# A Note on the Generator Subgraph of a Graph

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**Abstract**

Graphs considered in this paper are finite simple graphs. Let $G = (V(G), E(G))$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$, for some positive integer $m$. The *edge space* of $G$, denoted by $\mathscr{E}(G)$, is a vector space over the field $\mathbb{Z}_2$. The elements of $\mathscr{E}(G)$ are all the subsets of $E(G)$. Vector addition is defined as $X + Y = X \,\Delta\, Y$, the symmetric difference of sets $X$ and $Y$, for $X, Y \in \mathscr{E}(G)$. Scalar multiplication is defined as $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for $X \in \mathscr{E}(G)$. Let $H$ be a subgraph of $G$. The *uniform set of* $H$ with respect to $G$, denoted by $E_H(G)$, is the set of all elements of $\mathscr{E}(G)$ that induces a subgraph isomorphic to $H$. The subspace of $\mathscr{E}(G)$ generated by $E_H(G)$ shall be denoted by $\mathscr{E}_H(G)$. If $E_H(G)$ is a generating set, that is $\mathscr{E}_H(G) = \mathscr{E}(G)$, then $H$ is called a *generator subgraph* of $G$. This study determines the dimension of subspace generated by the set of all subsets of $E(G)$ with even cardinality and the subspace generated by the set of all $k-$ subsets of $E(G)$, for some positive integer $k$, $1 \leq k \leq m$. Moreover, this paper determines all the generator subgraphs of star graphs. Furthermore, it gives a characterization for a graph $G$ so that star is a generator subgraph of $G$.

*Keywords:* Edge Space, Even Edge Space, Edge-Induced Subgraph, Uniform Set, Generator Subgraph
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## 1. Introduction

Graphs considered in this paper are finite simple graphs, which has no loops and multiple edges. For $x, y \in V(G)$, we denote by $[x, y]$ if and only if $x$ and $y$ are adjacent in $G$. For other basic concepts in graph theory, reader may refer to the book written by Chartrand & Zhang [1].

Let $G$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$, for some positive integer $m$. The *edge space* of $G$, denoted by $\mathscr{E}(G)$, is a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$. The elements of $\mathscr{E}(G)$ are all the subsets of $E(G)$. Vector addition is defined as $X + Y = X \ \Delta \ Y$, the symmetric difference of sets $X$ and $Y$, for $X, Y \in \mathscr{E}(G)$. Scalar multiplication is defined as $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for $X \in \mathscr{E}(G)$. The set $S \subseteq \mathscr{E}(G)$ is called a generating set if every element of $\mathscr{E}(G)$ is a linear combination of the elements of $S$.

It can be verified that the set $\mathscr{A} = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$ forms a basis of $\mathscr{E}(G)$. Hence, $\dim \mathscr{E}(G) = m$, the size of $G$. Valdez, Gervacio and Bengo [4] called this set the natural basis for the edge space of $G$.

For a non-empty set $X \subseteq E(G)$, the smallest subgraph of $G$ with edge set $X$ is called the edge-induced subgraph of $G$, which we denote by $G[X]$. In this paper, when we say induced subgraph, we mean an edge-induced subgraph of a graph.

Let $H$ be a subgraph of $G$. The *uniform set of* $H$ with respect to $G$, denoted by $E_H(G)$, is the set of all elements of $\mathscr{E}(G)$ that induces a subgraph isomorphic to $H$. The subspace of $\mathscr{E}(G)$ generated by $E_H(G)$ shall be denoted by $\mathscr{E}_H(G)$. If $E_H(G)$ is a generating set, that is $\mathscr{E}_H(G) = \mathscr{E}(G)$, then $H$ is called a *generator subgraph* of $G$.

Clearly, $\mathscr{E}_H(G) \subseteq \mathscr{E}(G)$. To show that a subgraph $H$ is a generator subgraph of $G$, it is sufficient to show that $\mathscr{E}(G) \subseteq \mathscr{E}_H(G)$. That is, the basis $\{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\} \subseteq \mathscr{E}_H(G)$. Equivalently, we have the following remark.

**Remark 1.** *Let $H$ be a subgraph of $G$. Then $H$ is a generator subgraph of $G$ if and only if for every $e \in E(G)$ the singleton $\{e\} \in \mathscr{E}_H(G)$.*

For example, let $G = K_4$, a complete graph of order 4, where $E(K_4) = \{e_1, e_2, \ldots, e_6\}$ as shown in Figure 1. Let $H = P_4$, a path of order 4. We show that $P_4$ is a generator subgraph of $K_4$.

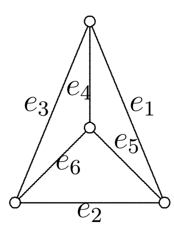


Figure 1: The labeling of $K_4$

First, we identify the elements of $E_{P_4}(K_4)$. Let $A_1 = \{e_2, e_4, e_5\}$. Then $A_1 \in E_{P_4}(K_4)$ since $G[A_1]$ is isomorphic to $P_4$, as shown in Figure 2.

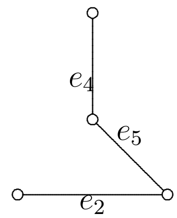


Figure 2: The graph $G[A_1]$

By enumerating all the elements of $E_{P_4}(K_4)$, we have the following:

| | | | |
|---|---|---|---|
| $A_1 =$ | $\{e_2, e_4, e_5\}$; | $A_7 =$ | $\{e_3, e_4, e_5\}$ |
| $A_2 =$ | $\{e_2, e_4, e_6\}$; | $A_8 =$ | $\{e_1, e_3, e_6\}$ |
| $A_3 =$ | $\{e_1, e_2, e_6\}$; | $A_9 =$ | $\{e_2, e_3, e_4\}$ |
| $A_4 =$ | $\{e_2, e_3, e_5\}$; | $A_{10} =$ | $\{e_1, e_2, e_4\}$ |
| $A_5 =$ | $\{e_1, e_5, e_6\}$; | $A_{11} =$ | $\{e_1, e_4, e_6\}$ |
| $A_6 =$ | $\{e_3, e_5, e_6\}$; | $A_{12} =$ | $\{e_1, e_3, e_5\}$ |

Next, we show that each singleton is an element of $\mathscr{E}_{P_4}(K_4)$. By trial and error, we have

$$\begin{aligned} A_1 + A_2 + A_5 &= (A_1 + A_2) + A_5 \\ &= (A_1 \Delta A_2)\Delta A_5 \\ &= (\{e_2, e_4, e_5\}\Delta\{e_2, e_4, e_6\})\Delta\{e_1, e_5, e_6\} \\ &= \{e_5, e_6\}\Delta\{e_1, e_5, e_6\} \\ &= \{e_1\}. \end{aligned}$$

Similarly,

$$\begin{aligned} \{e_2\} &= A_2 + A_6 + A_7 \\ \{e_3\} &= A_5 + A_7 + A_{11} \\ \{e_4\} &= A_2 + A_4 + A_6 \\ \{e_5\} &= A_8 + A_{11} + A_{12} \\ \{e_6\} &= A_1 + A_5 + A_{10} \end{aligned}$$

This shows that $P_4$ is a generator subgraph of $K_4$ by Remark 1.

The problem on generator subgraph of a graph was introduced by Gervacio in 2008. There are some researchers who worked on the generator subgraph problem. They investigated the generator subgraphs of a particular graph. In [2], a characterization of the generator subgraphs of the complete graph was established. Ruivivar [7] identified some generator subgraphs of the complete bipartite graph $K_{m,n}$. Valdez Bengo, and Gervacio [4] identified some generator subgraphs of wheels and fans.

Prior to the introduction of the generator subgraph problem, Gervacio and Mame [5], introduced the universal and primitive graphs. The study focused on the determination whether the given graph $G$ is a universal graph or a primitive graph. It is related to the problem on generator subgraphs in the sense that the term universal graphs later became the generator graphs described in [2], and at present called the generator subgraph of complete graphs [3]. A characterization of the primitive graphs was found. There is no characterization for universal graphs but one significant result found was a necessary condition for universal graphs. It was shown that if $G$ is universal then the size of $G$ is odd. This result gives rise to the fundamental theorem on generator subgraph that any generator subgraph has an odd number of edges. Since then, in identifying generator subgraphs of a graph $G$, we only consider the subgraphs with odd number of edges. Formally, we have the following theorem.

**Theorem 1.** *Let $H$ be a subgraph of the graph $G$. If $H$ is a generator subgraph of $G$, then $|E(H)|$ is odd.*

For a nonempty graph $G$ and considering the path $P_2$, it can be observed that $E_{P_2}(G)$ is precisely the set of all singletons in $\mathscr{E}(G)$, which is a basis of $\mathscr{E}(G)$. Consequently, we have the following theorem.

**Theorem 2.** *Let $G$ be a graph with $|E(G)| = m > 0$. Then the path $P_2$ is a generator subgraph of $G$.*

Let $G$ be a graph and consider a subgraph $H$ of $G$ that contain an isolated vertex. It is obvious that $E_H(G) = \emptyset$. Thus, $\mathscr{E}_H(G) = \emptyset$. A useful remark is stated below.

**Remark 2.** *If $H$ is a generator subgraph of $G$, then $H$ contains no isolated vertex.*

The next theorem is equivalent to the known theorem in linear algebra about dimension of a subspace of a vector space over a field.

**Theorem 3.** *Let $G$ be a graph with $|E(G)| = m$. If $H$ is a generator subgraph of $G$, then $|E_H(G)| \geq m$.*

The converse of the above theorem is not true. For instance, let $G = W_4$, a wheel of order 5 and $H = S_3$, a star graph of order 4. It can be shown that $|E_{S_3}(W_4)| = 8 = \dim \mathscr{E}(W_4)$. It can be verified that the subspace generated by $E_{S_3}(W_n)$ has dimension 7. Hence, $E_{S_3}(W_n)$ does not span $\mathscr{E}(W_4)$ so $S_3$ is not a generator subgraph of $W_4$.

## 2. Results

First we investigated the subspace of $\mathscr{E}(G)$ generated by some classes of subsets of $E(G)$.

### *2.1. Even Edge Space of a Graph*

By $\mathscr{E}^*(G)$, we mean the set of all subsets of $E(G)$ with even cardinality. The first result gives a relation between $\mathscr{E}^*(G)$ and $\mathscr{E}(G)$.

**Theorem 4.** *Let $G$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$. Then $\mathscr{E}^*(G)$ is a subspace $\mathscr{E}(G)$. Moreover,* $\dim \mathscr{E}^*(G) = m - 1$.

*Proof.* Clearly, $\mathscr{E}^*(G)$ is a subset of $\mathscr{E}(G)$ and $\mathscr{E}^*(G)$ is not empty since $\emptyset \in \mathscr{E}^*(G)$. Let $X_1, X_2 \in \mathscr{E}^*(G)$, then $X_1 + X_2 \in \mathscr{E}^*(G)$ since $|X_1 + X_2| = |X_1 \Delta X_2| = |X_1| + |X_2| - 2|X_1 \cap X_2|$ is even. Further, let $c \in \mathbb{Z}_2$ and $X \in \mathscr{E}^*(G)$, then either $c \cdot X = \emptyset$ or $c \cdot X = X$. In both cases, $|c \cdot X|$ is even so $c \cdot X \in \mathscr{E}^*(G)$. Hence, $\mathscr{E}^*(G)$ is a subspace of $\mathscr{E}(G)$.

Now, we find the dimension of $\mathscr{E}^*(G)$. Let $\mathscr{E}'(G) = \{X \in \mathscr{E}(G) : |X| \text{ is odd}\}$. We know that $\mathscr{E}(G)$ is the power set of a non-empty set $E(G)$. Klasar [6] showed that if $S$ is a non-empty set and $\mathscr{P}(S)$ is the power set of $S$ then the number of elements of $\mathscr{P}(S)$ with even cardinality is equal to the number of elements of $\mathscr{P}(S)$ with odd cardinality. Thus, $|\mathscr{E}^*(G)| = |\mathscr{E}'(G)| = \frac{1}{2}|\mathscr{E}(G)| = 2^{m-1}$. Now, let $\dim \mathscr{E}^*(G) = k$ and let $\mathscr{B} = \{X_1, X_2, \ldots, X_k\}$ be a basis for $\mathscr{E}^*(G)$. Then any vector in $\mathscr{E}^*(G)$ is of the form

$$c_1X_1 + c_2X_2 + \ldots + c_kX_k$$

and every vector is uniquely expressible in this form. Since $c_i$ is either 0 or 1 for each $i$, the total number of vectors in $\mathscr{E}^*(G)$ must be $2^k$. Since $|\mathscr{E}^*(G)| = 2^{m-1}$, it follows that $k = m - 1$. $\square$

In this paper, we shall call the vector space $\mathscr{E}^*(G)$ the *even edge space* of a graph $G$.

The following remark is a known result in linear algebra.

**Remark 3.** *If $A \subseteq \mathscr{E}^*(G)$, then the set of all linear combinations of the elements of $A$ is a subspace of $\mathscr{E}^*(G)$.*

Consequently, we have the next theorem.

**Theorem 5.** *Let $H$ be a subgraph of $G$. If $|E(H)|$ is even, then $\mathscr{E}_H(G) \subseteq \mathscr{E}^*(G)$.*

*Proof.* Since $|E(H)|$ is even, each $A \in E_H(G)$ has even cardinality. Thus $E_H(G) \subseteq \mathscr{E}^*(G)$. By Remark 3, $\mathscr{E}_H(G) \subseteq \mathscr{E}^*(G)$. $\square$

We now identify a basis for $\mathscr{E}^*(G)$. Let $G$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and define $\mathscr{B} = \{X_1, X_2, \ldots, X_{m-1}\}$, where $X_1 = \{e_1, e_2\}, X_2 = \{e_1, e_3\}, \ldots, X_{m-1} = \{e_1, e_m\}$. Since $X \in \mathscr{E}^*(G)$ can be expressed as a union of disjoint sets $\{e_i, e_j\} = \{e_1, e_i\}\Delta\{e_1, e_j\}$ , where $1 \leq i, j \leq m$, then $\mathscr{B}$ spans $\mathscr{E}^*(G)$. Since $|\mathscr{B}| = m - 1 = \dim \mathscr{E}^*(G)$, it follows that $\mathscr{B}$ forms a basis for $\mathscr{E}^*(G)$.

It is easily seen that $\mathscr{E}^*(G)$ is a maximal proper subspace of $\mathscr{E}(G)$.

*2.2. The $\mathscr{E}_k(G)$ Subspace*

Here we determine the dimension of the subspace of $\mathscr{E}(G)$ generated by the set of all $k$-subsets of $E(G)$.

**Definition 1.** *Let $G$ be graph with $m > 0$ edges. For a positive integer $k$, denote by $E_k(G)$ the set of all $k$-subsets of $E(G)$ and let $\mathscr{E}_k(G)$ denote the subspace of $\mathscr{E}(G)$ generated by $E_k(G)$.*

For instance, let $G$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ for some positive integer $m$. Then $E_1(G) = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$. Note that $E_1(G)$ is the natural basis for $\mathscr{E}(G)$ so $\mathscr{E}_1(G) = \mathscr{E}(G)$. Thus, $\dim \mathscr{E}_1(G) = m$. The set $E_m(G)$ contains exactly one element, the edge set of $G$. Since $E(G)$ is non-empty, $\dim \mathscr{E}_m(G) = 1$.

The following result shows the relation between $\mathscr{E}_k(G)$ and $\mathscr{E}^*(G)$.

**Lemma 1.** *Let $G$ be a graph with size $m > 0$ and let $k$ be a positive integer where $1 \leq k \leq m - 1$. Then $\mathscr{E}^*(G) \subseteq \mathscr{E}_k(G)$.*

*Proof.* Let $G$ be a graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let $k$ be a positive integer where $1 \leq k \leq m - 1$. Clearly, $\mathscr{E}^*(G) \subseteq \mathscr{E}_1(G)$ so we may assume that $k > 1$. Let $e_i$ be an element of $E(G)$ for some $i$, $1 \leq i \leq m$. Let $A \in E_k(G)$ such that $e_i \in A$. Since $k < m$, there exists $e_j \in E(G)$ such that $e_j \notin A$ for some $j$, $1 \leq j \leq m$ and $j \neq i$. Define $B = \{e_j\} \cup A \backslash \{e_i\}$. Obviously, $B \in E_k(G)$. Thus, $\{e_i, e_j\} = A \Delta B \in \mathscr{E}_k(G)$. In particular, the set $\mathscr{B} = \{\{e_1, e_2\}, \{e_1, e_3\}, \ldots, \{e_1, e_m\}\}$ is a subset of $\mathscr{E}_k(G)$. Since $\mathscr{B}$ forms a basis for $\mathscr{E}^*(G)$, it follows that $\mathscr{E}^*(G) \subseteq \mathscr{E}_k(G)$. □

The next result gives the dimension of $\mathscr{E}_k(G)$ for all values of $k$.

**Theorem 6.** *Let $G$ be a graph with size $m > 0$ and let $k$ be a positive integer where $1 \leq k \leq m$. Then*

$$\dim \mathscr{E}_k(G) = \begin{cases} 1 & \text{if } k = m, \\ m - 1 & \text{if } k \text{ is even, and} \\ m & \text{if } k \text{ is odd.} \end{cases}$$

*Proof.* Let $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let $k$ be an integer where $1 \leq k \leq m$. We know earlier that $\dim \mathscr{E}_k(G) = 1$ if $k = m$ and $\dim \mathscr{E}_k(G) = m$ if $k = 1$. We now assume that $1 < k \leq m-1$. Consider the two cases: Case 1, $k$ is even. Then $E_k(G)$ consists of sets with even cardinality. Thus, $\mathscr{E}_k(G) \subseteq \mathscr{E}^*(G)$ in view of Remark 3. By Lemma 1, $\mathscr{E}^*(G) \subseteq \mathscr{E}_k(G)$. Therefore $\mathscr{E}_k(G) = \mathscr{E}^*(G)$. It follows that $\dim \mathscr{E}_k(G) = m-1$. Case 2, $k$ is odd. Let $e_i \in E(G)$, $1 \leq i \leq m$. Then there exists $A \in E_k(G)$ such that $e_i \in A$. Define $B = A \backslash \{e_i\}$. Since $|A| = k$ is odd, $|B|$ is even so $B \in \mathscr{E}^*(G)$. By Lemma 1, $B \in \mathscr{E}_k(G)$. Now, $\{e_i\} = A \Delta B \in \mathscr{E}_k(G)$. Meaning, $E_k(G)$ is a generating set for $\mathscr{E}(G)$. Hence, $\mathscr{E}(G) \subseteq \mathscr{E}_k(G)$. But we know that $\mathscr{E}_k(G) \subseteq \mathscr{E}(G)$. Therefore $\mathscr{E}_k(G) = \mathscr{E}(G)$. It follows that $\dim \mathscr{E}_k(G) = m$.

□

The next result determines another basis for the edge space of $G$.

**Theorem 7.** *Let $G$ be a graph with size $m > 0$. If $m$ is even, then the set $E_{m-1}(G)$ forms a basis for $\mathscr{E}(G)$.*

*Proof.* Let $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $A_i = E(G) \backslash \{e_i\}$ where $1 \leq i \leq m$. Then $E_{m-1}(G) = \{A_1, A_2, \ldots, A_m\}$. Since $m$ is even, $m-1$ is odd. By Lemma 1, $\mathscr{E}_{m-1}(G) = \mathscr{E}(G)$. Thus, $E_{m-1}(G)$ spans $\mathscr{E}(G)$. Since $|E_{m-1}(G)| = m = \dim \mathscr{E}(G)$, it follows that $E_{m-1}(G)$ forms a basis for $\mathscr{E}(G)$. □

**Corollary 1.** *Let $G$ be a graph with size $m > 0$. If $m$ is odd, then the set $E_{m-1}(G)$ is a linearly dependent set.*

*2.3. Generator Subgraphs of Stars*

By a *star* of order $n+1$, denoted by $S_n$, we mean a graph which consists of an independent set of $n$ vertices each of which is adjacent to a common vertex called the *central vertex*. The size of $S_n$ is $n$. Hence $\dim \mathscr{E}(S_n) = n$ and $\dim \mathscr{E}^*(S_n) = n-1$. Here we determine all generator subgraphs of star graphs.

Let $E(S_n) = \{e_1, e_2, \ldots, e_n\}$. For a positive integer $q$, we can view $E_{S_q}(S_n)$ as $E_q(S_n)$, the set of all $q$-subsets of $E(S_n)$, since for each $A \in E_q(S_n)$, $S_n[A] \simeq S_q$. In fact, it is easy to verify that $E_{S_q}(S_n) = E_q(S_n)$. However, this equality holds only for some graphs.

First we establish a relation between $\mathscr{E}_{S_q}(S_n)$ and $\mathscr{E}^*(S_n)$.

**Lemma 2.** *Let $S_q$ be a subgraph of $S_n$ for some positive integers $q$ and $n$. If $q < n$, then $\mathscr{E}^*(S_n) \subseteq \mathscr{E}_{S_q}(S_n)$.*

*Proof.* Let $S_q$ be a subgraph of $S_n$ where $q < n$. We know earlier that $\mathscr{E}_{S_q}(S_n) = \mathscr{E}_q(S_n)$. Thus, by Lemma 1, $\mathscr{E}^*(S_n) \subseteq \mathscr{E}_{S_q}(S_n)$. □

The next theorem gives a family of generator subgraphs of $S_n$.

**Theorem 8.** *For positive integers $q$ and $n$ where $q < n$, the star $S_q$ is a generator subgraph of $S_n$ if and only if $q$ is odd.*

*Proof.* The necessary condition of the theorem follows directly from Theorem 1. Conversely, assume that $q$ is odd. We know that $\mathscr{E}_{S_q}(S_n) = \mathscr{E}_q(S_n)$. Thus, by Theorem 6, $\mathscr{E}_{S_q}(S_n) = \mathscr{E}(S_n)$. Therefore $S_q$ is a generator subgraph of $S_n$. □

The following theorem is a special case of Theorem 6.

**Theorem 9.** *Let $S_q$ be a subgraph of $S_n$ for some positive integers $q$ and $n$ where $q < n$. If $q$ is even, then $\dim \mathscr{E}_{S_q}(S_n) = n-1$.*

The next result determines the dimension of the subspace generated by the uniform sets of the subgraphs of star $S_n$.

**Theorem 10.** *Let $H$ be a subgraph of $S_n$, $n > 0$. If $H$ contains an isolated vertex then $\dim \mathscr{E}_H(S_n) = 0$. Moreover, if $H$ does not contain an isolated vertex, then*

$$\dim \mathscr{E}_H(S_n) = \begin{cases} 1 & \text{if } |E(H)| = n, \\ n-1 & \text{if } |E(H)| \text{ is even, and} \\ n & \text{if } |E(H)| \text{ is odd.} \end{cases}$$

*Proof.* Let $H$ be a subgraph of $S_n$. Then either $H$ contains an isolated vertex or $H$ does not contain an isolated vertex. Suppose $H$ contains an isolated vertex, then $E_H(S_n) = \emptyset$ in view of Remark 2. It follows that $\dim \mathscr{E}_H(S_n) = 0$. If $H$ does not contain an isolated vertex, then $H \simeq S_q$ for some positive integer $q$ where $1 \leq q \leq n$. Consider the following three cases: Case 1, $1 \leq q < n$ and $q$ is odd. By Theorem 8, $H$ is a generator subgraph of $S_n$ so $\dim \mathscr{E}_H(S_n) = n$. Case 2, $1 \leq q < n$ and $q$ is even. By Theorem 9, $\dim \mathscr{E}_H(S_n) = n - 1$. Case 3, $q = n$. Then $E_H(S_n)$ contains exactly one element, the edge set of $S_n$. Hence, $\dim \mathscr{E}_H(S_n) = 1$. □

*2.4. Star as a Generator Subgraph of Some Graphs*

This section determines some properties of graphs wherein star is one of its generator subgraphs.

**Theorem 11.** *Let $p > 0$ be an odd integer. If $G$ is a graph such that for every edge $[a, b]$ in $G$ either $\deg(a) > p$ or $\deg(b) > p$, then star $S_p$ is a generator subgraph of $G$.*

*Proof.* Let $[a, b]$ be an edge of $G$. We show that $\{[a, b]\} \in \mathscr{E}_{S_p}(G)$. Without loss of generality, assume that $\deg(a) = r > p$ for some integer $r$. Let $A = \{e_1, e_2, \ldots, e_r\}$ be the set of all edges in $G$ incident with $a$. Let $B \subseteq A$ with $|B| = p$. Then $G[A] \simeq S_r$ and $G[B] \simeq S_p$. Since $p$ is odd, $G[B]$ is a generator subgraph of $G[A]$ in view of Theorem 8. Thus, $\{e_i\} \in \mathscr{E}_{S_p}(G[A]) \subseteq \mathscr{E}_{S_p}(G)$ for all $i$, $1 \leq i \leq r$. Since $[a, b]$ is one of the $e_i's$, it follows that $\{[a, b]\} \in \mathscr{E}_{S_p}(G)$. Therefore $S_p$ is a generator subgraph of $G$. □

Below is an immediate consequence of Theorem 11.

**Corollary 2.** *Let $p > 0$ be odd. If $G$ is $k$- regular and $k > p$ then star $S_p$ is a generator subgraph of $G$.*

The converse of Theorem 11 is not true for $p = 1$ since a star $S_1 \simeq P_2$ is a generator subgraph of the graph $G = kP_2$, a graph consisting of $k$ vertex-disjoint copies of $P_2$. If $p \neq 1$, we have the following result.

**Theorem 12.** *Let $p > 1$ be odd. Then $S_p$ is a generator subgraph of $G$ if and only if for every edge $[a, b]$ in $G$, either $\deg(a) > p$ or $\deg(b) > p$.*

*Proof.* Assume that $S_p$ is a generator subgraph of $G$. Suppose, on the contrary, $\deg(a) \leq p$ and $\deg(b) \leq p$ for some $[a, b] \in E(G)$. Partition $E(G)$ into two sets $A$ and $B$ where $A = \{[a,b] \in E(G) : \deg(a) \leq p \textit{ and } \deg(b) \leq p\}$ and $B = \{[a,b] \in E(G) : \deg(a) > p \textit{ or } \deg(b) > p\}$. Clearly, $E_{S_p}(G[A]) \cap E_{S_p}(G[B]) = \emptyset$ and $E_{S_p}(G) = E_{S_p}(G[A]) \cup E_{S_p}(G[B])$. Now, let us consider the subgraph $G[A]$. Partition $V(G[A])$ into two sets $X$ and $Y$ where $X = \{x \in V(G[A]) : \deg(x) = p\}$ and $Y = \{y \in V(G[A]) : \deg(y) < p\}$. Observe that $|E_{S_p}(G[A])| = |X|$ and $|X|$ is maximum if $Y = \emptyset$. Let us assume that $Y = \emptyset$. Then $G[A]$ is $p$-regular. Thus, $\sum_{v \in V(G[X])} \deg(v) = p|X| = 2|E(G[A])|$. Since $p > 1$ is odd, $|X| = |E_{S_p}(G[A])| < |E(G[A])| = \dim \mathscr{E}(G[A])$. By Theorem 3, $S_p$ is not a generator subgraph of $G[A]$. Meaning, there exists $e \in E(G[A]) \subseteq E(G)$ such that $\{e\} \notin \mathscr{E}_{S_p}(G[A])$. It follows that $\{e\} \notin \mathscr{E}_{S_p}(G)$. This is a contradiction to the fact that $S_p$ is a generator subgraph of $G$. Therefore, for every edge $[a,b]$ in $G$, either $\deg(a) > p$ or $\deg(b) > p$. For the converse of the theorem, it follows by Theorem 11. □

The following result determines all graphs whose generator subgraph is the path $P_2$ only.

**Theorem 13.** *Let $G$ be a graph with size $m > 0$. If $m \leq 3$, then the only generator subgraph of $G$ is the path $P_2$.*

*Proof.* Let $G$ be a graph with size $m$ where $1 \leq m \leq 3$. We know by Theorem 2 that $P_2$ is a generator subgraph of $G$. Suppose there exists another generator subgraph of $G$, say $H$. Then $1 \leq |E(H)| \leq 3$. By Theorem 1, $|E(H)|$ is odd. Thus, either $|E(H)| = 1$ or $|E(H)| = 3$. Suppose $|E(H)| \neq 1$, then $|E(H)| = 3$. This implies that the size of $G$ is 3. Hence, $E_H(G) = \{E(G)\}$. It follows that $\dim \mathscr{E}_H(G) = 1 < 3 = \dim \mathscr{E}(G)$. This is a contradiction to Theorem 3. Therefore $|E(H)| = 1$. But $H$ does not contain isolated vertex by Remark 2. It follows that $H$ is isomorphic to $P_2$. □

Equivalently, we have the following remark.

**Remark 4.** *Let $G$ be a graph with size $m$. If $G$ has a generator subgraph which is not isomorphic to $P_2$, then $m \geq 4$.*

## 3. Summary and Conclusion

All generator subgraphs of star graphs were identified and a characterization for a graph $G$ so that star graph is a generator subgraph of $G$ was established. Moreover, the concept of even edge space was introduced here and found to be a maximal proper subspace of the edge space of a graph.

Finally, the dimension of even edge space and the dimension of the subspace generated by $k - subsets$ of $E(G)$ were determined.

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