ON THE ERDÖS-KO-RADO PROPERTY OF GROUPS OF ORDER A PRODUCT OF THREE PRIMES

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Abstract. Let G be a subgroup of the symmetric group $\mathbb{S}_n$. The group G has the Erdös-Ko-Rado (EKR) property, if the size of any intersecting subset of G is bounded above by the size of a point stabilizer in G. The aim of this paper is to investigate the EKR and the strict EKR properties of the groups of order pqr where p, q, r are three prime numbers.

Keywords: Cayley graph, prermutation groups, EKR property.

1. Introduction

Let $[n] = \{1, 2, \cdots n\}$, the Erdös-Ko-Rado (EKR) Theorem [5] is based on the largest family of subsets of size r from the set $[n]$ such that the intersection of each pair of subsets is non-empty. Let $n \geq 2r$, the size of largest collection is $\binom{n-1}{r-1}$ and in this case, the only collections of this size are the collections of all subsets that contain a fixed element from $[n]$. Suppose $G \leq \mathbb{S}_n$ is a permutation group. A subset S of G is said to be intersecting if for any pair of permutations $\sigma, \tau \in S$ there exists $i \in [n]$ such that $\sigma\tau^{-1}(i) = i$. A group G has the Erdös-Ko-Rado (EKR) property, if for any intersecting subset $S \subseteq G$, $|S|$ is bounded above by the size of the largest point stabilizer in G. The maximal intersecting set is one with maximum size. The group G has the strict EKR property if every maximal intersecting set is the coset of the stabilizer of a point. It is clear from the definition that if a group has the strict EKR property then it has the EKR property. A group can have the (strict) EKR property under one action while it fails to have this property under another action. In [3] the author defined a new version of the EKR property. Let the action of G on the set X is transitive, we say

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this action satisfies the weak EKR property if the cardinality of any intersecting set is bounded above by the cardinality of the stabilizers and G has the weak EKR property, if all transitive actions of G have the weak EKR property.

In 1977 Frankl and Deza [6] proved that $\mathbb{S}_n$ has the EKR property and conjectured that it has also the strict EKR property, see for more details [4, 9, 10, 17]. In [16] it is shown that $\mathbb{A}_n$ has the strict EKR property and in [17] the EKR property for some Coxeter groups is investigated. Meagher and Spiga [11, 12] stablished that the projective special group $PGL_2(q)$ has the strict EKR property, while the group $PGL_3(q)$ does not have this property. Ahmadi and Meagher [2] showed the strict EKR property of cyclic, dihedral and Frobenius groups. Recently, they also investigated the EKR property of Mathieu groups and all $2-$transitive groups with degree no more than 20, see [1]. Jalali-Rad and Ashrafi [14] investigated the EKR and the strict EKR properties of the group G where $G \in \{V_{8n}, U_{6n}, T_{4n}, SD_n\}$. We refer to [5] for background information about the history of this intresting problem. The aim of this paper is to investigate the EKR property of groups of order a product of three primes.

2. Definitions and Preliminaries

Let G be a finite group. A symmetric subset of group G is a subset $S \subseteq G$, where $e \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = Cay(G, S)$ with respect to S is a graph whose vertex set is $V(\Gamma) = G$ and two vertices $x, y \in V(\Gamma)$ are adjacent if and only if $yx^{-1} \in S$. A symmetric subset by these properties is called a Cayley set or a connection set. It is a well-known fact that a Cayley graph is connected if and only if $G = \langle S \rangle$. Recall that the Cayley graph $\Gamma = Cay(G, S)$ is normal if S is a symmetric normal subset of G.

A derangement is a permutation with no fixed points. The subset $\mathcal{D}$ of permutation group is derangement if all elements of $\mathcal{D}$ are derangement. Suppose G is a permutation group and $\mathcal{D} \subseteq G$ is a derangement set. The derangement graph $\Gamma_G = Cay(G, \mathcal{D})$ has the elements of G as its vertices and two vertices are adjacent if and only if they do not intersect. Since $\mathcal{D}$ is a union of conjugacy classes, Γ_G is a normal Cayley graph.

A clique of graph Γ is a complete subgraph of Γ. A set of vertices of Γ that induces an empty subgraph of Γ is called an independent set. The size of the largest clique and the size of the largest independent set in graph Γ are denoted by $\omega(\Gamma)$ and $\alpha(\Gamma)$, respectively. In particular for any permutation group $G \leq \mathbb{S}_n$, an independent set in Cayley graph Γ is a set of permutations in which every two permutations agree at least on one point.

Let G is a permutation group with derangement graph $\Gamma_G = Cay(G, \mathcal{D})$. By these notations, two permutations in G are intersecting if and only if their corresponding vertices are not adjacent in Γ_G. Therefore, the problem of classifying the maximum intersecting subsets of G is equivalent to characterizing the maximum independent sets of vertices in Γ_G.

The group G has the EKR property if and only if

$$\alpha(\Gamma) = max\{|G_x| \; |x \in G\}. \tag{2.1}$$

We recall that for given graph Γ, the independence number of Γ is equal with the clique number of $\bar{\Gamma}$. This means that the Eq.(2.1) can be reformulated as

$$\omega(\bar{\Gamma}) = max\{|G_x| \; |x \in G\}. \tag{2.2}$$

For independent number and clique number of a graph, we have the following result.

Lemma 2.1. [4] *If graph Γ is a vertex-transitive graph, then $\omega(\Gamma)\alpha(\Gamma) \leq |V(\Gamma)|$. Moreover, equality holds, if every maximum independent set and every maximum clique intersect.*

The group G has the strict EKR property if and only if the cosets of the largest stabilizer of a point are the only independent sets of maximum size.

Lemma 2.2. [2] *For any permutation $\sigma \in \mathbb{S}_n$, the cyclic group G generated by σ has the strict EKR property.*

Suppose G acts on X. For each $g \in G$ there is a map $\varphi_g : X \longrightarrow X$ defined by $\varphi_g : x \rightarrow xg^{-1}$ which is a permutation on X. Moreover, the map $\varphi : G \longrightarrow \mathbb{S}_X$ defined by $g \mapsto \varphi_g$ is a homomorphism; it is called the permutation representation of G corresponding to the group action G on X.

Suppose H is a subgroup of G and suppose $X = G/H$ denote to the set of right cosets of H in G. The map $\varphi : G \hookrightarrow S_X$, for every $g \in G$ is the homomorphism $\varphi(g) = \varphi_g$, where for all $a \in G$ we have $\varphi_g(Ha) = Hag^{-1}$. Let $H \leq G$, the core of H in G is defined as $cor_G(H) = \cap_{g \in G} g^{-1}Hg$. We say that H is core-free if $cor_G(H) = \{e\}$.

For the subgroup $H \leq G$ and $g \in G$, the conjugate of subgroup H in G is denoted by $H^g = g^{-1}Hg$. Suppose $G \leq Sym(n)$ is a transitive permutation group, then G is called a Frobenius group if it has a non-trivial subgroup H where $H \cap H^g = \{id\}$, for all $g \in G \setminus H$. The kernel of Frobenius group G is defined as

$$K = (G \setminus \cup_{g \in G} H^g) \cup \{id\}.$$

It is not difficult to see that all non-identity elements of K are all derangement elements of G.

Theorem 2.3. [15] (*Frobenius Theorem*) *Suppose H is a proper non-identity subgroup of G such that for all $g \in G \setminus H$.*

$$H \cap g^{-1}Hg = \{e\}. \tag{2.3}$$

Let $K = G \setminus \cup_{g \in G} g^{-1}(H \setminus \{e\})g$, then

$$K \lhd G,\ G = KH \text{ and } H \cap K = \{e\}.$$

Proposition 2.4. [2] *Let $G = KH$ be a Frobenius group with kernel K. Then G has the EKR property. Furthermore, G has the strict EKR property if and only if $|H| = 2$.*

Let us to show the direct product of two groups G, H by $G \times H$.

Corollary 2.5. [2] *Suppose $G_1 \leq Sym(n_1), ..., G_k \leq Sym(n_k)$ and $G = G_1 \times \cdots \times G_k$ then G_i $(1 \leq i \leq k)$ has the (strict) EKR property if and only if G has the (strict) EKR property.*

Theorem 2.6. [3] *Let G be a finite group which is either nilpotent or a subgroup of a direct product of groups of squrae-free order. Then G has the weak EKR property.*

3. Main results

In this section, we study the EKR property of groups of order a product of three primes. Let p be a prime number and $p > q$ where $q|p-1$. A non-abelian group of order pq has the following presentation:

$$F_{p,q} = \langle a, b : \ a^p = b^q = e, b^{-1}ab = a^u \rangle \tag{3.1}$$

where u is an element of order q in multiplicative group $\mathbb{Z}_p^*$, see [8].

It is not difficult to see that this group is a Frobenius group and hence by Proposition 2.4, it has the EKR property.

3.1. **Groups of order** pqr**.** Let $p > q > r$ be three prime numbers, in [7] the structures of all groups of order pqr are verified as follows:

- $G_1 = \mathbb{Z}_{pqr}$,
- $G_2 = F_{p,qr}(qr|p-1)$,
- $G_3 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$,
- $G_4 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$,
- $G_5 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$,
- $G_{5+d} = \langle a, b, c : a^p = b^q = c^r = e, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^d} \rangle$, where $r|p-1, q-1, q|p-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \leq d \leq r-1)$.

By Theorem 2.6, every group of order pqr (similarly every group of order p^3) has the weak EKR property and thus it has the EKR property. So we investigate the strict EKR property of them.

Theorem 3.1. *Group* $G = G_{5+d}$ $(1 \leq d \leq r-1)$ *is a Frobenius group.*

Proof. Let $H = \langle c \rangle$ and $K = \langle a, b \rangle$. Let $g = a^i b^j c^k \in G \setminus H$ and suppose that $H \cap g^{-1}Hg \neq \{e\}$. So there exist $1 \leq l, t \leq r-1$ such that $c^{-k}b^{-j}a^{-i}c^t a^i b^j c^k = c^l$. Hence $c^{-k+t}a^{-v^t i+i}b^{j-u^t j}c^k = c^l$ and so $c^t a^{v^k(i-v^t i)}b^{u^k(j-u^t j)} = c^l$. This yields that

$$\begin{cases} i - v^t i \equiv 0 \ (mod\ p), \\ j - u^t j \equiv 0 \ (mod\ q), \\ t = l. \end{cases}$$

and so $i = j = 0$, which means that $g \in H$, a contradiction. Hence, $H \cap H^g = \{e\}$ and Theorem 2.3 implies that G is a Frobenius group. Let $p, q \neq 2$, by Lemma 2.2, Proposition 2.4, Corollary 2.5 and Theorem 3.1 we conclude the following theorem.

Theorem 3.2. *Among all groups of order* pqr *only the group* G_1 *has the strict* EKR *property.*

3.2. **Groups of order** p^2q**.** According to [13] the structures of groups of order p^2q, where $p < q$ are as follows:

- $L_1 = \mathbb{Z}_{p^2q}$,
- $L_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,

- $L_3 = \mathbb{Z}_p \times F_{q,p}$ $(p|q-1)$,
- $L_4 = F_{q,p^2}$ $(p^2|q-1)$,
- $L_5 = \langle a, b : a^{p^2} = b^q = e, a^{-1}ba = b^u,\ u^p \equiv 1\ (mod\ q)\rangle$ $(p^2|q-1)$.

And the structures of groups of order p^2q where, $p > q$ are as follows:

- $Q_1 = \mathbb{Z}_{p^2q}$,
- $Q_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
- $Q_3 = \mathbb{Z}_p \times F_{p,q}$ $(q|p-1)$,
- $Q_4 = \langle a, b : a^q = b^{p^2} = 1, a^{-1}ba = b^\alpha,\ \alpha^q \equiv 1\ (mod\ p^2)\rangle$ $(q|p-1)$,
- $Q_5 = \langle a, b, c \mid a^q = b^p = c^p = 1, a^{-1}ba = c, a^{-1}ca = b^{-1}c^{2\alpha}, bc = cb, (\alpha + \sqrt{\alpha^2-1})^q = 1\ (mod\ p)\ \rangle,\ (q|p+1)$, $\alpha^2 - 1$ is not perfect square.
- $Q_{5+i} = \langle a, b, c \mid a^q = b^p = c^p = 1, a^{-1}ba = b^\beta, a^{-1}ca = c^{\beta^i}, bc = cb,\ \beta^q \equiv 1\ (mod\ p)\rangle\ (q|p-1),\ i = 1, 2, 3, \dots, \frac{q-1}{2}$ *and* $q-1$,

Lemma 3.3. *Groups $L_1, \cdots, L_4$ have the EKR property.*

Proof. By Lemma 2.2, every cyclic group has the EKR property and thus L_1 has the EKR property. The group L_4 is a Frobenius group and it has the EKR property. Also, by Proposition 2.4 and Corollary 2.5, L_2 and L_3 have the EKR property. □

Lemma 3.4. *No subgroup of $L = L_5$ is core-free.*

Proof. It is not difficult to see that all non-conjugate subgroups of L are $K_1 = \langle e\rangle$, $K_2 = \langle a^p\rangle$, $K_3 = \langle a\rangle$, $K_4 = \langle b\rangle$, $K_5 = \langle a^p, b\rangle$ and $K_6 = L$. We have $N_L(K_2) = L$, so K_2 is a normal subgroup of L. Also by Sylow theorem, K_4 is a normal subgroup of L. Since $[L : K_5] = p$ and p is the smallest prime number that divides the order of group, K_5 is normal subgroup of L. Let s be an arbitary integer, then $b^{-s}a^ib^s = a^ib^{s(1-u^i)}$ which yields that K_3 is not core-free. □

Lemma 3.4 implies that the action of L on the set $\{1, 2, \cdots p^2q\}$ is faithful. In the following, by $\bar{u}$ we mean u in modula q.

Lemma 3.5. *Let $L = L_5$ and $H = \{e\}$. The permutation peresentation of generetors of L are*

$$\begin{aligned} a^{-1} &= (1, 2, \cdots, p^2)(p^2+1, p^2+q+\overline{u}-1, \ \cdots, p^2q-q+\overline{u^{p^2-1}}-1) \\ \cdots & \ (p^2+q, p^2+2q+\overline{u}-2, \cdots, p^2+\overline{u^{p^2-1}}) \cdots (p^2q, p^2q+\overline{(q-1)u}, \cdots, p^2q-2q \\ + & \ 2+\overline{(q-1)u^{p-1}}), \\ b &= (1, p^2+q-1, \cdots, p^2+1)(2, p^2+2q-2, \cdots, p^2+q) \cdots (p^2, pq, \cdots, p^2q-q+2). \end{aligned}$$

Proof. Assign a labeling to the elements of L/H as follows:

$$\begin{aligned} H &\rightarrow 1, Ha \rightarrow 2, \cdots, Ha^{p^2-1} \rightarrow p^2, \\ Hb &\rightarrow p^2+1, \cdots, Hb^{q-1} \rightarrow p^2+q-1, \\ Hab &\rightarrow p^2+q, \cdots, Hab^{q-1} \rightarrow p^2+2q-2, \\ Ha^2b &\rightarrow p^2+2q-1, \cdots, Ha^{p^2-1}b^{q-1} \rightarrow p^2q. \end{aligned}$$

With regards to the relation of L and the action of L on the set of right cosets L/H, we have $\varphi_{a^{-1}}(Ha^ib^j) = Ha^{i+1}b^{uj}$ and $\varphi_b(Ha^ib^j) = Ha^ib^{j-1}$. Indeed, the permutation representation of a^{-1} is composed of q cycles of order p^2 and the permutation representation of b is composed of p^2 cycles of order q. □

Lemma 3.6. *The group $L = L_5$ has the EKR property.*

Proof. First, we show that any non-identity element in L has no fixed point. Assume that $a^{-i}b^j(Ha^rb^s) = Ha^rb^s, (1 \leq i, r \leq p^2, 1 \leq j, s \leq q)$. Hence $a^{-i}b^j(Ha^rb^s) = a^{-i}(Ha^rb^{s+j}) = Ha^{r+i}b^{u^i(s+j)} = Ha^rb^s$, then

$$\begin{cases} i+r \equiv r \ (mod \ p^2) \\ u^i(s+j) \equiv s \ (mod \ q) \end{cases}.$$

Thus, $i \equiv 0(mod \ p^2)$ and $j \equiv 0(mod \ q)$. Consequently, $\mathcal{D}_L = L - \{e\}$, so the derangement graph $\Gamma_L = Cay(L, \mathcal{D}_L)$ is complete graph and this completes the proof. □

Theorem 3.7. *Suppose G is a group of order p^2q where $p < q$. Then G has the EKR property.*

Proof. Use Lemmas 3.3 and 3.6. □

Theorem 3.8. *Suppose G is a group of order p^2q where $p > q$. Then G has the EKR property.*

Proof. By Lemma 2.2, Proposition 2.4 and Corollary 2.5, all groups Q_1, Q_2, Q_3 have the EKR property. Let $H = \langle a \rangle$ and $K = \langle b \rangle$. It is clear that $K \lhd Q_4$. On the other hand, $|Syl_q(Q_4)| = p^2$ and the intersection of Sylow q-subgroups is trivial. This yields that the group $Q_4 = KH$ is a Frobenius group. By a similar argument we can prove that groups Q_5 and $Q_5 + i$, $(i = 1, 2, 3, \ldots, \frac{q-1}{2}$ *and* $q - 1)$ are Frobenius groups and by Proposition 2.4, have the EKR property. ☐

Corollary 3.9. *A group of order p^2q has the EKR property.*

By Lemma 2.2 and Corollary 2.5, groups L_1, L_2, Q_1 and Q_2 have the strict EKR property. Also, by Lemma 3.6, derangement graph of group L_5 is complete graph and it has the strict EKR property. Let $p, q \neq 2$, by Proposition 2.4 and Corollary 2.5, the other groups of order p^2q dont have the strict EKR property. So we proved the following theorem.

Theorem 3.10. *Among all groups of order p^2q only the groups L_1, L_2, L_5, Q_1 and Q_2 have the strict the EKR property.*

3.3. **Groups of order p^3.** let p be an odd prime number. Then there are three abelian groups $\mathbb{Z}_{p^3}$, $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ of order p^3 and two non-abelian groups of order p^3 as follows

$$\begin{aligned} K_1 &= \langle a, b | a^p = b^{p^2} = e, a^{-1}ba = b^{p+1} \rangle, \\ K_2 &= \langle a, b, c | a^p = b^p = c^p = e, [a, b] = c, [a, c] = [b, c] = e \rangle. \end{aligned}$$

According to Lemma 2.2 and Corollary 2.5, three abelian groups have the strict EKR property. It is remained to check the strict EKR property for groups K_1 and K_2. First consider group K_1. To do this we have the following Lemma.

Lemma 3.11. *Let $H = \langle a \rangle \leq K_1$, then H is the only core-free subgroup of K_1.*

Proof. The group K_1 has three non-conjugate subgroups of order p, namely $H_1 = \langle e \rangle$, $H_2 = \langle a \rangle$ and $H_3 = Z(K_1)$. It is not difficult to see that H_2 is core free. On the other hand, there are $p + 1$ subgroups of order p^2 containing $Z(K_1)$ and we denote them by $H_4, \ldots, H_{p+4}$. All of them are normal in K_1. So for $i \neq 2$, $cor_G(H_i) = H_i$.

□

According to Lemma 3.11, the action of K_1 on the set $\{1, 2, \cdots p^2\}$ is faithful.

Theorem 3.12. *Let $H = \langle a \rangle \leq K_1$, the permutation peresentation of generators of K_1 are*

$$\begin{aligned} a^{-1} &= (2, (p+1)^2+1, \cdots, (p+1)^{p-1}+1) \cdots (3, 2(p+1)+1, \cdots, 2(p+1)^{p-1}+1) \\ &\cdots (p, (p-1)(p+1)+1, \cdots, (p-1)(p+1)^{p-1}+1), \\ b &= (p^2, p^2-1, \cdots, 1). \end{aligned}$$

Proof. First, consider the labeling $H \to 1, Hb \to 2, \cdots, Hb^{p^2-1} \to p^2$. It is not difficult to see that $\varphi_{a^{-1}}(Hb^i) = Hb^{(p+1)i}$ and $\varphi_b(Hb^i) = Hb^{i-1}$. The permutation representation of a^{-1} is a product of $p-1$ cycles of order p. Note that b has no fixed point. Since $\varphi_{a^{-1}}(Hb^i) = Hb^i$ if and only if $b^{pi} \in H$, we conclude a^{-1} fixes all points $1, p+1, 2p+1, \cdots, p^2+1$. Thus, every element $a^i b^{tp} (1 \leq i, t \leq p-1)$ has $p+1$ fixed points. □

Theorem 3.13. *The derangement graph $\Gamma_{K_1} = Cay(K_1, \mathcal{D}_{K_1})$ is isomorphic with the graph depicted in Figure 1.*

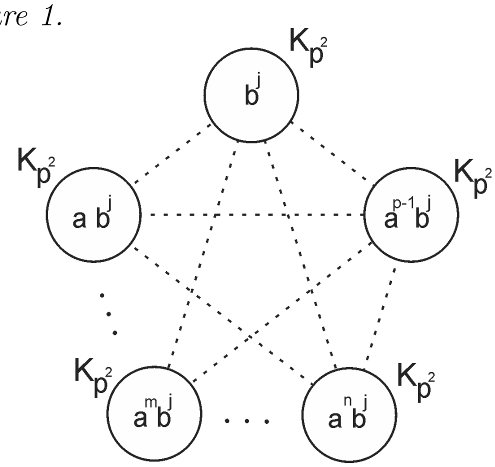

FIGURE 1. Cayley graph $\Gamma_{K_1} = Cay(K_1, \mathcal{D}_{k_1})$, $(1 \leq j \leq p^2)$.

Proof. The derangement set of Γ_{K_1} is $\mathcal{D}_{K_1} = K_1 - \{e, a^i b^j\}$, where $1 \leq i \leq p-1$ and $j = tp$ $(0 \leq t \leq p-1)$. Hence it is a regular graph and the degree of every vertex is $p^3 - p - (p-1)^2$. Since $a^i b^j$ is adjacent with $a^r b^s$ if and only if $a^r b^{s-j} a^{-i} \in S$ if and only if $a^{r+p-i} b^{(p+1)^{p-i}(s-j)} \in S$, we conclude that $a^i b^j$ is adjacent to $a^r b^s$ if and only if

$$(3.2)\begin{cases} r \neq i \ (mod\ p) \\ (1-pi)(s-j) \neq tp \ (mod\ p^2) \end{cases} or \begin{cases} r = i \ (mod\ p) \\ j \neq s \end{cases} \begin{cases} r \neq i \ (mod\ p) \\ j = s \end{cases}$$

where $1 \leq t \leq p-1$. The equation $(1-pi)(s-j) \neq tp \ (mod\ p^2)$, for every t has only one soloution, where $s \neq tp + j$ which yields Γ_{K_1} has p non-disjoint cliques of order p^2. □

Lemma 3.14. *The independent number of $\Gamma_{K_1} = Cay(K_1, \mathcal{D}_{K_1})$ is p.*

Proof. By using Lemma 2.1 and Theorem 3.13 one can see that $\alpha(\Gamma_{K_1}) \leq p$. Since $\langle a \rangle$ is an independent set of Γ_{K_1}, we have $\alpha(\Gamma_{K_1}) = p$.

By Eq. (3.2), and permutation peresentation of generators of K_1, we conclude that $S = \{e, ab^p, a^2b^p, \cdots, a^{p-1}b^p\}$ is a maximum independent set in $\Gamma = Cay(K_1, S)$. This yields that the group K_1 does not have the strict EKR property.

Lemma 3.15. *Let $H = \langle a \rangle \leq K_2$, then H is the largest core-free subgroup of K_2.*

Proof. All non-conjugate subgroups of K_2 are $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle a^i b^j \rangle$ and the number of such subgroups is $p-1+3 = p+2$. Let us show them by $H_1, \ldots, H_{p+2}$. For $2 \leq i \leq p+2$, $|N_G(H_i)| = p^2$ and $N_G(H_3) = G$. On the other hand, all non-conjugate subgroups of order p^2 are $\langle a^i, b^j \rangle (1 \leq i, j \leq p-1)$ and $\langle b, c \rangle$. Hence, there are $p-1+2 = p+1$ non-conjugate subgroups of this form. We call them by $G_1, \ldots, G_{p+1}$. For $1 \leq i \leq p+1$, we have $N_G(G_i) = G$. Since for $1 \leq i \leq p+1$, G_i is a normal subgroup of G and $cor_G(G_i) = G_i$. By the relations in group K_2 we have $b^{-1}a^i b = a^i c^{-i}$ which yields $cor_{K_2}(H) = \{e\}$. □

According to Lemma 3.15, the action of K_2 on the set $\{1, 2, \cdots p^2\}$ is faithful.

Theorem 3.16. *Let $H = \langle a \rangle \leq K_2$. The permutation peresentations of generators of K_2 are*

$$\begin{aligned} a^{-1} &= (2, 2p, \cdots, 3p-2)(3, 3p, \cdots, 4p-4) \cdots (p, p^2, \cdots, p^2-p+2), \\ b &= (1, p, \cdots, 2)(p+1, p^2-p+2, \cdots, 2p) \cdots (2p-1, p^2, \cdots, 3p-2), \\ c &= (1, 2p-1, \cdots, p+1)(2, 3p-2, \cdots, 2p) \cdots (p, p^2, \cdots, p^2-p+2). \end{aligned}$$

Proof. First, asign the following labels to the elements of coset K_2/H.

$$\begin{aligned} H &\to 1, Hb \to 2, \cdots, Hb^{p-1} \to p, \\ Hc &\to p+1, \cdots, Hc^{p-1} \to 2p-1, \\ Hbc &\to 2p, \cdots, Hbc^{p-1} \to 3p-2, \\ Hb^2c &\to 3p-1, \cdots, Hb^{p-1}c^{p-1} \to p^2. \end{aligned}$$

With regards to the relations of K_2 and the action of K_2 on the right coset K_2/H, we have $\varphi_{a^{-1}}(Hb^ic^j) = Hb^ic^{i+j}$, $\varphi_b(Hb^ic^j) = Hb^{i-1}c^j$ and $\varphi_c(Hb^ic^j) = Hb^ic^{j-1}$. Indeed, the permutation representation of a^{-1} is composed of $p-1$ cycles of order p and the permutation representations of b and c are composed of p cycles of order p. One can see that a^{-1} fixes all points $1, p+1, \cdots 2p-1$ and b, c move all points. □

Theorem 3.17. *The Cayley graph $\Gamma_{K_2} = Cay(K_2, \mathcal{D}_{K_2})$ is isomorphic with the graph depicted in Figure 2.*

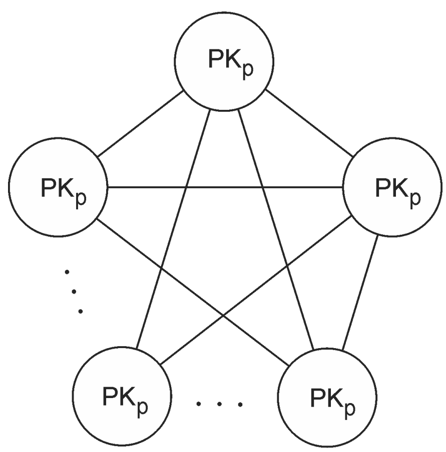

FIGURE 2. Cayley graph $\Gamma_{K2} = Cay(K_2, \mathcal{D}_{k_2})$.

Proof. The derangement set of Γ_{K_2} is $\mathcal{D}_{K_2} = K_2 - \{e, a^ic^k\}$, where $1 \leq i \leq p-1$, $0 \leq k \leq p-1$. Hence Γ_{K_2} is a regular graph of degree $p^3 - p(p-1) - 1$. Since $a^ib^jc^k$ is adjacent with $a^rb^sc^t$ if and only if $a^rb^{s-j}a^{-i}c^{t-k} \in \mathcal{D}_{K_2}$ if and only if $a^{r+p-i}b^{s-j}c^{-si+ij+t-k} \in \mathcal{D}_{K_2}$, we conclude that $a^ib^jc^k$ is not adjacent to $a^rb^sc^t$ if and only if $s = j$ $(mod\ p)$ and $r \neq i$ $(mod\ p)$. By these relations, the derangement graph Γ_{K_2} is composed of p subgraphs $G_1, \cdots, G_p$ where $G_i \cong pK_p(i = 1, 2, \cdots, p)$ and $pK_p = \cup_{i=1}^{p} K_P$. For two distinct integers $i, j \in [p]$

all vertices of G_i are adjacent with all vertices of G_j. Hence, the Cayley graph $\Gamma_{K_2} = Cay(K_2, \mathcal{D}_{K_2})$ is as depicted in Figure 2 and thus the independent number of Γ_{K_2} is p. □

By the proof of Theorem 3.17 and permutation peresentation of generators of K_2, we conclude that $S = \{e, ac, a^2c, \cdots, a^{p-1}c\}$ is a maximum independent set in $\Gamma = Cay(K_2, S)$. Then the group K_2 does not have the strict EKR property.

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