



Minimally 4-restricted edge connected graphs

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Abstract

Suppose that G is minimally 4-restricted edge connected graph without triangle, $\delta(G) \geq 2$ and $\alpha_4(G) \geq 6$. Suppose that A is a λ_4 -atom of G such that for each path of length 3 in A , say $P_3 = xyz$, we have $d_A(y) - 1 = d_{A - \{x,y,z\}}(y)$, where $x \sim y \sim z$. In this paper, under these assumptions, we show that all atoms of G are trivial.

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1. Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For each vertex $v \in V(G)$, the *neighborhood* $N(v)$ of v is defined as the set of all vertices adjacent to v and $d(v) = |N(v)|$ is the *degree* of v . For two vertices $u, v \in V(G)$, we write $u \sim v$ (or $uv \in E(G)$) when u and v are adjacent. For two disjoint vertex set $U_1, U_2 \subseteq V(G)$, denote by $[U_1, U_2]_G$ the set of edge of G with one end in U_1 and the other end in U_2 , $G[U]$ is the subgraph of G induced by vertex set $U \subseteq V(G)$, $\overline{U} = V(G) - U$ is the complement of U , $w_G(U) = |[U, \overline{U}]_G|$ is the number of edges between U and $\overline{U}$. When the graph under consideration is obvious, we omit the subscript G . For a subset $S \subseteq G$ write $d_S(U) = |[U, S - U]|$ and $d_S(u) = d_S\{u\}$.

A *m -restricted edge cut* is an edge cut of a connected graph that separates this graph into components each having order at least m vertices. The size of a minimum m -restricted edge cut of graph G is called its *restricted edge connectivity* and denoted by $\lambda_m(G)$. A m -restricted edge cut S with $|S| = \lambda_m(G)$ is called a λ_m -cut. Let $\varepsilon_m(G) = \min\{|w(U)|, |U| = m \text{ and } G[U] \text{ is connected}\}$.

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A graph G is λ_m -graph (or λ_m -optimal) if $\lambda_m(G) = \varepsilon_m(G)$. There is much research on sufficient conditions for a graph to be λ_m -optimal, such as symmetric conditions [12, 14, 16, 17] degree conditions [14, 15, 18]. For more information on this topic we refer the reader to the [5].

A graph G is λ_4 -independent if each component of G has at most three vertices. A connected graph of order at most three is λ_4 -trivial. A graph is called λ_4 -non-trivial if it has a component which contains at least four vertices.

For two disjoint non-empty subsets X and Y of $V(G)$, the following inequality is well known (see Lovász 1979, Problem 6.48).

$$w(X \cap Y) + w(X \cup Y) \leq w(X) + w(Y). \quad (1)$$

A vertex set U is called a λ_m -fragment if $[U, \bar{U}]$ is a minimum m -restricted edge cut of graph G . λ_m -restricted edge fragment with the least cardinality is called a λ_m -atom. The cardinality of λ_m -atom is denoted by $\alpha_m(G)$. Clearly $m \leq \alpha_m(G) \leq |V(G)|/2$. An atom is said to be *trivial* if it is a single cycle. In this paper we give another type of sufficient condition called a minimally restricted edge connected condition. A graph G is a *minimally m -restricted edge connected* (minimally λ_m -graph for short) if $\lambda_m(G - e) < \lambda_m(G)$ for each $e \in E(G)$. If e is a pending edge then $G - e$ doesnot have λ_m -cut for $m \geq 2$. So we always assume $\delta(G) \geq 2$ where G is a minimally λ_m -graph for some $m \geq 2$ and $\delta(G)$ is the minimum degree of G . A minimally λ_1 -graph is exactly a minimally edge connected graph, which has been shown to be λ -optimal ([10, Exercise 49]). In [6] the authors have proved that every minimally λ_2 -graph is λ_2 -optimal. Also in [7] the authors have proved that every minimally λ_3 -graph is always λ_3 -optimal except the 3-cube. For more information on this topic we refer the reader to the [1, 2, 3, 4, 8, 9, 11, 13]. In this paper with the similar methods we show that all atoms of a minimally 4-restricted edge connected graph without triangle are trivial (see Figure 1).

2. Main result

Throughout in this section we assume that G is a λ_4 -connected graph, $\delta(G) \geq 2$ and $\alpha_4(G) \geq 6$. First with the similar arguments in the proofs of Lemmas 1 – 5 in [7] we have the following five results.

Lemma 2.1. Suppose that F is a subset of G such that $G[\bar{F}]$ is connected. If one of the following conditions is satisfied:

- (i) $w(F) < \lambda_4(G)$, or
 - (ii) $w(F) = \lambda_4(G)$ and $|F| < \alpha_4(G)$,
- then $G[F]$ is λ_4 -independent.

Proof. Suppose that C is λ_4 -non-trivial component of $G[F]$. Since $G[\bar{C}]$ is connected, we have $w(F) \geq w(C) \geq \lambda_4(G)$ which contradicts the first condition. Also if condition (ii) occurs then we have $w(F) = w(C) = \lambda_4(G)$. Hence $V(C)$ is a $\lambda_4(G)$ -fragment. But we have $|F| \geq |C| \geq \alpha_4(G)$ which contradicts the second condition. Thus $G[F]$ is λ_4 -independent, as wanted. $\square$

Lemma 2.2. Suppose that A is a λ_4 -atom of G and B is a λ_4 -fragment of G .

- (i) If U is a subset of A such that $G[U]$ is connected and $G[A \setminus U]$ has λ_4 -non-trivial component,

then $d_A(U) > d_{\bar{A}}(U)$.

(ii) If U is a subset of B such that $G[U]$ is connected and $G[B \setminus U]$ has λ_4 -non-trivial component, then $d_B(U) \geq d_{\bar{B}}(U)$.

(iii) $\delta(G[A]) \geq 2$.

Proof. (i) Suppose to contrary that $d_A(U) \leq d_{\bar{A}}(U)$. Now we have $w(A - U) = w(A) + d_A(U) - d_{\bar{A}}(U) \leq w(A) = \lambda_4(G)$. Since $|A - U| < |A| = \alpha_4(G)$, it follows by Lemma 2.1, $G[A - U]$ is λ_4 -independent, a contradiction.

(ii) The proof is similar to the first case.

(iii) If $A - x$ is λ_4 -independent for each $x \in A$, then we have $d_A(x) \geq 2$. Thus we may suppose that $A - x$ has a λ_4 -non-trivial component for some $x \in A$. Set $U = x$. Now $d_A(x) > d_{\bar{A}}(x)$ and since $d_A(x) > 1/2(d_A(x) + d_{\bar{A}}(x))$ we see that $d_A(x) \geq 2$, as wanted. $\square$

Similar to Lemma 2.2, we can prove the following lemma. The key observation to the proof, as well as some proofs after it, is that for any edge $e \in E(G)$, if $\lambda_4(G - e) < \lambda_4(G)$, then any λ_4 -fragment of $G - e$ contains exactly one end of e , and is a λ_4 -fragment of G .

Lemma 2.3. Suppose that $e = uv$ is an edge of G , $\lambda_4(G - e) < \lambda_4(G)$ and A is a λ_4 -atom of $G - e$ with $u \in A$, $v \notin A$.

(i) If U is a subset of A such that $G[U]$ is connected and $G[A] - U$ has λ_4 -non-trivial component and e is not incident with U then $d_A(U) > d_{\bar{A}}(U)$.

(ii) $d_{G[A]}(x) \geq 2$ for each $x (\neq u) \in A$.

Lemma 2.4. Suppose that A is a λ_4 -atom and B is a λ_4 -fragment of G and $A \cap B \neq \emptyset$. Then for any component C of $G[A \cap B]$ either $G[A] - C$ or $G[B] - C$ is λ_4 -independent.

Proof. Suppose to contrary that $G[A] - C$ and $G[B] - C$ have a λ_4 -non-trivial component. Set $U = G[C]$. By Lemma 2.2 we have $d_A(U) > d_{\bar{A}}(U) \geq d_{B-A}(U) = d_B(U)$ and $d_B(U) > d_{\bar{B}}(U) \geq d_{A-B}(U) = d_A(U)$, a contradiction. $\square$

With the similar arguments with proof of Lemma 2.4 we have the following lemma.

Lemma 2.5. Suppose that $e = uv$ is an edge of G , $\lambda_4(G - e) < \lambda_4(G)$, A is a λ_4 -atom of $G - e$ with $u \in A$, $v \notin A$, B is a λ_4 -fragment of G , $A \cap B \neq \emptyset$ and e is not incident with B . Then for any component C of $G[A \cap B]$ either $G[A] - C$ or $G[B] - C$ is λ_4 -independent.

Lemma 2.6. Suppose that A is a λ_4 -atom of G , $e = uv$ is an edge of $G[A]$ and B is a λ_4 -atom of $G - e$. Then $G[A \cap B]$, $G[A - B]$ and $G[B - A]$ are connected.

Proof. Suppose to contrary that $G[A \cap B]$ is not connected. Suppose that C_1 and C_2 are two components of $G[A \cap B]$ and one of them has at least three elements, say C_2 . By Lemma 2.4, $G[A] - C_1$ has λ_4 -non-trivial component. This contradicts Lemma 2.4 contradicting. Thus all components of $G[A \cap B]$ have at most 2 vertices. Suppose that C is a component of $G[A \cap B]$. Thus $G[A] - C_1$ and $G[B] - C_2$ are λ_4 -nontrivial, a contradiction and so $G[A \cap B]$ is connected. Now suppose that $G[A - B]$ is not connected. Suppose that C_1 and C_2 are two components of

$G[A - B]$. Suppose that one of the component has at least four elements, say C_2 . Then $G[A] - C$ and $G[\overline{B}] - C$ is λ_4 -non-trivial. This contradicts Lemma 2.4. Now suppose that all components has at most three elements. Now $G[A] - C$ and $G[\overline{B}] - C$ are λ_4 -non-trivial, where C is a component of G , a contradiction. Similarly we can show that $G[B - A]$ is connected. $\square$

Lemma 2.7. *Suppose that A is a λ_4 -atom of G , $e = uv$ is an edge of $G[A]$ and B is a λ_4 -atom of $G - e$. Then $|A - B| = 3$ and $|A \cap B| = 3$.*

Proof. Suppose that $|A \cap B| \leq 2$. Thus $G[A - B]$ and $G[B - A]$ are λ_4 -non-trivial. This contradict Lemma 2.4. Now suppose that $|A \cap B| \geq 4$. Thus $G[A \cap B]$ is λ_4 -non-trivial. Also since $G[\overline{A}]$ and $G[\overline{B}]$ are connected, it follows that $G[\overline{A \cup B}] = G[\overline{A \cap B}]$ is connected. Taking $F = A \cap B$ in Lemma 2.1. Noting that $|F| = |A \cap B| < |A| = \alpha_4(G)$ and since A is an λ_4 -atom and $e \in A$ we have $w(A) = w(B) = \lambda_4(G)$. Since A is an atom, it follows that A is connected and there is an edge between $A - B$ and $\overline{A - B}$. Thus $w(A \cap B) > \lambda_4(G)$. Now by equation (1) $w(A \cap B) + w(A \cup B) \leq w(A) + w(B)$, and so $w(\overline{A \cup B}) = w(A \cup B) < \lambda_4(G)$. Taking $F = \overline{A \cup B}$ in Lemma 2.1. We see that $G[\overline{A \cup B}]$ is λ_4 -independent and so $G[\overline{A \cup B}]$ is composed of some components of cardinality less than or equal 3.

Claim $d_A(C) = d_B(C)$ and $d_{A \cap B}(C) = 0$, where C is a component of $G[\overline{A \cup B}]$.

Suppose that $1 \leq |C| \leq 2$. Thus $G[\overline{A}] - C$ is λ_4 -non-trivial and so by Lemma 2.2, $d_{\overline{A}}(C) \geq d_A(C)$. Also since C is a component of $G[\overline{A \cup B}]$, it implies that $||[C, \overline{A} - C]| = |[C, B - A - C]|$ and hence $d_{\overline{A}}(C) = d_{B-A}(C)$. Therefore

$$d_A(C) \leq d_{\overline{A}}(C) = d_{B-A}(C) = d_B(C) - d_{A \cap B}(C).$$

Similarly,

$$d_B(C) \leq d_{\overline{B}}(C) = d_{A-B}(C) = d_A(C) - d_{A \cap B}(C).$$

Thus $d_A(C) = d_B(C)$ and $d_{A \cap B}(C) = 0$. Now suppose that $|C| = 3$. Assume that $|\overline{A \cup B}| = 4$ and $|A - B| = 1$. Put $A' = (A - B) \cup C$. Then $G[A']$ and $G[\overline{A'}]$ are subgraphs of order at least 4 and $w(A') = w(\overline{A'}) = w(B \cup (\cup C_i)) = w(B) + d_{A-B}(\cup C_i) - d_{B-A}(\cup C_i) = \lambda_4(G)$, where C_i are components of $G[\overline{A \cup B}]$ and $1 \leq |C_i| \leq 2$. Thus A' is an atom and $|A'| < |A|$, a contradiction. Therefore we may suppose that $|\overline{A \cup B}| \geq 5$ or $|A - B| \geq 2$. Now with the similar arguments as above for every component C of $G[\overline{A \cup B}]$, $G[\overline{A}] - C$ has at least order 4, a contradiction. Thus $|A \cap B| = 3$. Also $|A - B| = |A| - |A \cap B| \geq 3$. If $|A - B| \geq 4$ then $A - B$ is a λ_4 -non-trivial, a contradiction. Thus $|A - B| = 3$. $\square$

Now by Lemmas 2.6 and 2.7, we have the following result.

Lemma 2.8. *Suppose that A is a λ_4 -atom of G , $e = uv$ is an edge of $G[A]$ and B is a λ_4 -atom of $G - e$. Then $G[A] \cong X$, where X is a subgraph of complete graph K_6 .*

For the rest of paper we may assume that G is K_3 -free or (without triangle), $A \cap B$ is a path of length 3, say $P_3 = xyz$, and $d_A(y) - 1 = d_{A - \{x, y, z\}}(y)$. Also we note that $x \sim y \sim z$.

Lemma 2.9. *Suppose that A is a λ_4 -atom of G , $e = uv$ is an edge of $G[A]$ and B is a λ_4 -atom of $G - e$. Then.*

- (i) $d_A(xyz) \geq d_{\overline{A}}(xyz)$.
- (ii) $d_{A-B}(xyz) = d_{B-A}(xyz)$.
- (iii) For each vertex a of A , $d_{\overline{A}}(a) = d_A(a) - 1$.
- (iv) $||[A \cap B, \overline{A \cup B}]| = |[A - B, B - A]| = 0$.

Proof. (i) By Lemma 2.2, $d_A(x) > d_{\overline{A}}(x)$ and so $d_{\overline{A}}(x) \leq d_A(x) - 1$. Now $d_A(xyz) = d_{A-\{x,y,z\}}(x) + d_{A-\{x,y,z\}}(y) + d_{A-\{x,y,z\}}(z) \geq d_{\overline{A}}(x) + d_{\overline{A}}(y) + d_{\overline{A}}(z) = d_{\overline{A}}(xyz)$.
 (ii) $d_{A-B}(xyz) = d_A(xyz) - d_{A \cap B}(xyz) = d_A(xyz) \geq d_{\overline{A}}(xyz) = d_{B-A}(xyz) + d_{\overline{A \cup B}}(xyz) = d_{B-A}(xyz)$. If $|B - A| = 3$ then B is λ_4 -atom of G and so $d_{B-A}(xyz) \geq d_{A-B}(xyz)$. Thus $d_{A-B}(xyz) = d_{B-A}(xyz)$. If $|B - A| \geq 4$ then $G[B - A]$ is λ_4 -non-trivial. Now by Lemma 2.2, $d_{B-A}(xyz) \geq d_{A-B}(xyz)$ and so $d_{A-B}(xyz) = d_{B-A}(xyz)$.
 (iii) Since $d_A(xyz) = d_{\overline{A}}(xyz)$ by part (i), $d_{A-\{x,y,z\}}(x) + d_{A-\{x,y,z\}}(y) + d_{A-\{x,y,z\}}(z) = d_{\overline{A}}(x) + d_{\overline{A}}(y) + d_{\overline{A}}(z)$. On the other hand $d_{A-\{x,y,z\}}(x) = d_A(x) - 1$, $d_{A-\{x,y,z\}}(y) = d_A(y) - 1$ and $d_{A-\{x,y,z\}}(z) = d_A(z) - 1$. Since $d_{\overline{A}}(a) \leq d_A(a)$, where $a \in \{x, y, z\}$, we have $d_{\overline{A}}(a) = d_A(a) - 1$ for $a \in \{x, y, z\}$. Now we considering the vertex set $\{x', y', z'\}$. By replace B by $\overline{B}$ we see that $d_{\overline{A}}(a) = d_A(a) - 1$ for $a \in \{x', y', z'\}$.
 (iv) By taking the place B by $\overline{B}$, $d_{B-A}(x'y'z') = 0$. Thus $||[A - B, B - A]| = 0$. On the other hand $||[A \cap B, \overline{A \cup B}]| = 0$. $\square$

Lemma 2.10. Suppose that A is a λ_4 -atom of G , $e = uv$ is an edge of $G[A]$ and B is a λ_4 -atom of $G - e$. Then $G[\overline{A \cup B}]$ is connected.

Proof. Suppose to contrary that $G[\overline{A \cup B}]$ is disconnected and C_1 and C_2 are two components of $G[\overline{A \cup B}]$. By Lemma 2.9, $d_A(x'y'z') \geq d_{\overline{A}}(x'y'z')$, where $x', y', z' \in A - B$. Thus $w(C_1) < w(\overline{A \cup B}) = w(A \cup B) = w(B) + d_{\overline{A \cup B}}(x'y'z') - d_{A \cup B}(x'y'z') = w(B) + (d_{\overline{A}}(x'y'z') - d_{B-A}(x'y'z')) - (d_{B-A}(x'y'z') + d_A(x'y'z')) = w(B) - d_A(x'y'z') + d_{\overline{A}}(x'y'z') - 2d_{B-A}(x'y'z') \leq w(B) = \lambda_4(G)$.

Now by Lemma 2.1, C_1 is λ_4 -independent and so $V(C_2) \leq 3$. Since $G[B \setminus A]$ is connected, $|B \setminus A| \geq 3$ and C_2 is connected to $B \setminus A$, we see that $G[\overline{A}] - C_1$ has a λ_4 -non-trivial component. Now by Lemma 2.2, we have

$$d_A(C_1) \leq d_{\overline{A}}(C_1) = d_{B-A}(C_1) = d_B(C_1).$$

Similarly

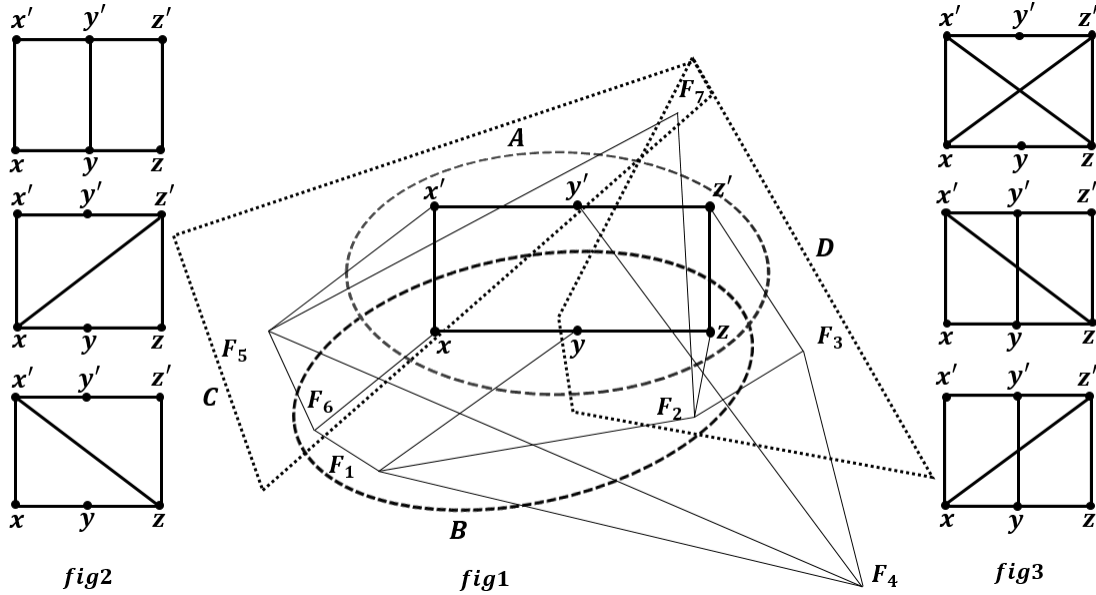
$$d_B(C_1) \leq d_{\overline{B}}(C_1) = d_{A-B}(C_1) = d_A(C_1).$$

Thus $d_A(C_1) = d_B(C_1)$. Put $A' = (A - B) \cup C_1$. Thus $G[A']$ and $G[\overline{A'}]$ are connected. Also by Lemma 2.9, $w(A') = w(A - B) + d_B(C_1) - d_{A-B}(C_1) = w(A - B) + d_B(C_1) - d_A(C_1) = w(A - B) =$

$w(A) + d_{A-B}(xyz) - d_{B-A}(xyz) = w(A) = \lambda_4(G)$. Thus A' is λ_4 -fragment and since $|A'| \leq |A|$ we see that A' is also a λ_4 -atom of G . Now $d_{\overline{A'}}(x'y'z') = d_B(x'y'z') + d_{\cup C_i}(x'y'z') = d_{B-A}(x'y'z') + d_{A \cap B}(x'y'z') + d_{\cup C_i}(x'y'z') \geq d_{A \cap B}(x'y'z') + d_{C_1}(x'y'z') = d_A(x'y'z') + d_{C_1}(x'y'z') > d_A(x'y'z')$, where the C_i^s are component of G , a contradiction. $\square$

Theorem 2.1. Suppose that G is the minimally λ_4 -connected graph. Then G is isomorphic to the Figure 1 and all the atoms of G are trivial.

Proof. Suppose that A is λ_4 -atom of G , $e = uv$ is an edge of $G[A]$ and B is a λ_4 -atom of $G - e$. We can apply Lemma 2.6 to A and B . Suppose that $A \cap B = \{x, y, z\}$ and $A - B = \{x', y', z'\}$. By Lemma 2.9 $d_{\bar{A}}(t) = d_A(t) - 1$ for each $t \in A$. Thus $\lambda_4(G) = w(A) = d_{\bar{A}}(x) + d_{\bar{A}}(y) + d_{\bar{A}}(z) + d_{\bar{A}}(x') + d_{\bar{A}}(y') + d_{\bar{A}}(z') = d_A(x) + d_A(y) + d_A(z) + d_A(x') + d_A(y') + d_A(z') - 6 - 2 + 2 = d_A(xyz) + d_A(x'y'z') + 2 = 2|[xyz, x'y'z']| + 2$.



Now suppose that D is λ_4 -atom of $G - xy$ and C is a λ_4 -atom of $G - y'z'$. We apply Lemma 2.10 to A, D and A, C . Without loss of generality we may suppose that $A \cap D = \{z', z, y\}$ and $A \cap C = \{x, x', y'\}$. Set $F_1 = ((B - A) - D) - C$, $F_2 = (B - A) \cap D$, $F_3 = (\overline{A \cup B}) \cap (D - C)$, $F_4 = (\overline{A \cup B}) - D - C$, $F_5 = ((\overline{A \cup B}) \cap (C - D))$, $F_6 = (B - A) \cap C$ and $F_7 = (C \cap D)$. By Lemma 2.9 we have $|\{x', y', z'\}, F_1 \cup F_2 \cup F_6 \cup F_7| = |\{x, y, z\}, F_3 \cup F_4 \cup F_5 \cup F_7| = |\{x, x', y'\}, F_3 \cup F_1 \cup F_2 \cup F_7| = |\{y, z, z'\}, F_4 \cup F_5 \cup F_6 \cup F_7| = |\{z, z', y'\}, F_1 \cup F_6 \cup F_5 \cup F_7| = |\{x', x, y\}, F_2 \cup F_3 \cup F_4 \cup F_7| = 0$.

Thus $|[x, F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup F_7]| = 0$ and so $|[x, F_6]| = |[x, \bar{A}] = d_{\bar{A}}(a) = d_A(a) - 1 \geq \delta_{G[A]} - 1 \geq 1$. With the same argument we have

$$|[x, F_6]| \geq 1, |[x', F_5]| \geq 1, |[y', F_4]| \geq 1, |[y, F_1]| \geq 1, |[z, F_2]| \geq 1, |[z', F_3]| \geq 1. \quad (2)$$

Thus non of F_1, F_2, F_3, F_4, F_5 and F_6 are empty and $[A, \bar{A}] = [x, F_6] \cup [y, F_1] \cup [z, F_2] \cup [x', F_5] \cup [y', F_4] \cup [z', F_3]$. Also note that since $|[F_5 \cup F_6, F_7]| \geq 1$ and $|[F_2 \cup F_3, F_7]| \geq 1$ we have $F_7 \neq \emptyset$. Since $G[F_1 \cup F_2 \cup F_6] = G[B - A]$, $G[F_5 \cup F_6 \cup F_7] = G[C - A]$, $G[F_2 \cup F_3 \cup F_7] = G[D - A]$, $G[F_1 \cup F_4 \cup F_5 \cup F_6] = G[\overline{A \cup D}]$ and $G[F_3 \cup F_4 \cup F_5 \cup F_7] = G[\overline{A \cup B}]$ are connected without loss of generality we have

$$|[F_1, F_2]| \geq 1, |[F_2, F_3]| \geq 1, |[F_1, F_4]| \geq 1, |[F_3, F_4]| \geq 1,$$

$$|[F_5, F_6]| \geq 1, |[F_1, F_6]| \geq 1, |[F_2, F_7]| \geq 1, [F_5, F_7] \geq 1. \quad (3)$$

Therefore we have

$$w(B) = |[xyz, x'y'z']| + |[F_1 \cup F_2 \cup F_6, F_3 \cup F_4 \cup F_5 \cup F_7]|, \quad (4)$$

$$w(D) = |[yzz', xx'y']| + |[F_2 \cup F_3 \cup F_7, F_1 \cup F_4 \cup F_6 \cup F_5]| + |[y, F_1]|, \quad (5)$$

$$w(C) = |[y', F_4]| + |[\{x, x', y'\}, \{y, z, z'\}]| + |[F_6 \cup F_5 \cup F_7, F_1 \cup F_2 \cup F_4 \cup F_3]|. \quad (6)$$

We know that $G[A]$ is isomorphic to the subgraph of K_6 . Thus either $G[A] \cong C_6$ or $G[A]$ is one of the graphs in Figures (2) and (3). Suppose that $G[A] \cong C_6$. Then $\lambda_4(G) = 2|[xyz, x'y'z']| + 2 = 6$ and so equalities hold in (2). Also by (3), (4), (5) and (6) we have $w(B) \geq 6$, $w(D) \geq 6$ and $w(C) \geq 6$. Since B, C and D are λ_4 -fragment, we have $w(B) = w(C) = w(D) = \lambda_4(G) = 6$. In particular, $[F_1, F_3] = 0$. Now by considering $[F_1, A] = [y, F_1] = 1$ we have $w(F_1) = 4 < \lambda_4(G)$. Now by Lemma 2.1, $G[F_1]$ is λ_4 -independent. Also since $G[F_1 \cup F_2 \cup F_6]$ is connected, without loss of generality we may suppose that $[F_1, F_2] = 1$. We see that $G[F_1]$ is connected and so $|F_1| \leq 3$. Suppose that $|F_1| = 2$ or $|F_1| = 3$. Put $F = F_1 \cup x \cup y$. We have $|F| < \alpha_4(G)$ and $w(F) = w(F_1) - |[x, F_1]| + d_A(x) - |[y, F_1]| + d_A(y) - |[x, y]| = 4 - 0 + 2 - 1 + 2 - 1 = 6 = \lambda_4(G)$. By Lemma 2.1, $G[F]$ is λ_4 -independent, a contradiction. Thus $|F_1| = 1$. Similarly, $|F_1| = |F_2| = |F_3| = |F_4| = |F_5| = |F_6| = |F_7| = 1$.

Now suppose that $G[A]$ is isomorphic to one of the graphs in Figure 2. We have $\lambda_4(G) = 2|[xyz, x'y'z']| + 2 = 8$ and $w(B) = w(D) = w(C) = \lambda_4(G) = 8$. Now by (6), $[F_5 \cup F_6 \cup F_7, F_1 \cup F_2 \cup F_3 \cup F_4] \leq 4$ and so by (3) without loss of generality we may suppose that $[F_1, F_6] \leq 1$. By (4), $[F_1 \cup F_2 \cup F_6, F_3 \cup F_4 \cup F_5 \cup F_7] = 5$ and so by (3) we have $[F_1, F_4] \leq 2$. Also by (5), $[F_2 \cup F_3 \cup F_7, F_1 \cup F_4 \cup F_5 \cup F_6] \leq 4$ and so by (3) we have $[F_1, F_2] \leq 2$. Therefore $w(F_1) = |[y, F_1]| + |[F_1, F_2]| + |[F_1, F_6]| + |[F_1, F_7]| \leq 6 < 8$. Now by Lemma 2.1, $G[F_1]$ is λ_4 -independent. Also since $[F_1, F_4] = 1$ and $G[F_1 \cup F_4]$ is connected we conclude that $G[F_1]$ is connected. Thus $|F_1| \leq 3$. Suppose that $|F_1| = 3$ and put $F = F_1 \cup y$. $G[F]$ and $G[\overline{F}]$ are connected and $|F| = 4 < \alpha_4(G)$ and $w(F) = w(F_1) + d_A(y) - |[y, F_1]| \leq 6 + 2 - 1 = 7 < \lambda_4(G)$. Thus $G[F]$ is λ_4 -independent, a contradiction. Thus we may suppose that $|F_1| = 1$ or $|F_1| = 2$. If $|F_i| = 1$ for $1 \leq i \leq 7$. Since G is simple by (3) we conclude that $[F_1, F_6] = [F_1, F_2] = [F_1, F_4] = 1$ and $w(F_1) = |[y, F_1]| + |[F_1, F_6]| + |[F_1, F_2]| + |[F_1, F_4]| = 4$. Thus if we put $F = F_1 \cup z' \cup y \cup z$ then $w(F) = 8$ and by $|F| < \alpha_4(G)$, a contradiction. Thus we may suppose that $|F_1| = |F_6| = 2$. We put $F = F_1 \cup F_6$. Thus $w(F) = w(F_1) + w(F_6) - |[F_1, F_6]| \leq |[y, F_1]| + |[F_1, F_6]| + |[F_1, F_2]| + |[F_1, F_4]| + |[F_1, F_6]| + |[F_5, F_6]| + |[F_6, x]| - |[F_1, F_6]| \leq 8$, a contradiction. Now suppose that $G[A]$ is isomorphic to one of the graphs in Figure 3. We have $\lambda(G) = 2|[xyz, x'y'z']| + 2 = 8 + 2 = 10$. Suppose that $|F_1| = 3$ and put $F = F_1 \cup y$. Thus $G[F]$ and $G[\overline{F}]$ are connected and $w(F) = w(F_1) - |[y, F_1]| + d_A(y) \leq 8 - 1 + 2 = 9 < \lambda_4(G) = 10$ and so F is λ_4 -independent a contradiction. Thus we may suppose that $|F_1| = 1$ or $|F_1| = 2$. If $|F_1| = 2$ then by (4) we have $[F_1 \cup F_2 \cup F_3 \cup F_4, F_5 \cup F_6 \cup F_7] = 5$, $[F_1 \cup F_4 \cup F_5 \cup F_6, F_2 \cup F_3 \cup F_7] = 5$ and $[F_1 \cup F_2 \cup F_6, F_3 \cup F_4 \cup F_5 \cup F_7] = 6$. By (3) without loss of generality we may suppose that $[F_1, F_6] = 1$, $[F_1, F_2] \leq 3$ and $[F_1, F_4] \leq 2$. Now $w(F_1) \leq 7$. Put $F = F_1 \cup x \cup y$. Thus $w(F) = w(F_1) - |[y, F_1]| + d_A(x) + d_A(y) - |[x, y]| \leq 7 - 1 + 2 + 3 = 11 > \lambda_4(G)$ and by $|F| < \alpha_4(G)$ we get a contradiction. Therefore we may suppose that $|F_i| = 1$ for $1 \leq i \leq 7$. Now since G is

simple we have $|[F_1, F_6]| = |[F_1, F_2]| = |[F_1, F_4]| = 1$ and so $w(F_1) = 4$. Put $F = F_1 \cup x \cup y \cup z$. Now $w(F) \leq 9 < \lambda_4(G)$, a contradiction. $\square$

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References

- [1] M. Bai, Y. Tian, and J. Yin, The super restricted edge-connectedness of direct product graphs, *Parallel Process. Lett.* **33(3)** (2023) 2350008:1-2350008:7.
- [2] H. Cheng, R. Varmazyar, and M. Ghasemi, On k -restricted connectivity of direct product of graphs, *Discrete Math. Algorithms Appl.* **15(8)** (2023) 2250175.
- [3] M. Ghasemi, Some results about the reliability of folded hypercubes. *Bull. Malaysian Math. Sci. Soc.* **44** (2021) 1093– 1099.
- [4] M. Ghasemi, On the reliability of modified bubble-sort graphs. *Trans. Combinatorics.* **11(2)** (2022) 77–83.
- [5] A. Hellwig and L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs: A survey, *Discrete Math.* **308** (2008) 3265–3296.
- [6] Y.M. Hong, Q.H. Liu, and Z. Zhang, Minimally restricted edge connected graphs, *App. Math. Lett.* **21** (2008) 820–823.
- [7] Q.G. Liu, Y.M. Hong, and Z. Zhang, Minimally 3-restricted edge connected graphs, *Discrete Appl. Math.* **157** (2009) 685–690.
- [8] X.M. Liu and J. Meng, The k -restricted edge-connectivity of the data center network DCell, *Appl. Math. Comput.* **396** (2021) 125941.
- [9] S. Liu, C. Ouyang, and J. Ou, Sufficient conditions for optimally and super m -restricted edge-connected graphs with given girth, *Linear Multilinear Algebra.* **70(6)** (2020) 1146–1158.
- [10] L. Lovász, Combinatorial Problems and Exercises, North-Holland Publishing Company, 1979.
- [11] T. Ma, J. Wang, and M. Zhang, The restricted edge-connectivity of kronecker product Graphs, *Parallel Process. Lett.* **29(3)** (2019) 1950012:1-1950012:7.
- [12] J.X. Meng, Optimally super-edge-connected transitive graphs, *Discrete Math.* **260** (2003) 239–248.
- [13] H. Tarakmi, H. Azanchilar, M. Ghasemi, and Gh. Azadi, n -restricted edge connectivity of m -barrel fullerene graphs, *Iran J. Sci. Technol Trans. A. Sci.* **45** (2021) 997–1004

- [14] Y.Q. Wang and Q. Li, Super-edge-connectivity properties of graphs with diameter 2, *J. Shanghai Jiaotong Univ.* **33** (6) (1999) 647–649(in Chinese)
- [15] Y.Q. Wang and Q. Li, Sufficient conditions for a graph to be maximally restricted edge-connected, *J. Shanghai Univ.* **35** (8) (2001) 1253–1255(in Chinese).
- [16] J.M. Xu and K.L. Xu, On restricted edge-connectivity of graphs, *Discrete Math.* **243** (2002) 291–298.
- [17] Z. Zhang and J.X. Meng, Restricted edge connectivity of edge transitive graphs, *Ars Combinatorica.* LXXVIII (2006) 297–308.
- [18] Z. Zhang and J.J. Yuan, Degree conditions for restricted-edge-connectivity and isoperimetric-edge-connectivity to be optimal, *Discrete Math.* **307** (2007) 293–298.