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# Perfect 3-colorings of the cubic graphs of order 10

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**Abstract**

In this paper we enumerate the parameter matrices of all perfect 3-colorings of the cubic graphs of order 10

*Keywords:* Perfect coloring, Equitable partition, Cubic graphs
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## 1. Introduction

The concept of a perfect $m$-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as ”equitable partition” (see[8] ).
The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the nonexistence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(4,2)$, $J(5,2)$, $J(6,2)$, $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(v,3)$ ($v$ odd) (see [1, 2, 3, 7]).
Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of $n$-dimensional hypercube $Q_n$ for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the $n$-dimensional cube with a given parameter matrix (see [4, 5, 6]).

## 2. Definition and Concepts

In this section, we give some basic definitions and concepts. In this paper all garaphs are assumed simple, connected and undirected. Let $G = (V, E)$ be an undirected graph. Two vertices $u, v \in V(G)$ are adjacent if there exists an edge $e = \{u, v\} \in E(G)$ to which they are both incident. The adjacent will be shown $u \leftrightarrow v$.

a cubic graph is a 3-regular graph. In [11], it is shown that the number of connected cubic graphs with 10 vertices is 19. Each graph is described by a drawing as shown in Figure 1.

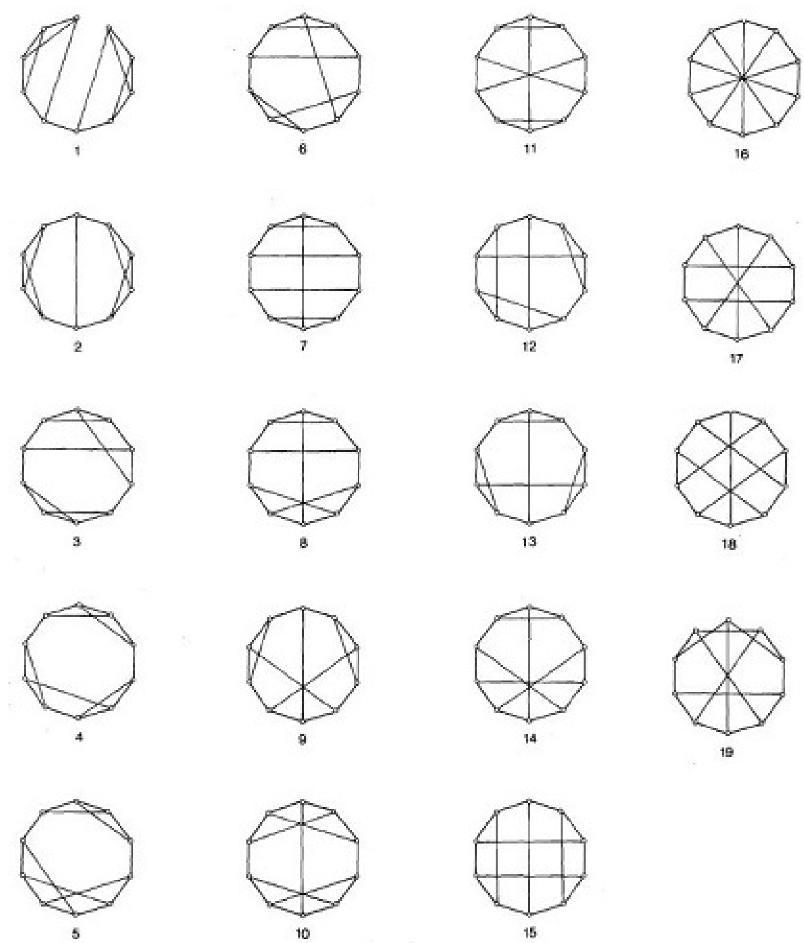


Figure 1. Connected cubic graphs of order 10

**Definition 2.1.** For a graph $G$ and an integer $m$, a mapping $T : V(G) \to \{1, \cdots, m\}$ is called a perfect $m$-coloring with matrix $A = (a_{ij})_{i,j\in\{1,\cdots,m\}}$, if it is surjective, and for all $i, j$, for every vertex of color $i$, the number of its neighbors of color $j$ is equal to $a_{ij}$. The matrix $A$ is called the *parameter matrix* of a perfect coloring. In the case $m = 3$, we call the first color *white*, the second color *black*, and the third color *red*. In this paper, we generally show a parameter matrix by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

*Remark* 2.2. In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e, we identify the perfect 3-coloring with the matrices

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix},$$
$$\begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix},$$

obtained by switching the colors with the original coloring.

## 3. Preliminaries and Analysis

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of a cubic connected graph of order 10 with a given parameter matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

The simplest necessary condition for the existence of perfect 3-colorings of a cubic connected graph with the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is

$$a + b + c = d + e + f = g + h + i = 3.$$

Also, it is clear that we cannot have $b = c = 0$, $d = f = 0$, or $g = h = 0$, since the graph is connected. In addition, $b = 0$, $c = 0$, $f = 0$ if $d = 0$, $g = 0$, $h = 0$, respectively.

The next proposition gives a formula for calculating the number of white, black and red vertices, in a perfect 3-coloring.

**Lemma 3.1.** *[10] If T is a perfect coloring of a graph G in m colors, then any eigenvalue of T is an eigenvalue of G.*

**Proposition 3.1.** *Let $T$ be a perfect 3-coloring of a graph G with the matrix* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

***1.*** *If* $b, c, f \neq 0$*, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d}+1+\frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b}+1+\frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f}+1+\frac{g}{c}}.$$

***2.*** *If* $b = 0$*, then*

$$|W| = \frac{|V(G)|}{\frac{c}{g}+1+\frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h}+1+\frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f}+1+\frac{g}{c}}.$$

***3.*** *If* $c = 0$*, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d}+1+\frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b}+1+\frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f}+1+\frac{dh}{bf}}.$$

***4.*** *If* $f = 0$*, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d}+1+\frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b}+1+\frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c}+1+\frac{bg}{cd}}.$$

*Proof.* **(1):** Consider the 3-partite graph obtained by removing the edges $uv$ such that $u$ and $v$ are the same color. By counting the number of edges between parts, we can easily obtain $|W|b = |B|d$, $|W|c = |R|g$, and $|B|f = |R|h$. Now, we can conclude the desired result from $|W| + |B| + |R| = |V(G)|$.
The proof of **(2), (3), (4)** is similar to **(1).** □

In this section, without restriction of generality, we assume $|W| \leq |B| \leq |R|$.
In the next lemma, under the condition $|W| = 1$, we enumerate all matrices that can be a parameter matrix for a cubic connected graph.

**Lemma 3.2.** *Let* $G$ *be a cubic connected graph. If* $T$ *be a perfect 3-coloring with the matrix* $A$*, and* $|W| = 1$*, then* $A$ *should be one of the following matrices,*

$$A_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$
$$A_5 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}, A_7 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$
$$A_9 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, A_{10} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, A_{11} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

*Proof.* Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a parameter matrix with $|W| = 1$. Consider the white vertex. It is clear that none of its adjacent vertices are white; i.e, $a = 0$. Therefore, we have two cases below.

(1) The adjacent vertices of the white vertex are the same color.
If they are black, then $b = 3$ and $c = 0$. From $c = 0$, we get $g = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

If the adjacent vertices of the white vertex are red, then $c = 3$, $b = 0$. From $b = 0$, we get $d = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Finally, by using Remark 2.2 and the fact that $|W| \leq |B| \leq |R|$, it is obvious that there are only six matrices in (1), as shown $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$.

(2) The adjacent vertices of the white vertex are different colors.
It immediately gives that $b, c \neq 0$. Also, it can be seen that $d = g = 1$. An easy computation, As in (1), shows that there are only five matrices that can be a parameter matrix in this case, as shown $A_7$, $A_8$, $A_9$, $A_{10}$, $A_{11}$.

□

The next lemma is useful to obtain the other parameter metrices.

**Lemma 3.3.** *Let $G$ be a cubic connected graph of order 10. Then*

***1.*** *If $T$ is a perfect 3-coloring with the matrix $A$, and $|W| = |B| = 2$, $|R| = 6$, then $A$ should be one of the following matrices*

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

***2.*** *If $T$ is a perfect 3-coloring with the matrix $A$, and $|W| = 2$, $|B| = 3$, $|R| = 5$, then $A$ should be one of the following matrices*

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

***3.*** *If $T$ is a perfect 3-coloring with the matrix $A$, and $|W| = 2$, $|B| = |R| = 4$, then $A$ should be one of the following matrices*

$$\begin{bmatrix}2&0&1\\0&1&2\\1&1&1\end{bmatrix}, \begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix}, \begin{bmatrix}1&0&2\\0&1&2\\2&1&0\end{bmatrix}, \begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix},$$
$$\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix}, \begin{bmatrix}2&1&0\\1&1&1\\0&2&1\end{bmatrix}, \begin{bmatrix}1&2&0\\2&0&1\\0&2&1\end{bmatrix}, \begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix}.$$

***4.*** *If $T$ is a perfect 3-coloring with the matrix $A$, and $|W| = 3$, $|B| = 3$, $|R| = 4$, then $A$ should be one of the following matrices*

$$\begin{bmatrix}0&1&2\\3&0&0\\1&0&2\end{bmatrix}, \begin{bmatrix}0&2&1\\1&2&0\\3&0&0\end{bmatrix}, \begin{bmatrix}2&0&1\\0&1&2\\1&1&1\end{bmatrix}, \begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix},$$
$$\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix}, \begin{bmatrix}1&0&2\\0&1&2\\2&1&0\end{bmatrix}, \begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix}, \begin{bmatrix}2&1&0\\2&0&1\\0&3&0\end{bmatrix}.$$

*Proof.* **(1):** First, suppose that $b, c \neq 0$. As $|W| = 2$, by Proposition 3.1, it follows that $\dfrac{b}{d} + \dfrac{c}{g} = 4$. Therfore $b = c = 2, d = g = 1$ and we get a contradiction of $b + c \leq 3$.
Second, suppose that $b = 0$ and, in consequence, $d = 0$. As $|R| = 4$, by Proposition 3.1, we have $\dfrac{g}{c} + \dfrac{h}{f} = \dfrac{2}{3}$. Therefore $c = f = 3$, $g = h = 1$, and consequently $A = \begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix}$.
Finally, suppose that $c = 0$ and, in consequence, $g = 0$. As $|B| = 2$, by Proposition 3.1, it follows that $\dfrac{d}{b} + \dfrac{f}{h} = 4$. Therefore $b = f = 2$, $d = h = 1$, or $b = 3$, $d = f = h = 1$ or $b = 3$,$d = 1$, $f = h = 2$. Hence $A = \begin{bmatrix}1&2&0\\1&0&2\\0&1&2\end{bmatrix}$, or $A = \begin{bmatrix}0&3&0\\1&1&1\\0&1&2\end{bmatrix}$, or $A = \begin{bmatrix}0&3&0\\1&0&2\\0&2&1\end{bmatrix}$.
The proof of **(2), (3), (4)** is similar to **(1).** □

By using the Lemma 3.3 and Proposition 3.1, it can be seen that only the following matrices can be parameter ones.

$$\begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix}, \begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix}, \begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix}, \begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix}, \begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix}, \begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix}.$$

By Remark 2.2, it is clear that the matrix $\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix}$ is the same as the matrix $\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix}$ and the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ is the same as the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ up to renaming the colors. Therfore, if $T$ is a perfect 3-coloring with the matrix $A$ , then $A$ should be one of the following matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

The next theorem can be useful to find the eigenvalues of a parameter matrix.

**Theorem 3.1.** *If* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *be a parameter matrix of a* $k$*-regular graph, then the eigenvalues of* $A$ *are*

$$\lambda_{1,2} = \frac{\operatorname{tr} A - k}{2} \pm \sqrt{\left(\frac{\operatorname{tr} A - k}{2}\right)^2 - \frac{\det A}{k}} \qquad , \qquad \lambda_3 = k.$$

*Proof.* By using the condition $a+b+c = d+e+f = g+h+i = k$, it is clear that one of the eigenvalues is $k$. Therefore $\det A = k\lambda_1\lambda_2$. From $\lambda_2 = \operatorname{tr} A - \lambda_1 - k$, we get

$$\det A = k\lambda_1(\operatorname{tr} A - \lambda_1 - k) = -k\lambda_1^2 + k(\operatorname{tr} A - k)\lambda_1.$$

By solving the equation $\lambda^2 + (k - \operatorname{tr} A)\lambda + \dfrac{\det A}{k} = 0$, we obtain

$$\lambda_{1,2} = \frac{\operatorname{tr} A - k}{2} \pm \sqrt{\left(\frac{\operatorname{tr} A - k}{2}\right)^2 - \frac{\det A}{k}}.$$

□

## 4. Main result

In this section we peresent the main theorem that shows the parameter matrices of all perfect 3-colorings of the cubic connected graphs of order 10.

**Theorem 4.1.** *The parameter matrices of cubic graphs of order* $10$ *are listed in the following table.*

| *graphs* | *matrix* $A_1$ | *matrix* $A_2$ | *matrix*$A_3$ | *matrix* $A_4$ |
|---|---|---|---|---|
| 1 | $\surd$ | $\times$ | $\surd$ | $\times$ |
| 2 | $\surd$ | $\times$ | $\surd$ | $\surd$ |
| 3 | $\times$ | $\times$ | $\times$ | $\times$ |
| 4 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 5 | $\times$ | $\times$ | $\times$ | $\times$ |
| 6 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 7 | $\times$ | $\times$ | $\times$ | $\times$ |
| 8 | $\times$ | $\times$ | $\times$ | $\times$ |
| 9 | $\times$ | $\times$ | $\times$ | $\times$ |
| 10 | $\times$ | $\times$ | $\times$ | $\times$ |
| 11 | $\times$ | $\times$ | $\times$ | $\times$ |
| 12 | $\times$ | $\times$ | $\times$ | $\times$ |
| 13 | $\surd$ | $\surd$ | $\times$ | $\surd$ |
| 14 | $\times$ | $\times$ | $\times$ | $\surd$ |
| 15 | $\times$ | $\times$ | $\times$ | $\times$ |
| 16 | $\times$ | $\times$ | $\times$ | $\times$ |
| 17 | $\times$ | $\times$ | $\times$ | $\times$ |
| 18 | $\times$ | $\times$ | $\surd$ | $\surd$ |
| 19 | $\times$ | $\times$ | $\surd$ | $\surd$ |

*Table* $1$

*Proof.* As it shown in section 3, only matrices $A_1$ , $A_2$, $A_3$ and $A_4$ can be parameter matrices. By using Theorem $3.1$ we see that the connected cubic graphs with 10 vertices can have perfect 3-coloring with matrices $A_1$, $A_2$, $A_3$ and $A_4$ shown in the following table

| graphs | matrix $A_1$ | matrix $A_2$ | matrix$A_3$ | matrix $A_4$ |
|---|---|---|---|---|
| 1 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 2 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 4 | $\surd$ | $\surd$ | $\times$ | $\times$ |
| 5 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 6 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 9 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 10 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 13 | $\times$ | $\times$ | $\surd$ | $\surd$ |
| 14 | $\times$ | $\times$ | $\surd$ | $\surd$ |
| 18 | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| 19 | $\times$ | $\times$ | $\surd$ | $\surd$ |

Now we check which graphs have perfect 3-colorings with these matrices. The graph 1 has perfect 3-colorings with the matrices $A_1$ and $A_3$. The vertices of graph 1 are labeled clockwise with $a_1, a_2, ..., a_{10}$, respectively. Consider two mapping as follows,

$$T_1(a_1) = T_1(a_{10}) = 1, T_1(a_4) = T_1(a_7) = 2,$$
$$T_1(a_2) = T_1(a_3) = T_1(a_5) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3.$$
$$T_2(a_5) = T_2(a_6) = 1, T_2(a_2) = T_2(a_3) = T_2(a_8) = T_2(a_9) = 2,$$
$$T_2(a_1) = T_2(a_4) = T_2(a_7) = T_2(a_3) = 3,$$

it is clear that $T_1$ and $T_2$ are perfect 3-coloring with the matrices $A_1$ and $A_3$, respectively. Now we show the graph 1 has no perfect 3-coloring with matrices $A_2$ and $A_4$.
According to the matrix $A_2$, each vertex with white color has a neighbor with white color, so the two vertices with white color are adjacent. In the case that $a_1 \leftrightarrow a_2$, $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$, $a_3 \leftrightarrow a_4$ and according to symmetry $a_7 \leftrightarrow a_8$, $a_7 \leftrightarrow a_9$, $a_8 \leftrightarrow a_{10}$ and $a_9 \leftrightarrow a_{10}$, they have Less than four adjacent vertices, these vertices are red color, it causes contradiction. So $a_5 \leftrightarrow a_6$, $a_4 \leftrightarrow a_5$ and its symmetric $a_6 \leftrightarrow a_7$ will be remain that are white color. In the case that $a_4 \leftrightarrow a_5$, the neighbors of $a_4$ and $a_5$ are red color and vertex $a_1$ that is their neighbor's is also red color has two neighbors with red color which it is not possible. If $a_5$ and $a_6$ are white color, adjacent vertices are red color and other vertices are black color, so each black color is adjacent with another black color vertex, that is a contradiction. So we conclude the graph 1 does not have a perfect 3-colring with matrix $A_2$.
According to the matrix $A_4$, each vertex with white color has three adjacent with black color. If $a_1$ is white color, then $a_2, a_3, a_5$ are black color, which is a contradiction with second row of matrix $A_4$. If $a_2$ is white color, then according to the matrix $A_4$, $a_1, a_3, a_4$ are black color, which is a contradiction with second row of matrix $A_4$. If $a_3$ is white color, then according to the matrix $A_4$, $a_1, a_2, a_4$ are black color, which is a contradiction with second row of matrix $A_4$. If $a_4$ is white color, then according to the matrix $A_4$, $a_2, a_3, a_5$ are black color, which is a contradiction with second row of matrix $A_4$. If $a_5$ is white color, then $a_3$ is a vertex that is black color and has three red color neighbors, which is a counteraction with second row of matrix $A_4$. According to the symmetric, vertices $a_6, a_7, a_8, a_9, a_{10}$ can't be white color. Therefore the graph 1 has no perfect 3-coloring with matrix $A_4$.
As it is stated in the before paragraphs, the graph 1 has no perfect 3-coloring with matrices $A_2$ and $A_4$.
Similary, we can prove the designed result. □

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