text{ is odd} \\ 2n-4, & \text{if } n \text{ is even.} \end{cases}$

Proof. 1. Since $C_{D_{2n}}(r^i) = \{r^i : 1 \leq i \leq n\}$, then, from Lemma 2.1, $deg_{\Gamma_\Omega}(r^i) = |D_{2n}| - |C_{D_{2n}}(r^i)| = 2n - n = n$.
2. If n is odd, then $C_{D_{2n}}(sr^i) = \{1, sr^i\}$ for all i, $1 \leq i \leq n$. This follows that $deg_{\Gamma_\Omega}(sr^i) = 2n-2$ for all $1 \leq i \leq n$. If n is even, then $C_{D_{2n}}(sr^i) = \{1, r^{\frac{n}{2}}, sr^i, sr^{\frac{n}{2}+i}\}$ for all $1 \leq i \leq n$. Thus, $deg_{\Gamma_\Omega}(sr^i) = 2n-4$ for all $1 \leq i \leq n$. □

Theorem 2.3. *Let Γ_Ω be a non-commuting graph on D_{2n}.*

1. *If $\Omega = \Omega_1$, then $\Gamma_\Omega = \overline{K}_l$, where $l = |\Omega_1|$.*
2. *If $\Omega = \Omega_2$, then*

$$\Gamma_\Omega = \begin{cases} K_n, & \text{if } n \text{ is odd} \\ K_n - \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$$

where $\frac{n}{2}K_2$ denotes $\frac{n}{2}$ copies of K_2.

Proof. 1. The centralizer of r^i, $1 \leq i \leq n$, is $C_{D_{2n}}(r^i) = \{r^i : 1 \leq i \leq n\}$ of size n, then there is no edge between any pair of vertices in Γ_{Ω_1}. Thus, $\Gamma_{\Omega_1} = \overline{K}_l$, where $l = |\Omega_1|$.
2. **When n is odd**. Since the element sr^i, where $i = 1, 2, ..., n$, has centralizer $C_{D_{2n}}(sr^i) = \{1, sr^i\}$ of size 2, so let $\Omega = \Omega_2 = \{sr, sr^2, ..., sr^n\}$. Then the subgraph $\Gamma_\Omega = K_n$ is complete.
When n is even. Since $C_{D_{2n}}(sr^i) = \{1, r^{\frac{n}{2}}, sr^i, sr^{\frac{n}{2}+i}\}$ for all $1 \leq i \leq n$. Then there is no edge between the vertices sr^i and $sr^{\frac{n}{2}+i}$ in Γ_Ω for all $1 \leq i \leq n$. Therefore, $\Gamma_\Omega = K_n - \frac{n}{2}K_2$ □

Theorem 2.4. *Let $n \geq 3$ be an odd integer and H be a subset of $D_{2n} - Z(D_{2n})$. Then $\Gamma_H = K_{1,n-1}$ if and only if $H = \{sr^i, r, r^2, \cdots, r^{n-1}\}$ for some i.*

Proof. Suppose that $\Gamma_H = K_{1,n}$. By Theorem 2.2, $H = \{sr^i, r, r^2, \cdots, r^{n-1}\}$ for some i. Conversely, suppose $H = \{sr^i, r, r^2, \cdots, r^{n-1}\}$. Then $C_H(sr^i) = \{sr^i\}$ and $C_H(r^j) = \{r, r^2, \cdots, r^{n-1}\}$ for $1 \leq j<n$. Thus, $\Gamma_H = K_{1,n-1}$. □

Corollary 2.5. *Let $n \geq 3$ be an odd integer and $\Omega = \Omega_1 \cup \Omega_2$. Then Γ_Ω is a split graph.*

Proof. The proof follows from Theorem 2.3 and Theorem 2.4. $\square$

Theorem 2.6. *Let Γ_Ω be a non-commuting graph on D_{2n}, where $\Omega = \Omega_1 \cup \Omega_2$. We have*

$$|E(\Gamma_\Omega)| = \begin{cases} \frac{3n(n-1)}{2}, & \text{if } n \text{ is odd;} \\ \frac{3n(n-2)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. It is clear that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = D_{2n} - Z(D_{2n}) = \Omega$. According to n, there are two cases to consider.
Case 1. If n is odd, then the subgraph induced by Ω_1 has no edges and the subgraph induced by Ω_2 is complete. Thus, the number of edges in Γ_Ω is sum of the number of edges in $\langle \Omega_2 \rangle$ and the number of edges from set of vertices in Ω_1 to set of vertices in Ω_2. Therefore, $|E(\Gamma_\Omega)| = \frac{n(n-1)}{2} + n(n-1) = \frac{3n(n-1)}{2}$.
Case 2. If n is even, then the subgraph induced by Ω_1 has no edges and the subgraph induced by Ω_2 has $\frac{n(n-1)}{2} - \frac{n}{2} = \frac{n(n-2)}{2}$ edges. Thus, the number of edges in Γ_Ω is sum of the number of edges in $\langle \Omega_2 \rangle$ and the number of edges from set of vertices in Ω_1 to set of vertices in Ω_2. Therefore, $|E(\Gamma_\Omega)| = \frac{n(n-2)}{2} + n(n-2) = \frac{3n(n-2)}{2}$. $\square$

3 Detour index, eccentric connectivity and total eccentricity polynomials of non- commuting graphs on D_{2n}

Theorem 3.1. *Let Γ_Ω be a non-commuting graph on D_{2n}, where $\Omega = \Omega_1 \cup \Omega_2$. Then for any $u, v \in \Gamma_\Omega$,*

$$D(u, v) = \begin{cases} 2n - 2, & \text{if } n \text{ is odd;} \\ 2n - 3, & \text{if } n \text{ is even.} \end{cases}$$

Proof. There are two cases. **When n is odd**. From Theorem 2.3 and Theorem 2.4, we see that no two vertices in Ω_1 are adjacent, any pair of distinct vertices in Ω_2 are adjacent, and each vertex in Ω_1 is adjacent to every vertex in Ω_2. Then for all $u, v \in \Omega$, there is a $u - v$ path of length $2n - 2$.
When n is even. Again, no two vertices in Ω_1 are adjacent, each vertex in Ω_1 is adjacent to every vertex in Ω_2, and any pair of distinct vertices u and v in Ω_2 are adjacent if $u, v \notin \{sr^i, sr^{\frac{n}{2}+i}\}$ for $1 \leq i \leq \frac{n}{2}$. So, for all $u, v \in \Omega$, there is a $u - v$ path of length $2n - 3$. $\square$

Theorem 3.2. *Let Γ_Ω be a non-commuting graph on D_{2n}, where $\Omega = \Omega_1 \cup \Omega_2$. Then*

$$D(\Gamma_\Omega, x) = \begin{cases} (n-1)(2n-1)x^{2n-2}, & \text{if } n \text{ is odd;} \\ (n-1)(2n-3)x^{2n-3}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. **Case 1**. n is odd. Since $|\Gamma_\Omega| = 2n-1$, there are $\binom{2n-1}{2} = (n-1)(2n-1)$ possibilities of distinct pairs of vertices. By Theorem 3.1, $D(u,v) = 2n-2$ for any $u, v \in \Gamma_\Omega$. Then $D(\Gamma_\Omega, x) = \sum_{\{u,v\}} x^{D(u,v)} = \binom{2n-1}{2} x^{2n-2} = (n-1)(2n-1)x^{2n-2}$.
Case 2. n is even. We have that $|\Gamma_\Omega| = 2n-2$ and the possibility of taking distinct pairs of vertices form Γ_Ω is $\binom{2n-2}{2} = (n-1)(2n-3)$. From Theorem 3.1, we deduce that $D(\Gamma_\Omega, x) = \sum_{\{u,v\}} x^{D(u,v)} = \binom{2n-2}{2} x^{2n-3} = (n-1)(2n-3)x^{2n-3}$. □

Corollary 3.3. *For the graph Γ_Ω,*

$$dd(\Gamma_\Omega) = \begin{cases} 2(n-1)^2(2n-1), & \text{if } n \text{ is odd;} \\ (n-1)(2n-3)^2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. It is clear that $dd(\Gamma_\Omega) = \frac{d}{dx}(D(\Gamma_\Omega, x))|_{x=1}$. From Theorem 3.2, the result follows. □

Theorem 3.4. *Let Γ_Ω be a non-commuting graph on D_{2n}, where $\Omega = \Omega_1 \cup \Omega_2$.*

1. *When n is odd, then*

$$ecc(v) = \begin{cases} 2, & \text{if } v \in \Omega_1; \\ 1, & \text{if } v \in \Omega_2. \end{cases}$$

2. *When n is even, then $ecc(v) = 2$ for each $v \in \Omega$.*

Proof. 1. **When n is odd**. There is no edge between any pair of vertices in Ω_1 and each vertex in Ω_2 is adjacent to every vertex in Ω. So the maximum distance between any vertex of Ω_1 and the other vertices in Ω is 2 and the maximum distance between any vertex of Ω_2 and the other vertices in Ω is 1.
2. **When n is even**. Again, There is no edge between any pair of vertices in Ω_1. Also, each vertex in Ω_1 is adjacent to every vertex in Ω_2. Thus, $ecc(v) = 2$ for each $v \in \Omega_1$. By Theorem 2.3, the subgraph Γ_{Ω_2} is not a complete graph because there is no edge between the vertices sr^i and $sr^{i+\frac{n}{2}}$. This means that the maximum distance between any vertex in Ω_2 and any other vertex in Ω is 2, so $ecc(v) = 2$ for each $v \in \Omega_2$. □

From above theorem, we can have the following.

Theorem 3.5. *Let Γ_Ω be a non-commuting graph on D_{2n}, where $\Omega = \Omega_1 \cup \Omega_2$. Then*

1.
$$\Xi(\Gamma_\Omega, x) = \begin{cases} n(n-1)x^2 + 2n(n-1)x, & \text{if } n \text{ is odd;} \\ 3n(n-2)x^2, & \text{if } n \text{ is even.} \end{cases}$$

2.
$$\Theta(\Gamma_\Omega, x) = \begin{cases} (n-1)x^2 + nx, & \text{if } n \text{ is odd;} \\ 2(n-1)x^2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. The proof follows directly from Theorem 2.2 and Theorem 3.4. □

From the above theorem, one can obtain the eccentric connectivity index and the total eccentricity of a graph Γ_Ω from their corresponding polynomials by computing their first derivatives at $x = 1$.

Corollary 3.6. *Let Γ_Ω be a non-commuting graph on D_{2n}, where $\Omega = \Omega_1 \cup \Omega_2$. Then*

$$\xi^c(\Gamma_\Omega) = \begin{cases} 4n(n-1), & \text{if } n \text{ is odd;} \\ 6n(n-2), & \text{if } n \text{ is even.} \end{cases}$$

4 The mean distance of the graph $\Gamma_{D_{2n}}$

Through this section we find the mean (average) distance of the graph $\Gamma_{D_{2n}}$.

Lemma 4.1. *In the graph $\Gamma_{D_{2n}}$, where n is odd, the transmission of each vertex r^i is $\sigma(r^i, \Gamma_{D_{2n}}) = 3n - 4$ for all $1 \le i \le n-1$ and the transmission of a vertex sr^i is $\sigma(sr^i, \Gamma_{D_{2n}}) = 2n - 2$ for all $1 \le i \le n$.*

Proof. The vertices set of the graph $\Gamma_{D_{2n}}$ is $V(\Gamma_{D_{2n}}) = \{r^i, sr^j : 1 \le i < n, 1 \le j \le n\}$. Then $|V(\Gamma_{D_{2n}})| = 2n - 1$, where n is odd. A vertex r^i is adjacent with all vertices sr^j for all $1 \le j \le n$, so, $d(r^i, sr^j) = 1$ for all $1 \le i \le n-1$ and all $1 \le j \le n$. While a vertex r^i is not adjacent with r^j for all $i \ne j$, $1 \le i \le n-1$ and $1 \le j \le n$, then $d(r^i, r^j) = 2$ for all $1 \le i \le n-1$, $1 \le j \le n$ and $i \ne j$. So,

$$\sigma(r^i, \Gamma_{D_{2n}}) = \Sigma_{\substack{1 \le j < n \\ j \ne i}} d(r^i, r^j) + \Sigma_{1 \le j \le n} d(r^i, sr^j) = 2(n-2) + n = 3n - 4$$

for all $1 \le i \le n-1$. On the other hand every vertex sr^i is adjacent with sr^j for all $i \ne j$, $1 \le i, j \le n$. Therefore, $d(sr^i, sr^j) = 1$, for all $i \ne j$, $1 \le i, j \le n$. Also, every vertex sr^i is adjacent with r^j, then $d(sr^i, r^j) = 1$ for all $1 \le i \le n$, $1 \le j \le n-1$. So,

$$\sigma(sr^i, \Gamma_{D_{2n}}) = \Sigma_{\substack{1 \le i,j \le n \\ i \ne j}} d(sr^i, sr^j) + \Sigma_{1 \le j < n} d(sr^i, r^j) = (n-1) + (n-1) = 2n-2,$$

for all $1 \leq i \leq n$. □

Lemma 4.2. *In the graph $\Gamma_{D_{2n}}$, where n is even, the transmission of each vertex r^i is $\sigma(r^i, \Gamma_{D_{2n}}) = 3n - 6$ for all $1 \leq i \leq n-1$ and the transmission of a vertex sr^i is $\sigma(sr^i, \Gamma_{D_{2n}}) = 2n - 2$ for all $1 \leq i \leq n$.*

Proof. Let $M = \{1, 2, \ldots, n-1\} - \{n/2\}$. Then the vertices set of the graph $\Gamma_{D_{2n}}$, where n is even, is $V(\Gamma_{D_{2n}}) = \{r^i, sr^j : i \in M, 1 \leq j \leq n\}$. So, $|V(\Gamma_{D_{2n}})| = 2n-2$. A vertex r^i is adjacent with all vertices sr^j for all $i \in M$ and all $1 \leq j \leq n$. Thus, $d(r^i, sr^j) = 1$ for all $i \in M$ and all $1 \leq j \leq n$. Notice that every two vertices r^i and r^j are non-adjacent for all $i, j \in M$ and $i \neq j$, then $d(r^i, r^j) = 2$ for all $i, j \in M$ and $i \neq j$. So,

$$\sigma(r^i, \Gamma_{D_{2n}}) = \Sigma_{\substack{j \in S \\ j \neq i}} d(r^i, r^j) + \Sigma_{1 \leq j \leq n} d(r^i, sr^j) = 2(n-3) + n = 3n - 6$$

for all $i \in M$. Also, every vertex sr^i is adjacent with sr^j for all $i \neq j$, $1 \leq i \leq n/2$, and all $j \in \{1, 2, \ldots, n-1\} - \{i + n/2\}$, then $d(sr^i, sr^j) = 1$, for all $j \in \{1, 2, \ldots, n-1\} - \{i + n/2\}$, and $d(sr^i, sr^{i+n/2}) = 2$, for all $1 \leq i \leq n/2$. Since each vertex sr^i is adjacent with all vertices r^j, for all $1 \leq i \leq n$, and $j \in M$, then $d(sr^i, r^j) = 1$. Therefore,

$$\sigma(sr^i, \Gamma_{D_{2n}}) = \Sigma_{\substack{1 \leq j \leq n \\ j \neq i}} d(sr^i, sr^j) + \Sigma_{j \in S} d(sr^i, r^j) = (n-2) + 2 + (n-2) = 2n-2,$$

for all $1 \leq i \leq n$. □

Theorem 4.3. *The mean distance of the graph $\Gamma_{D_{2n}}$, where n is odd, is $\mu(\Gamma_{D_{2n}}) = \frac{5n-4}{4n-2}$.*

Proof. By Lemma 4.1, we see that the transmission of the graph $\Gamma_{D_{2n}}$ is

$$\begin{aligned} \sigma(\Gamma_{D_{2n}}) &= \Sigma_{i=1}^{n-1} \sigma(r^i, \Gamma_{D_{2n}}) + \Sigma_{i=1}^{n} \sigma(sr^i, \Gamma_{D_{2n}}) \\ &= (n-1)(3n-4) + n(2n-2) \\ &= 5n^2 - 9n + 4. \end{aligned}$$

Notice that $|V(\Gamma_{D_{2n}})| = 2n - 1$. Therefore, $\mu(\Gamma_{D_{2n}}) = \frac{\sigma(\Gamma_{D_{2n}})}{|V(\Gamma_{D_{2n}})|(|V(\Gamma_{D_{2n}})|-1)} = \frac{5n^2-9n+4}{(2n-1)(2n-2)} = \frac{5n-4}{4n-2}$. □

Theorem 4.4. *The mean distance of the graph $\Gamma_{D_{2n}}$, where n is even, is $\mu(\Gamma_{D_{2n}}) = \frac{5n^2-14n+12}{(2n-2)(2n-3)}$.*

Proof. By using Lemma 4.2, we can find the transmission of the graph $\Gamma_{D_{2n}}$ which is

$$\begin{aligned}\sigma(\Gamma_{D_{2n}}) &= \Sigma_{\substack{i=1\\ i\neq n/2}}^{n-1}\sigma(r^i,\Gamma_{D_{2n}}) + \Sigma_{i=1}^{n}\sigma(sr^i,\Gamma_{D_{2n}})\\ &= (n-2)(3n-6) + n(2n-2)\\ &= 5n^2 - 14n + 12.\end{aligned}$$

Notice that $|V(\Gamma_{D_{2n}})| = 2n-2$. Therefore, $\mu(\Gamma_{D_{2n}}) = \frac{\sigma(\Gamma_{D_{2n}})}{|V(\Gamma_{D_{2n}})|(|V(\Gamma_{D_{2n}})|-1)} = \frac{5n^2-14n+12}{(2n-2)(2n-3)}$. □

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