Electronic Journal of Graph Theory and Applications **9** (2) (2021), 451–468

Electronic Journal of
Graph Theory and Applications

On maximum packings of λ-fold complete 3-uniform hypergraphs with triple-hyperstars of size 4

Amber Armstrong[a], Ryan C. Bunge[b], William Duncan[b], Saad I. El-Zanati[b], Kristin Koe[b], Rachel Stutzman[c]

[a]*St. Paul Academy and Summit School, St. Paul, MN*
[b]*Illinois State University, Normal, IL 61790-4520*
[c]*Greenville University, Greenville, IL*

rcbunge@ilstu.edu, saad@ilstu.edu

Abstract

A symmetric triple-hyperstar is a connected, 3-uniform hypergraph where, for some edge $\{a, b, c\}$, vertices a, b, and c all have degree $k > 1$ and all other edges contain exactly 2 vertices of degree 1. Let H denote the symmetric triple-hyperstar with 4 edges and, for positive integers λ and v, let ${}^{\lambda}K_v^{(3)}$ denote the λ-fold complete 3-uniform hypergraph on v vertices. We find maximum packings of ${}^{\lambda}K_v^{(3)}$ with copies of H.

Keywords: hypergraph decomposition, maximum packing, hypergraph design
Mathematics Subject Classification : 05C65, 05C51
DOI: 10.5614/ejgta.2021.9.2.17

1. Introduction

A *hypergraph* H consists of a finite, nonempty set V of *vertices* and a finite collection $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of V called *hyperedges* or simply *edges*. For a given hypergraph H, we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of H, respectively. We call $|V(H)|$ and $|E(H)|$ the *order* and *size* of H, respectively. A hypergraph H is *simple* if no edge appears more than once in $E(H)$. If for each $e \in E(H)$ we have $|e| = t$, then H is said to be t-*uniform*. Thus t-uniform hypergraphs are generalizations of the concept

Received: 10 September 2019, Revised: 12 February 2021, Accepted: 28 February 2021.

of a graph (where $t = 2$). Graphs with repeated edges are often called *multigraphs*. If H is a simple hypergraph and λ is a positive integer, then λ-*fold* H, denoted ${}^{\lambda}H$, is the multi-hypergraph obtained from H by repeating each edge exactly λ times. The hypergraph with vertex set V and edge set the set of all t-element subsets of V is called the *complete* t-*uniform hypergraph on* V and is denoted by $K_V^{(t)}$. If $v = |V|$, then ${}^{\lambda}K_v^{(t)}$ is called the λ-*fold complete* t-*uniform hypergraph of order* v and is used to denote any hypergraph isomorphic to ${}^{\lambda}K_V^{(t)}$. When $t = 2$, we will use ${}^{\lambda}K_v$ in place of ${}^{\lambda}K_v^{(2)}$. Similarly, if $\lambda = 1$, then we will use $K_v^{(t)}$ in place of ${}^{1}K_v^{(t)}$. If H' is a subhypergraph of H, then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H'. We may refer to $H \setminus H'$ as the hypergraph H with a *hole* H'. The vertices in H' may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A *decomposition* of a multigraph K is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of subgraphs of K such that $\{E(G_1), E(G_2), \ldots, E(G_s)\}$ is a partition of $E(K)$. If each element of Δ is isomorphic to a fixed graph G, then Δ is called a G-*decomposition* of K. If exactly one element $L \in \Delta$ is not isomorphic to G, then Δ is called a G-*packing* of K with *leave* L. Such a G-packing is *maximum* if no other possible G-packing of K has a leave of a smaller size than that of L. Clearly, if $|E(L)| < |E(G)|$, then the G-packing is maximum. Moreover, a G-decomposition of K can be viewed as a maximum G-packing with an empty leave.

A G-decomposition of ${}^{\lambda}K_v$ is also known as a G-*design of order* v *and index* λ. A K_k-design of order v and index λ is usually known as a 2-(v, k, λ) *design* or as a *balanced incomplete block design of index* λ or a (v, k, λ)-***BIBD***. The problem of determining all v for which there exists a G-design of order v is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of K such that $\{E(H_1), E(H_2), \ldots, E(H_s)\}$ is a partition of $E(K)$. Any element of Δ isomorphic to a fixed hypergraph H is called an H-*block*. If all elements of Δ are H-blocks, then Δ is called an H-*decomposition* of K. If exactly one element $L \in \Delta$ is not an H-block, then Δ is called an H-*packing* of K with *leave* L, where we again define such a packing to be *maximum* if L has the fewest edges possible. An H-decomposition of ${}^{\lambda}K_v^{(t)}$ is called an H-*design of order* v *and index* λ. The problem of determining all v for which there exists an H-design of order v and index λ is called the λ-*fold spectrum problem for* H-*designs*.

A $K_k^{(t)}$-design of order v and index λ is a generalization of 2-(v, k, λ) designs and is known as a t-(v, k, λ) *design* or simply as a t-*design*. A summary of results on t-designs appears in [16]. A t-$(v, k, 1)$ design is also known as a *Steiner system* and is denoted by $S(t, v, k)$ (see [9] for a summary of results on Steiner systems). Keevash [15] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of v. Similar results were obtained by Glock, Kühn, Lo, and Osthus [10, 11] and extended to include the corresponding asymptotic results for H-designs of order v for all uniform hypergraphs H. These results for t-uniform hypergraphs mirror the celebrated results of Wilson [24] for graphs. Although these asymptotic results assure the existence of H-designs for sufficiently large values of v for any uniform hypergraph H, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on G-decompositions of K_v where G is a graph with a relatively small number of edges (see [1] and [6] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [5] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [5], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered H-designs where H is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let T, O, and I denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph T is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [12]. In another paper [13], Hanani settled the spectrum problem for O-designs and gave necessary conditions for the existence of I-designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer m, let $S_m^{(3)}$ denote the 3-uniform hypergraph of size m that consists of one vertex of degree m and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S_m^{(3)}$-decompositions of $K_v^{(3)}$ are given in [22] for $m \in \{4, 5, 6\}$ and settled in [19] for any m. Some results on maximum $S_m^{(3)}$-packings of $K_v^{(3)}$ are given in [20]. Perhaps the best known general result on decompositions of complete t-uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m. There are, however, several articles on decompositions of complete t-uniform hypergraphs (see [2] and [21]) and of t-uniform t-partite hypergraphs (see [17] and [23]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [14] and [18]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum H-packings of ${}^{\lambda}K_v^{(3)}$, where H is a 3-uniform symmetric triple-hyperstar with 4 edges. A *triple-hyperstar* is a connected 3-uniform hypergraph where, for some edge $\{a, b, c\}$, vertices a, b, and c all have degree greater than 1 and all other edges contain exactly two vertices of degree 1. That is, if the degrees of vertices a, b, and c in the triple-hyperstar are $m_1 + 1$, $m_2 + 1$, and $m_3 + 1$, respectively, then the removal of edge $\{a, b, c\}$ would result in the hypergraph consisting of three components, namely $S_{m_1}^{(3)}$, $S_{m_2}^{(3)}$, and $S_{m_3}^{(3)}$. We call such a triple-hyperstar *symmetric* if $m_1 = m_2 = m_3 = m$. Thus a symmetric triple-hyperstar has $6m + 3$ vertices and $3m + 1$ edges. We are interested in the case $m = 1$.

Let $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$ denote the symmetric triple-hyperstar H with vertex set $\big\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\big\}$ and edge set $\big\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\big\}$ as seen Figure 1. Here we show that for all $v \geq 9$ and $\lambda \geq 1$, there exists a maximum H-packing of ${}^{\lambda}K_v^{(3)}$ where the leave has fewer than 4 edges.

1.1. Additional Notation and Terminology

Let $\mathbb{Z}_n$ denote the group of integers modulo n. We next define some notation for certain types of 3-uniform hypergraphs.

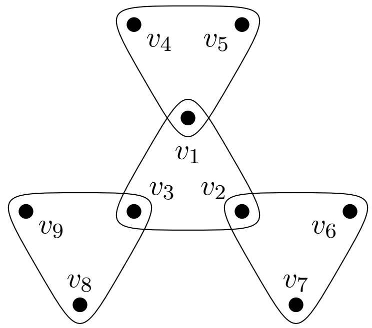

Figure 1. The symmetric triple-hyperstar H of size 4, denoted by $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$.

Let U_1, U_2, U_3 be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of U_1, U_2, U_3 is denoted by $K^{(3)}_{U_1,U_2,U_3}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of U_1, U_2 is denoted by $L^{(3)}_{U_1,U_2}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K^{(3)}_{u_1,u_2,u_3}$ or $L^{(3)}_{u_1,u_2}$ to denote any hypergraph that is isomorphic to $K^{(3)}_{U_1,U_2,U_3}$ or $L^{(3)}_{U_1,U_2}$, respectively.

2. Main Results

2.1. Decompositions and Packings of Simple Hypergraphs

We begin by giving necessary conditions for the existence of an H-decomposition of $K^{(3)}_v$. An obvious necessary condition is that 4 must divide the number of edges in $K^{(3)}_v$, and thus we must have $v \equiv 0$, 1, 2, 4, or 6 $\pmod 8$. Since $K^{(3)}_1$ and $K^{(3)}_2$ contain no edges, it is vacuously true that H decomposes $K^{(3)}_1$ and $K^{(3)}_2$. Also, since H has order 9, there is no H-decomposition of $K^{(3)}_4$, $K^{(3)}_6$, or $K^{(3)}_8$. Hence, we have the following.

Lemma 1. *There exists an H-decomposition of $K^{(3)}_v$ only if $v \equiv 0$, 1, 2, 4, or 6 $\pmod 8$ and $v \notin \{4, 6, 8\}$.*

We intend to prove that the above conditions are sufficient by showing how to construct H-decompositions of $K^{(3)}_v$ for all $v \equiv 0$, 1, 2, 4, or 6 $\pmod 8$ with $v \geq 9$. Our constructions are dependent on the many small examples given in the Appendix. We begin by proving a lemma that is fundamental to our constructions.

Lemma 2. *Let n, x, and r be nonnegative integers such that $nx + r \geq 3$. There exists a decomposition of $K^{(3)}_{nx+r}$ that is comprised of isomorphic copies of each of the following under the given conditions:*

- $K^{(3)}_r$ *if* $x = 0$,
- $K^{(3)}_{n+r}$ *if* $x \geq 1$,

- $K^{(3)}_{n+r} \setminus K^{(3)}_r$ *if* $x \geq 2$,
- $K^{(3)}_{r,n,n} \cup L^{(3)}_{n,n}$ *if* $x \geq 2$,
- $K^{(3)}_{n,n,n}$ *if* $x \geq 3$.

Furthermore, if $x \geq 1$ *and* $r \geq 3$*, then the decomposition contains exactly one isomorphic copy of* $K^{(3)}_{n+r}$.

Proof. If $x \in \{0, 1\}$, the decomposition is trivial. Similarly, if $n = 0$, then $r \geq 3$, and the result is trivial because $K^{(3)}_r = K^{(3)}_{n+r} = K^{(3)}_{nx+r}$ while $K^{(3)}_{n+r} \setminus K^{(3)}_r$, $K^{(3)}_{r,n,n} \cup L^{(3)}_{n,n}$, and $K^{(3)}_{n,n,n}$ are all empty (i.e., contain no edges). For the remainder of the proof, we assume that $x \geq 2$ and $n \geq 1$.

Let $V_0, V_1, \dots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$ and $|V_1| = |V_2| = \cdots = |V_x| = n$. Then, the decomposition of $K^{(3)}_{nx+r}$ results from the fact that the complete 3-uniform hypergraph on the vertex set $V_0 \cup V_1 \cup \cdots \cup V_x$, which is $nx + r$ vertices, can be viewed as the (edge-disjoint) union

$$K^{(3)}_{V_1 \cup V_0} \ \cup \bigcup_{2 \leq i \leq x} \left(K^{(3)}_{V_i \cup V_0} \setminus K^{(3)}_{V_0} \right) \ \cup \bigcup_{1 \leq i < j \leq x} \left(K^{(3)}_{V_0, V_i, V_j} \cup L^{(3)}_{V_i, V_j} \right) \ \cup \bigcup_{1 \leq i < j < k \leq x} \left(K^{(3)}_{V_i, V_j, V_k} \right).$$

In addition, if $r \geq 3$, the single isomorphic copy of $K^{(3)}_{n+r}$ in the decomposition is $K^{(3)}_{V_1 \cup V_0}$. □

We now give our main results.

Theorem 3. *There exists an* H*-decomposition of* $K^{(3)}_v$ *if and only if* $v \equiv 0$, 1, 2, 4, *or* 6 $\pmod 8$ *and* $v \notin \{4, 6, 8\}$.

Proof. The necessary conditions for the existence of an H-decomposition of $K^{(3)}_v$ are established in Lemma 1. Thus we need only to establish their sufficiency. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{1, 2, 4, 6, 8\}$. By Lemma 2 it suffices to find H-decompositions of $K^{(3)}_{8+r}$, $K^{(3)}_{8+r} \setminus K^{(3)}_r$, $K^{(3)}_{r,8,8} \cup L^{(3)}_{8,8}$, and $K^{(3)}_{8,8,8}$. We note that if $r \in \{1, 2\}$ then $K^{(3)}_{8+r} \setminus K^{(3)}_r$ is isomorphic to $K^{(3)}_{8+r}$. Also, $K^{(3)}_{3,8,8}$ decomposes $K^{(3)}_{6,8,8}$, and $K^{(3)}_{4,8,8}$ decomposes $K^{(3)}_{8,8,8}$. Thus, it suffices to find H-decompositions of $K^{(3)}_9$, $K^{(3)}_{10}$, $K^{(3)}_{12}$, $K^{(3)}_{14}$, $K^{(3)}_{16}$, $K^{(3)}_{12} \setminus K^{(3)}_4$, $K^{(3)}_{14} \setminus K^{(3)}_6$, $K^{(3)}_{16} \setminus K^{(3)}_8$, $K^{(3)}_{1,8,8} \cup L^{(3)}_{8,8}$, $K^{(3)}_{2,8,8} \cup L^{(3)}_{8,8}$, $K^{(3)}_{3,8,8}$, $K^{(3)}_{4,8,8}$, and $L^{(3)}_{8,8}$, which are each shown to exist within Examples 1–16. □

Theorem 4. *If* $v \geq 9$ *is an integer, then there exists a maximum* H*-packing of* $K^{(3)}_v$ *where the leave has fewer than four edges.*

Proof. If $v \equiv 0$, 1, 2, 4, or 6 $\pmod 8$, then the result follows from the H-decomposition result in Theorem 3, which translates to a maximum H-packing with an empty leave. Hence, we need only consider when $v \equiv 3$, 5, or 7 $\pmod 8$. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find

- a maximum H-packing of $K^{(3)}_{8+r}$ with a leave consisting of fewer than four edges and
- H-decompositions of $K^{(3)}_{8+r} \setminus K^{(3)}_r$, $K^{(3)}_{r,8,8} \cup L^{(3)}_{8,8}$, and $K^{(3)}_{8,8,8}$.

We note that an H-decomposition of $K_{11}^{(3)} \setminus K_3^{(3)}$ is a subset of an H-packing of $K_{11}^{(3)}$ with a leave consisting of the single edge in the hole, which is necessarily then a maximum H-packing of $K_{11}^{(3)}$. Also, $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, it suffices to find maximum H-packings (with leaves of fewer than four edges) of $K_{11}^{(3)}$, $K_{13}^{(3)}$, and $K_{15}^{(3)}$, which are each shown to exist in Examples 17–19, and H-decompositions of $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, $K_{5,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which are each shown to exist within Examples 6–15. □

2.2. Results for any Positive Index

We show here the necessary conditions for an H-decomposition of λ-fold $K_v^{(3)}$ for any positive integer λ. This will inform our choice on which combinations of λ and v we search for decompositions of ${}^{\lambda}K_v^{(3)}$ versus finding maximum packings.

Lemma 5. *Let $v \geq 9$ be an integer. There exists an H-decomposition of λ-fold $K_v^{(3)}$ only if the following hold:*

- *if* $\gcd(\lambda, 4) = 1$, *then* $v \equiv 0,\ 1,\ 2,\ 4$, *or* $6 \pmod 8$;
- *if* $\gcd(\lambda, 4) = 2$, *then* $v \equiv 0,\ 1$, *or* $2 \pmod 4$;
- *if* $\gcd(\lambda, 4) = 4$, *then* $v \geq 9$.

Proof. Suppose there exists an H-decomposition of ${}^{\lambda}K_v^{(3)}$. Since $|E(H)| = 4$, we must have $4 \mid \lambda\binom{v}{3} = \lambda v(v-1)(v-2)/6$, and thus $8 \mid \lambda v(v-1)(v-2)$. First, if $\gcd(\lambda, 4) = 1$, then $8 \mid v(v-1)(v-2)$, and thus $v \equiv 0$, 1, 2, 4, or $6 \pmod 8$. Second, if $\gcd(\lambda, 4) = 2$, then $4 \mid v(v-1)(v-2)$, and thus $v \equiv 0$, 1, or $2 \pmod 4$. Finally, if $\gcd(\lambda, 4) = 4$, then $2 \mid v(v-1)(v-2)$, which is true for any $v \geq 9$. □

Next, we settle the decomposition and maximum packing results for some small values of λ.

Theorem 6. *Let $v \geq 9$ be an integer. There exists an H-decomposition of 2-fold $K_v^{(3)}$ if $v \equiv 0$, 1, or $2 \pmod 4$.*

Proof. If $v \equiv 0, 1, 2, 4$, or $6 \pmod 8$, then the result follows from 2 copies of an H-decomposition of $K_v^{(3)}$, which exists by Theorem 3. Hence, we need only consider when $v \equiv 5 \pmod 8$.

First, we consider when $v = 13$. Let $v_1, v_2, \ldots, v_9 \in V\left(K_v^{(3)}\right)$. By Example 18, there exist both a maximum H-packing, say Δ_1, of $K_{13}^{(3)}$ with a leave consisting of two edges that share a single vertex and a maximum H-packing, say Δ_2, of $K_{13}^{(3)}$ with a leave consisting of two vetex-disjoint edges. Let L_1 and L_2 be the leaves of Δ_1 and Δ_2, respectively. Without loss of generality, we may assume that

$$E(L_1) = \big\{\{v_1, v_2, v_3\},\ \{v_1, v_4, v_5\}\big\},$$
$$E(L_2) = \big\{\{v_2, v_6, v_7\},\ \{v_3, v_8, v_9\}\big\}.$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, L' is isomorphic to H, and the (multi-)set

$$\big(\Delta_1 \setminus \{L_1\}\big) \cup \big(\Delta_2 \setminus \{L_2\}\big) \cup \{L'\}$$

is a collection of H-blocks such that each edge of $K_{13}^{(3)}$ is represented exactly twice. Therefore, we have an H-decomposition of ${}^2K_{13}^{(3)}$.

Now, let $v = 8x + 5$ where $x \geq 2$. By Lemma 2 it suffices to find H-decompositions of (2-fold) $K_{13}^{(3)}$, $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{5,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. We note that $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$, and we already have that H decomposes ${}^2K_{13}^{(3)}$. Thus, we need only additionally find H-decompositions of $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{5,8,8}^{(3)}$, $L_{8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$, which exist by Examples 13, 11, 6, and 10, respectively. □

Theorem 7. *If $v \geq 9$ is an integer, then there exists a maximum H-packing of 2-fold $K_v^{(3)}$ where the leave has no edges or two vertex-disjoint edges.*

Proof. If $v \equiv 0,\, 1,$ or $2 \pmod 4$, then the result follows from the H-decomposition result in Theorem 6, which translates to a maximum H-packing with an empty leave. Hence, we need only consider when $v \equiv 3 \pmod 4$.

First, we consider when $v = 11$. Let Δ_1 and Δ_2 be maximum H-packings of $K_{11}^{(3)}$ with leaves L_1 and L_2, respectively, which exist by Example 17. Now, let L' be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, L' consists of two edges. In fact, we further note that L' can be any hypergraph with two edges, including ${}^2K_3^{(3)}$. Hence, the (multi-)set

$$\big(\Delta_1 \setminus \{L_1\}\big) \cup \big(\Delta_2 \setminus \{L_2\}\big) \cup \{L'\}$$

is a maximum H-packing of ${}^2K_{11}^{(3)}$ with a leave, L', consisting of two (possibly vertex-disjoint) edges.

Second, we consider when $v = 15$. Let $v_1, v_2, \ldots, v_9 \in V\Big(K_v^{(3)}\Big)$. By Example 19, there exist maximum H-packings of $K_{15}^{(3)}$ where the leaves consist of three disjoint edges. Let Δ_1 and Δ_2 be such H-packings of $K_{15}^{(3)}$ with leaves L_1 and L_2, respectively. Without loss of generality, we may assume that

$$E(L_1) = \big\{\{v_1, v_2, v_3\},\, \{v_4, v_5, v_6\},\, \{v_7, v_8, v_9\}\big\},$$
$$E(L_2) = \big\{\{v_1, v_4, v_5\},\, \{v_2, v_6, v_7\},\, \{v_3, v_8, v_9\}\big\}.$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that L' is decomposable into copies of $K_3^{(3)}$ and H. That is, if we let L'' be the hypergraph with edge set $\big\{\{v_4, v_5, v_6\},$ $\{v_7, v_8, v_9\}\big\}$, then $L' \setminus L''$ is isomorphic to H, and the (multi-)set

$$\big(\Delta_1 \setminus \{L_1\}\big) \cup \big(\Delta_2 \setminus \{L_2\}\big) \cup \{L' \setminus L'',\, L''\}$$

is a maximum H-packing of ${}^2K_{15}^{(3)}$ with a leave, L'', consisting of two (disjoint) edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 7\}$. By Lemma 2 it suffices to find

- a maximum H-packing of (2-fold) $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and
- H-decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum H-packing results. Also, we note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$, Thus, we need only additionally find H-decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which exist by Examples 17, 15, 9, 10, and 6, respectively. □

Theorem 8. *If $v \geq 9$ is an integer, then there exists a maximum H-packing of 3-fold $K_v^{(3)}$ where the leave has fewer than four edges.*

Proof. If $v \equiv 0, 1, 2, 4$, or $6 \pmod 8$, then the result follows from the H-decomposition result in Theorem 3, which translates to a maximum H-packing with an empty leave. Hence, we need only consider when $v \equiv 3, 5$, or $7 \pmod 8$.

First, we consider when $v = 11$. Let Δ_1 be a maximum H-packing of $K_{11}^{(3)}$ with leave L_1 consisting of a single edge, which exists by Example 17, and let Δ_2 be a maximum H-packing of ${}^2K_{11}^{(3)}$ with leave L_2 consisting of two edges, which exists by Theorem 7, Now, let L' be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, L' consists of three edges. In fact, we further note that L' can be any hypergraph with three edges, including ${}^3K_3^{(3)}$. Hence, the (multi-)set

$$\big(\Delta_1 \setminus \{L_1\}\big) \cup \big(\Delta_2 \setminus \{L_2\}\big) \cup \{L'\}$$

is a maximum H-packing of ${}^3K_{11}^{(3)}$ with a leave, L', consisting of three edges.

Second, we consider when $v = 13$. Let Δ_1 be a maximum H-packing of $K_{13}^{(3)}$ with leave L_1 consisting of two edges, which exists by Example 18, and let Δ_2 be an H-decomposition of ${}^2K_{13}^{(3)}$, which exists by Theorem 6, Hence, the (multi-)set $\Delta_1 \cup \Delta_2$ is a maximum H-packing of ${}^3K_{13}^{(3)}$ with a leave, L_1, consisting of two edges.

Third, we consider when $v = 15$. Let $v_1, v_2, \ldots, v_9 \in V\Big(K_v^{(3)}\Big)$, let Δ_1 be a maximum H-packing of $K_{15}^{(3)}$ with leave L_1 consisting of a three vertex-disjoint edges, which exists by Example 19, and let Δ_2 be a maximum H-packing of ${}^2K_{15}^{(3)}$ with leave L_2 consisting of two vertex-disjoint edges, which exists by Theorem 7, Without loss of generality, we may assume that

$$E(L_1) = \big\{\{v_1, v_4, v_5\},\ \{v_2, v_6, v_7\},\ \{v_3, v_8, v_9\}\big\},$$
$$E(L_2) = \big\{\{v_1, v_2, v_3\},\ \{v_4, v_5, v_6\}\big\}.$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that L' is decomposable into copies of $K_3^{(3)}$ and H. That is, if we let L'' be the hypergraph with the single edge $\{v_4, v_5, v_6\}$, then $L' \setminus L''$ is isomorphic to H, and the (multi-)set

$$\big(\Delta_1 \setminus \{L_1\}\big) \cup \big(\Delta_2 \setminus \{L_2\}\big) \cup \{L' \setminus L'',\ L''\}$$

is a maximum H-packing of ${}^3K_{15}^{(3)}$ with a leave, L'', consisting of one edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find

- a maximum H-packing of (3-fold) $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and
- H-decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum H-packing results. Also, we note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$, Thus, we need only additionally find H-decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{5,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which exist by Examples 17, 13, 15, 9, 11, 10, and 6, respectively. □

Theorem 9. *Let $v \geq 9$ be an integer. There exists an H-decomposition of 4-fold $K_v^{(3)}$.*

Proof. If $v \equiv 0$, 1, or 2 $\pmod 4$, then the result follows from 2 copies of an H-decomposition of ${}^2K_v^{(3)}$, which exists by Theorem 6. Hence, we need only consider when $v \equiv 3 \pmod 4$. Let $v_1, v_2, \ldots, v_9 \in V\left(K_v^{(3)}\right)$.

First, we consider when $v = 11$. For $i \in \{1,2,3,4\}$, let Δ_i be a maximum H-packing of $K_{11}^{(3)}$ with leave L_i consisting of a single edge, which exists by Example 17, Without loss of generality, we may assume that

$$E(L_1) = \big\{\{v_1, v_2, v_3\}\big\}, \qquad E(L_2) = \big\{\{v_1, v_4, v_5\}\big\},$$
$$E(L_3) = \big\{\{v_2, v_6, v_7\}\big\}, \qquad E(L_4) = \big\{\{v_3, v_8, v_9\}\big\}.$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2) \cup E(L_3) \cup E(L_4)$. Hence, L' is isomorphic to H, and the (multi-)set

$$L' \cup \bigcup_{i=1}^{4} \big(\Delta_i \setminus \{L_i\}\big)$$

is a collection of H-blocks such that each edge of $K_{11}^{(3)}$ is represented exactly four times. Therefore, we have an H-decomposition of ${}^4K_{11}^{(3)}$.

Second, we consider when $v = 15$. Let Δ_1 be a maximum H-packing of $K_{15}^{(3)}$ with leave L_1 consisting of a three vertex-disjoint edges, which exists by Example 19, and let Δ_2 be a maximum H-packing of ${}^3K_{15}^{(3)}$ with leave L_2 consisting of a single edge, which exists by Theorem 8, Without loss of generality, we may assume that

$$E(L_1) = \big\{\{v_1, v_4, v_5\},\ \{v_2, v_6, v_7\},\ \{v_3, v_8, v_9\}\big\},$$
$$E(L_2) = \big\{\{v_1, v_2, v_3\}\big\}.$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, L' is isomorphic to H, and the (multi-)set

$$\big(\Delta_1 \setminus \{L_1\}\big) \cup \big(\Delta_2 \setminus \{L_2\}\big) \cup \{L'\}$$

is a collection of H-blocks such that each edge of $K_{15}^{(3)}$ is represented exactly four times. Therefore, we have an H-decomposition of ${}^4K_{15}^{(3)}$.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 7\}$. By Lemma 2 it suffices to find H-decompositions of (4-fold) $K^{(3)}_{8+r}$, $K^{(3)}_{8+r} \setminus K^{(3)}_r$, $K^{(3)}_{r,8,8} \cup L^{(3)}_{8,8}$, and $K^{(3)}_{8,8,8}$. We note that $K^{(3)}_{7,8,8}$ is decomposable into copies of $K^{(3)}_{3,8,8}$ and $K^{(3)}_{4,8,8}$. Also, $K^{(3)}_{4,8,8}$ decomposes $K^{(3)}_{8,8,8}$, and we already have that H decomposes ${}^4K^{(3)}_{11}$ and ${}^4K^{(3)}_{15}$. Thus, we need only additionally find H-decompositions of $K^{(3)}_{11} \setminus K^{(3)}_3$, $K^{(3)}_{15} \setminus K^{(3)}_7$, $K^{(3)}_{3,8,8}$, $K^{(3)}_{4,8,8}$, and $L^{(3)}_{8,8}$, which exist by Examples 17, 15, 9, 10, and 6, respectively. □

Finally, we show that the necessary conditions for the existence of an H-decomposition of λ-fold $K^{(3)}_v$ are sufficient.

Theorem 10. *Let λ and v be positive integers with $v \geq 9$. There exists an H-decomposition of λ-fold K_v if and only if the following hold:*

- *if* $\gcd(\lambda, 4) = 1$, *then* $v \equiv 0, 1, 2, 4,$ *or* $6 \pmod 8$*;*
- *if* $\gcd(\lambda, 4) = 2$, *then* $v \equiv 0, 1,$ *or* $2 \pmod 4$*;*
- *if* $\gcd(\lambda, 4) = 4$, *then* $v \geq 9$.

Proof. The necessary conditions are established in Lemma 5. For sufficiency, we consider the following cases.

Case 1. $\lambda \equiv 0 \pmod 4$
Let $\lambda = 4t$ for some positive integer t. Then the result follows from t copies of an H-decomposition of ${}^4K^{(3)}_v$, which exists by Theorem 9.

Case 2. $\lambda \equiv 1$ or $3 \pmod 4$
Since $\gcd(\lambda, 4) = 1$, we have that $v \equiv 0, 1, 2, 4,$ or $6 \pmod 8$. Let $\lambda = 4t + r$ for some integers $t \geq 0$ and $r \in \{1, 3\}$. Then the result follows from t copies of an H-decomposition of ${}^4K^{(3)}_v$, which exists by Theorem 9, and r copies of an H-decomposition of $K^{(3)}_v$, which exists by Theorem 3.

Case 3. $\lambda \equiv 2 \pmod 4$
Since $\gcd(\lambda, 4) = 2$, we have that $v \equiv 0, 1,$ or $2 \pmod 4$. Let $\lambda = 4t + 2$ for some nonnegative integer t. Then the result follows from t copies of an H-decomposition of ${}^4K^{(3)}_v$, which exists by Theorem 9, and 1 copy of an H-decomposition of ${}^2K^{(3)}_v$, which exists by Theorem 6. □

Theorem 11. *If $v \geq 9$ is an integer, then there exists a maximum H-packing of λ-fold $K^{(3)}_v$ where the leave has fewer than four edges.*

Proof. If $1 \leq \lambda \leq 3$, then the result follows from Theorems 4, 7, and 8. If $\lambda = 4$, then the result follows from the H-decomposition result in Theorem 9, which translates to a maximum H-packing with an empty leave. For the remainder of the proof, we assume that $\lambda \geq 5$. Let $\lambda = 4t + r$ for some integers $t \geq 1$ and $r \in \{1, 4\}$. Then the result follows from t copies of an H-decomposition of ${}^4K^{(3)}_v$, which exists by Theorem 9, and 1 copy of a maximum H-packing of r-fold $K^{(3)}_v$. □

Acknowledgments

This research is supported by grant number A1659815 from the Division of Mathematical Sciences at the National Science Foundation. This work was done while the first, third, fifth, and sixth authors were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

References

[1] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[2] R.F. Bailey and B. Stevens, Hamiltonian decompositions of complete k-uniform hypergraphs, *Discrete Math.* **310** (2010), 3088–3095.

[3] Zs. Baranyai, On the factorization of the complete uniform hypergraph, in: Infinite and finite sets, *Colloq. Math. Soc. János Bolyai* **10**, North-Holland, Amsterdam, 1975, 91–108.

[4] J.C. Bermond, A. Germa, and D. Sotteau, Hypergraph-designs, *Ars Combinatoria* **3** (1977), 47–66.

[5] D. Bryant, S. Herke, B. Maenhaut, and W. Wannasit, Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs, *Australas. J. Combin.* **60** (2014), 227–254.

[6] D.E. Bryant and T.A. McCourt, Existence results for G-designs, `http://wiki.smp.uq.edu.au/G-designs/`

[7] R.C. Bunge, S.I. El-Zanati, L. Haman, C. Hatzer, K. Koe, and K. Spornberger, On loose 4-cycle decompositions of complete 3-uniform hypergraphs, submitted.

[8] C.J. Colbourn and J.H. Dinitz (Editors), *Handbook of Combinatorial Designs*, 2nd ed., Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

[9] C.J. Colbourn and R.Mathon, "Steiner systems," in [8], pp. 102–110.

[10] S. Glock, D. Kühn, A. Lo, and D. Osthus, The existence of designs via iterative absorption, arXiv:1611.06827v2, (2017), 63 pages.

[11] S. Glock, D. Kühn, A. Lo, and D. Osthus, Hypergraph F-designs for arbitrary F, arXiv:1706.01800, (2017), 72 pages.

[12] H. Hanani, On quadruple systems, *Canad. J. Math.*, **12** (1960), 145–157.

[13] H. Hanani, Decomposition of hypergraphs into octahedra, Second International Conference on Combinatorial Mathematics (New York, 1978), pp. 260–264, *Ann. New York Acad. Sci.*, 319, New York Acad. Sci., New York, 1979.

[14] H. Jordon, and G. Newkirk, 4-cycle decompositions of complete 3-uniform hypergraphs, *Australas. J. Combin.* **71** (2018), 312–323.

[15] P. Keevash, The existence of designs, arXiv:1401.3665v2, (2018), 39 pages.

[16] G.B. Khosrovshahi, and R. Laue, "t-designs with $t \geq 3$," in [8], pp. 79–101.

[17] J. Kuhl and M.W. Schroeder, Hamilton cycle decompositions of k-uniform k-partite hypergraphs, *Australas. J. Combin.* **56** (2013), 23–37.

[18] D. Kühn, and D. Osthus, Decompositions of complete uniform hypergraphs into Hamilton Berge cycles, *J. Combin. Theory Ser. A* **126** (2014), 128–135.

[19] Z. Lonc, Solution of a delta-system decomposition problem, *J. Combin. Theory, Ser. A* **55** (1990), 33–48.

[20] Z. Lonc, Packing, covering and decomposing of a complete uniform hypergraph into delta-systems, *Graphs Combin.* **8** (1992), 333–341.

[21] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australas. J. Combin.* **45** (2009), 291–302.

[22] A.F. Mouyart and F. Sterboul, Decomposition of the complete hypergraph into delta-systems II, *J. Combin. Theory, Ser. A* **41** (1986), 139–149.

[23] M.W. Schroeder, On Hamilton cycle decompositions of r-uniform r-partite hypergraphs, *Discrete Math.* **315** (2014), 1–8.

[24] R.M. Wilson, Decompositions of Complete Graphs into Subgraphs Isomorphic to a Given Graph, in "Proc. Fifth British Combinatorial Conference" (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), pp. 647–659, *Congr. Numer.* XV, 1975.

Appendix: Some Small Examples

We give several examples of H-decompositions and H-packings that are used in proving our main result.

Decomposition Examples

Example 1. Let $V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$B = \big\{H[0,1,4,5,6,\infty_1,3,\infty_2,2],\ H[\infty_1,\infty_2,0,3,6,1,2,4,5],\ H[0,2,5,\infty_2,4,\infty_1,1,6,3]\big\}.$$

Then an H-decomposition of $K_9^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1 \pmod 7$.

Example 2. Let $V\left(K_{10}^{(3)}\right) = \mathbb{Z}_{10}$ and let

$$B = \{H[0,2,4,8,9,3,6,5,1],\ H[0,2,7,1,6,5,8,9,3],\ H[0,1,5,7,9,2,4,8,3]\}.$$

Then an H-decomposition of $K_{10}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $j \mapsto j+1 \pmod{10}$.

Example 3. Let $V\left(K_{12}^{(3)}\right) = \mathbb{Z}_{11} \cup \{\infty\}$ and let

$$B = \{H[0,1,3,8,10,2,5,6,7],\ H[0,1,5,\infty,6,2,8,10,3],\ H[0,6,9,2,5,10,3,\infty,8],\ H[\infty,0,3,8,10,2,4,6,9],\ H[0,1,2,\infty,7,5,10,3,8]\}.$$

Then an H-decomposition of $K_{12}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{11}$.

Example 4. Let $V\left(K_{14}^{(3)}\right) = \mathbb{Z}_{13} \cup \{\infty\}$ and let

$$B = \{H[0,1,3,10,12,2,5,6,7],\ H[0,1,5,7,12,2,10,6,11],\ H[\infty,4,6,0,1,2,3,5,12],\ H[\infty,4,8,0,3,7,12,11,1],\ H[\infty,6,11,12,5,8,10,2,7],\ H[0,2,7,6,10,4,11,12,1],\ H[0,2,5,8,11,6,12,3,9]\}.$$

Then an H-decomposition of $K_{14}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{13}$.

Example 5. Let $V\left(K_{16}^{(3)}\right) = \mathbb{Z}_{15} \cup \{\infty\}$ and let

$$B_1 = \{H[0,1,3,5,6,2,14,4,9],\ H[0,2,5,3,11,4,14,8,12],\ H[0,1,4,\infty,7,2,13,8,12],\ H[0,2,6,3,9,4,13,\infty,11],\ H[0,2,8,7,14,4,11,\infty,10],\ H[0,1,7,4,9,2,10,\infty,13],\ H[0,1,5,3,6,2,12,\infty,8],\ H[0,2,7,\infty,1,4,12,9,11],\ H[0,3,8,4,10,6,13,\infty,12]\},$$

$$B_2 = \{H[0,5,10,1,2,6,7,11,12],\ H[1,6,11,2,3,7,8,12,13],\ H[2,7,12,3,4,8,9,13,14],\ H[3,8,13,4,5,9,10,14,0],\ H[4,9,14,5,6,10,11,0,1]\}.$$

Then an H-decomposition of $K_{16}^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{15}$ along with the H-blocks in B_2.

Example 6. Let $V\left(L_{8,8}^{(3)}\right) = \mathbb{Z}_{16}$ with vertex partition $\{\{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\}$ and let

$$B = \{H[0,1,2,7,9,4,14,8,13],\ H[0,1,3,5,14,2,15,8,10],\ H[0,4,5,7,15,8,3,12,13],\ H[0,6,13,3,7,12,15,1,4],\ H[0,1,7,9,15,3,14,2,4],\ H[0,1,6,10,15,8,12,13,2],\ H[0,1,4,3,11,2,14,12,15]\}.$$

Then an H-decomposition of $L_{8,8}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $j \mapsto j+1 \pmod{16}$.

Example 7. Let $V\left(L^{(3)}_{8,8} \cup K^{(3)}_{1,8,8}\right) = \mathbb{Z}_{16} \cup \{\infty\}$ with vertex partition $\big\{\{\infty\}, \{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$\begin{aligned} B = \big\{ & H[0,1,3,5,14,2,15,8,10],\ H[0,4,5,7,15,8,3,12,13],\ H[0,6,13,3,7,12,15,1,4], \\ & H[0,1,7,9,15,3,14,2,4],\ H[0,1,6,10,15,8,12,13,2],\ H[0,1,4,3,11,2,14,12,15], \\ & H[\infty,0,9,10,13,7,14,15,4],\ H[\infty,0,11,3,4,1,2,5,8]\big\}. \end{aligned}$$

Then an H-decomposition of $L^{(3)}_{8,8} \cup K^{(3)}_{1,8,8}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{16}$.

Example 8. Let $V\left(L^{(3)}_{8,8} \cup K^{(3)}_{2,8,8}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2\}$ with vertex partition $\big\{\{\infty_1,\infty_2\}, \{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$\begin{aligned} B = \big\{ & H[0,1,3,5,14,2,15,8,10],\ H[0,4,5,7,15,8,3,12,13],\ H[\infty_1,0,15,3,10,1,4,11,14], \\ & H[0,1,7,9,15,3,14,2,4],\ H[0,1,6,10,15,8,12,13,2],\ H[\infty_1,0,3,9,14,5,13,8,11], \\ & H[0,6,13,3,7,12,15,1,4],\ H[\infty_2,0,9,5,6,14,15,11,2],\ H[\infty_2,0,13,5,10,3,6,2,7]\big\}. \end{aligned}$$

Then an H-decomposition of $L^{(3)}_{8,8} \cup K^{(3)}_{2,8,8}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1 \pmod{16}$.

Example 9. Let $V\left(K^{(3)}_{3,8,8}\right) = \mathbb{Z}_{16} \cup \{\infty_1,\infty_2,\infty_3\}$ with vertex partition $\big\{\{\infty_1,\infty_2,\infty_3\}, \{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$\begin{aligned} B = \big\{ & H[\infty_1,0,1,2,5,\infty_2,7,\infty_3,6],\ H[\infty_2,0,1,2,5,\infty_3,7,\infty_1,6], \\ & H[\infty_3,0,1,2,5,\infty_1,7,\infty_2,6]\big\}. \end{aligned}$$

Then an H-decomposition of $K^{(3)}_{3,8,8}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2,3\}$, and $j \mapsto j+1 \pmod{16}$.

Example 10. Let $V\left(K^{(3)}_{4,8,8}\right) = \mathbb{Z}_{16} \cup \{\infty_1,\infty_2,\infty_3,\infty_3,\infty_4\}$ with vertex partition $\big\{\{\infty_1,\infty_2,\infty_3,\infty_4\}, \{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$\begin{aligned} B = \big\{ & H[\infty_1,0,1,2,5,\infty_2,7,\infty_3,6],\ H[\infty_2,0,1,2,5,\infty_3,7,\infty_4,6], \\ & H[\infty_3,0,1,2,5,\infty_4,7,\infty_1,6],\ H[\infty_4,0,1,2,5,\infty_1,7,\infty_2,6]\big\}. \end{aligned}$$

Then an H-decomposition of $K^{(3)}_{4,8,8}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,\dots,4\}$, and $j \mapsto j+1 \pmod{16}$.

Example 11. Let $V\left(K^{(3)}_{5,8,8}\right) = \mathbb{Z}_{16} \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$ with vertex partition $\big\{\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}, \{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$\begin{aligned} B = \big\{ & H[\infty_1,0,1,2,5,\infty_2,7,\infty_3,6],\ H[\infty_2,0,1,2,5,\infty_3,7,\infty_4,6], \\ & H[\infty_3,0,1,2,5,\infty_4,7,\infty_5,6],\ H[\infty_4,0,1,2,5,\infty_5,7,\infty_1,6], \\ & H[\infty_5,0,1,2,5,\infty_1,7,\infty_2,6]\big\}. \end{aligned}$$

Then an H-decomposition of $K_{5,8,8}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \dots, 5\}$, and $j \mapsto j+1 \pmod{16}$.

Example 12. Let $V\left(K_{12}^{(3)} \backslash K_4^{(3)}\right) = \mathbb{Z}_{12}$ with $0, 3, 6, 9$ being the vertices in the hole and let

$$B_1 = \{H[0,3,7,2,5,6,11,9,4],\ H[0,2,6,1,11,4,10,8,9],\ H[0,1,6,7,11,2,8,10,5],\ H[0,1,4,8,11,3,5,9,2]\},$$

$$B_2 = \{H[7,8,10,1,4,2,5,0,9],\ H[1,2,4,7,10,8,11,3,6],\ H[8,9,11,0,4,6,7,2,5],\ H[1,10,11,5,9,4,7,0,2],\ H[2,3,5,6,10,0,1,8,11],\ H[4,5,7,1,10,6,8,11,3]\}.$$

Then an H-decomposition of $K_{12}^{(3)} \backslash K_4^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $j \mapsto j+1 \pmod{12}$ along with the H-blocks in B_2.

Example 13. Let $V\left(K_{13}^{(3)} \setminus K_5^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5$ being the vertices in the hole and let

$$B_1 = \{H[\infty_3,\infty_5,0,1,3,\infty_2,7,\infty_1,\infty_4],\ H[\infty_4,\infty_5,0,\infty_3,6,\infty_1,7,2,3],\ H[\infty_2,\infty_4,0,5,7,1,4,\infty_5,3],\ H[\infty_4,0,2,4,5,\infty_1,7,\infty_5,3]\},$$

$$\begin{aligned}B_2 = \{&H[\infty_1,\infty_3,0,3,5,1,2,4,6],\ H[\infty_1,\infty_3,1,4,6,2,3,5,7],\\ &H[\infty_1,\infty_3,2,5,7,3,4,6,0],\ H[\infty_1,\infty_3,3,6,0,4,5,7,1],\\ &H[\infty_1,\infty_3,4,7,1,5,6,0,2],\ H[\infty_1,\infty_3,5,0,2,6,7,1,3],\\ &H[\infty_1,\infty_3,6,1,3,7,0,2,4],\ H[2,4,7,\infty_2,\infty_3,\infty_1,1,5,6],\\ &H[3,5,0,\infty_2,\infty_3,\infty_1,2,6,7],\ H[4,6,1,\infty_2,\infty_3,\infty_1,3,7,0],\\ &H[5,7,2,\infty_2,\infty_3,\infty_1,4,0,1],\ H[6,0,3,\infty_2,\infty_3,\infty_1,5,1,2],\\ &H[7,1,4,\infty_2,\infty_3,\infty_1,6,2,3],\ H[0,2,5,\infty_2,\infty_3,\infty_1,7,3,4],\\ &H[2,3,7,\infty_3,5,\infty_1,\infty_2,1,6],\ H[3,4,0,\infty_3,6,\infty_1,\infty_2,2,7],\\ &H[4,5,1,\infty_3,7,\infty_1,\infty_2,3,0],\ H[5,6,2,\infty_3,0,\infty_1,\infty_2,4,1],\\ &H[6,7,3,\infty_3,1,\infty_1,\infty_2,5,2],\ H[7,0,4,\infty_3,2,\infty_1,\infty_2,6,3],\\ &H[0,1,5,\infty_3,3,\infty_1,\infty_2,7,4],\ H[\infty_2,\infty_3,1,0,4,\infty_1,7,2,6],\\ &H[\infty_2,1,5,\infty_1,2,\infty_3,0,6,4],\ H[1,3,6,\infty_3,4,5,7,\infty_2,2],\\ &H[\infty_1,0,3,2,4,5,6,\infty_2,7],\ H[\infty_2,0,1,3,6,\infty_1,4,\infty_3,5],\\ &H[\infty_2,1,2,4,7,\infty_1,5,\infty_3,6],\ H[\infty_2,2,3,5,0,\infty_1,6,\infty_3,7],\\ &H[\infty_2,3,4,6,1,\infty_1,7,\infty_3,0],\ H[\infty_2,4,5,7,2,\infty_4,0,\infty_5,1],\\ &H[\infty_2,5,6,0,3,\infty_4,1,\infty_5,2],\ H[\infty_2,6,7,1,4,\infty_4,2,\infty_5,3],\\ &H[\infty_2,7,0,2,5,\infty_4,3,\infty_5,4],\ H[\infty_5,0,2,1,3,4,5,6,7],\\ &H[\infty_5,3,5,2,4,7,0,6,1],\ H[\infty_5,4,6,5,7,0,1,2,3],\ H[\infty_5,1,7,0,6,2,5,3,4]\}.\end{aligned}$$

Then an H-decomposition of $K_{13}^{(3)} \setminus K_5^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \dots, 5\}$, and $j \mapsto j+1 \pmod 8$ along with the H-blocks in B_2.

Example 14. Let $V\left(K_{14}^{(3)} \backslash K_6^{(3)}\right) = \mathbb{Z}_{12} \cup \{\infty_1, \infty_2\}$ with $0, 3, 6, 9, \infty_1, \infty_2$ being the vertices in the hole and let

$$\begin{aligned} B_1 = \big\{ & H[0,1,5,7,11,2,10,6,9],\ H[\infty_1,0,1,6,8,10,11,\infty_2,2],\ H[\infty_2,0,4,1,3,9,11,\infty_1,8], \\ & H[0,1,6,\infty_2,7,2,8,\infty_1,11],\ H[0,2,5,7,10,4,8,9,11],\ H[0,2,4,3,8,5,9,6,11] \big\}, \\ B_2 = \big\{ & H[\infty_1,2,8,5,11,1,4,7,10],\ H[\infty_2,5,11,2,8,4,7,1,10],\ H[0,1,3,4,8,\infty_1,7,2,5], \\ & H[3,4,6,7,11,\infty_1,10,5,8],\ H[6,7,9,2,10,\infty_2,1,8,11],\ H[0,9,10,2,11,1,5,\infty_2,4], \\ & H[\infty_1,\infty_2,1,2,5,8,11,4,7],\ H[\infty_1,\infty_2,2,1,4,7,10,5,8], \\ & H[\infty_1,\infty_2,4,5,8,2,11,7,10],\ H[\infty_1,\infty_2,5,4,7,1,10,8,11], \\ & H[\infty_1,\infty_2,7,8,11,2,5,1,10],\ H[\infty_1,\infty_2,8,7,10,1,4,2,11], \\ & H[\infty_1,\infty_2,10,2,11,5,8,1,4],\ H[\infty_1,\infty_2,11,1,10,4,7,2,5] \big\}. \end{aligned}$$

Then an H-decomposition of $K_{14}^{(3)} \backslash K_6^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1 \pmod{12}$ along with the H-blocks in B_2.

Example 15. Let $V\left(K_{15}^{(3)} \setminus K_7^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7$ being the vertices in the hole and let

$$\begin{aligned} B_1 = \big\{ & H[\infty_1,\infty_4,0,\infty_5,5,3,4,\infty_6,\infty_7],\ H[\infty_2,\infty_4,0,\infty_5,5,2,4,\infty_7,3], \\ & H[\infty_3,\infty_4,0,\infty_5,5,\infty_7,4,\infty_6,3],\ H[\infty_4,\infty_5,0,4,7,\infty_6,1,\infty_3,2], \\ & H[\infty_3,\infty_6,0,\infty_7,6,3,4,\infty_1,1],\ H[\infty_5,\infty_7,0,5,6,\infty_1,4,\infty_6,2], \\ & H[\infty_2,\infty_6,0,\infty_7,6,\infty_4,2,\infty_5,3],\ H[\infty_7,3,5,0,1,\infty_1,\infty_6,\infty_2,7] \big\}, \\ B_2 = \big\{ & H[\infty_1,\infty_3,0,3,5,1,2,4,6],\ H[\infty_1,\infty_3,1,4,6,2,3,5,7], \\ & H[\infty_1,\infty_3,2,5,7,3,4,6,0],\ H[\infty_1,\infty_3,3,6,0,4,5,7,1], \\ & H[\infty_1,\infty_3,4,7,1,5,6,0,2],\ H[\infty_1,\infty_3,5,0,2,6,7,1,3], \\ & H[\infty_1,\infty_3,6,1,3,7,0,2,4],\ H[2,4,7,\infty_2,\infty_3,\infty_1,1,5,6], \\ & H[3,5,0,\infty_2,\infty_3,\infty_1,2,6,7],\ H[4,6,1,\infty_2,\infty_3,\infty_1,3,7,0], \\ & H[5,7,2,\infty_2,\infty_3,\infty_1,4,0,1],\ H[6,0,3,\infty_2,\infty_3,\infty_1,5,1,2], \\ & H[7,1,4,\infty_2,\infty_3,\infty_1,6,2,3],\ H[0,2,5,\infty_2,\infty_3,\infty_1,7,3,4], \\ & H[2,3,7,\infty_3,5,\infty_1,\infty_2,1,6],\ H[3,4,0,\infty_3,6,\infty_1,\infty_2,2,7], \\ & H[4,5,1,\infty_3,7,\infty_1,\infty_2,3,0],\ H[5,6,2,\infty_3,0,\infty_1,\infty_2,4,1], \\ & H[6,7,3,\infty_3,1,\infty_1,\infty_2,5,2],\ H[7,0,4,\infty_3,2,\infty_1,\infty_2,6,3], \\ & H[0,1,5,\infty_3,3,\infty_1,\infty_2,7,4],\ H[\infty_2,\infty_3,1,0,4,\infty_1,7,2,6], \\ & H[\infty_2,1,5,\infty_1,2,\infty_3,0,6,4],\ H[1,3,6,\infty_3,4,5,7,\infty_2,2], \\ & H[\infty_1,0,3,2,4,5,6,\infty_2,7],\ H[\infty_2,0,1,3,6,\infty_1,4,\infty_3,5], \\ & H[\infty_2,1,2,4,7,\infty_1,5,\infty_3,6],\ H[\infty_2,2,3,5,0,\infty_1,6,\infty_3,7], \end{aligned}$$

$H[\infty_2,3,4,6,1,\infty_1,7,\infty_3,0],\ H[\infty_2,4,5,7,2,\infty_4,0,\infty_5,1],$
$H[\infty_2,5,6,0,3,\infty_4,1,\infty_5,2],\ H[\infty_2,6,7,1,4,\infty_4,2,\infty_5,3],$
$H[\infty_2,7,0,2,5,\infty_4,3,\infty_5,4],\ H[\infty_6,0,4,2,6,5,7,1,3],\ H[\infty_6,1,5,3,7,6,0,2,4],$
$H[\infty_7,2,6,0,4,7,1,3,5],\ H[\infty_7,3,7,1,5,0,2,4,6],\ H[\infty_5,0,2,1,3,4,5,6,7],$
$H[\infty_5,3,5,2,4,7,0,6,1],\ H[\infty_5,4,6,5,7,0,1,2,3],\ H[\infty_5,1,7,0,6,2,5,3,4]\big\}.$

Then an H-decomposition of $K_{15}^{(3)} \setminus K_7^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,\dots,7\}$, and $j \mapsto j+1 \pmod 8$ along with the H-blocks in B_2.

Example 16. Let $V\left(K_{16}^{(3)}\backslash K_8^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6,\infty_7,\infty_8\}$ with $\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6,\infty_7,\infty_8$ being the vertices in the hole and let

$$\begin{aligned}B_1 = \big\{&H[\infty_1,\infty_2,0,\infty_3,1,\infty_5,2,\infty_4,\infty_8],\ H[\infty_2,\infty_3,0,\infty_4,1,\infty_6,2,\infty_1,\infty_5],\\ &H[\infty_3,\infty_4,0,\infty_5,1,\infty_7,2,\infty_1,\infty_8],\ H[\infty_4,\infty_5,0,\infty_6,1,\infty_8,2,\infty_3,\infty_7],\\ &H[\infty_5,\infty_6,0,\infty_7,1,\infty_1,2,\infty_2,\infty_8],\ H[\infty_6,\infty_7,0,\infty_8,1,\infty_2,2,\infty_1,\infty_4],\\ &H[\infty_7,\infty_8,0,\infty_1,1,\infty_3,2,\infty_2,\infty_6],\ H[\infty_5,0,7,2,5,\infty_6,3,\infty_8,4],\\ &H[\infty_6,0,1,2,4,\infty_1,6,\infty_7,3],\ H[\infty_7,0,1,3,6,\infty_3,5,\infty_4,4],\\ &H[\infty_8,0,1,2,4,\infty_4,6,\infty_3,3],\ H[0,1,4,\infty_1,3,2,7,\infty_2,6],\ H[0,2,4,\infty_2,5,3,7,6,1]\big\},\end{aligned}$$

$$\begin{aligned}B_2 = \big\{&H[0,1,2,\infty_1,4,\infty_5,3,\infty_2,6],\ H[1,2,3,\infty_1,5,\infty_5,4,\infty_2,7],\\ &H[2,3,4,\infty_1,6,\infty_5,5,\infty_2,0],\ H[3,4,5,\infty_1,7,\infty_5,6,\infty_2,1],\\ &H[4,5,6,\infty_3,0,\infty_5,7,\infty_4,2],\ H[5,6,7,\infty_3,1,\infty_5,0,\infty_4,3],\\ &H[6,7,0,\infty_3,2,\infty_5,1,\infty_4,4],\ H[7,0,1,\infty_3,3,\infty_5,2,\infty_4,5],\\ &H[0,1,3,\infty_5,4,\infty_6,5,\infty_1,2],\ H[1,2,4,\infty_5,5,\infty_6,6,\infty_1,3],\\ &H[2,3,5,\infty_5,6,\infty_6,7,\infty_1,4],\ H[3,4,6,\infty_5,7,\infty_6,0,\infty_1,5],\\ &H[4,5,7,\infty_7,0,\infty_8,1,\infty_1,6],\ H[5,6,0,\infty_7,1,\infty_8,2,\infty_1,7],\\ &H[6,7,1,\infty_7,2,\infty_8,3,\infty_1,0],\ H[7,0,2,\infty_7,3,\infty_8,4,\infty_1,1],\\ &H[\infty_2,0,1,5,6,\infty_3,7,\infty_4,2],\ H[\infty_3,0,1,5,6,\infty_4,7,\infty_2,2],\\ &H[\infty_4,0,1,5,6,\infty_2,7,\infty_3,2],\ H[\infty_2,3,4,6,7,\infty_3,2,\infty_4,5],\\ &H[\infty_3,3,4,6,7,\infty_4,2,\infty_2,5],\ H[\infty_4,3,4,6,7,\infty_2,2,\infty_3,5]\big\}.\end{aligned}$$

Then an H-decomposition of $K_{16}^{(3)}\backslash K_8^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,\dots,8\}$, and $j \mapsto j+1 \pmod 8$ along with the H-blocks in B_2.

Maximum Packing Examples

Example 17. Let $V\left(K_{11}^{(3)}\right) = \mathbb{Z}_{10} \cup \{\infty\}$ and let

$$B_1 = \big\{H[0,2,7,1,4,\infty,9,3,6],\ H[0,3,6,1,5,\infty,9,7,2],\ H[0,2,5,1,3,\infty,4,7,8]\big\},$$

$$B_2 = \{H[\infty,0,1,8,9,2,6,5,7],\ H[\infty,1,2,0,9,3,7,6,8],\ H[\infty,2,3,5,6,4,8,7,9],\ H[\infty,3,4,6,7,5,9,8,0],\ H[\infty,4,5,7,8,6,0,9,1],\ H[3,5,7,2,4,\infty,0,8,9],\ H[4,6,8,3,5,\infty,1,0,2],\ H[5,7,9,4,6,\infty,2,0,1],\ H[0,6,8,2,4,5,7,\infty,3],\ H[1,7,9,2,3,6,8,\infty,4],\ H[0,1,2,8,9,3,5,4,6]\}.$$

Then a maximum H-packing of $K_{11}^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{10}$ along with the H-blocks in B_2 and a leave consisting of the edge $\{1,3,9\}$.

Example 18. Let $V\left(K_{13}^{(3)}\right) = \mathbb{Z}_{13}$ and let

$$B_1 = \{H[0,3,7,6,10,5,11,9,1],\ H[0,2,11,1,7,5,12,3,8],\ H[0,3,5,8,10,7,1,9,11],\ H[0,1,5,8,12,2,7,10,11],\ H[0,1,3,10,12,2,5,6,7]\},$$

$$B_2 = \{H[0,4,8,1,12,5,6,9,10],\ H[1,5,9,2,3,6,7,10,11],\ H[2,6,10,3,4,7,8,11,12],\ H[3,4,5,7,11,8,12,10,1],\ H[7,8,9,11,2,12,3,0,4],\ H[11,12,0,2,6,3,7,5,9]\}.$$

Then a maximum H-packing of $K_{13}^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $j \mapsto j+1 \pmod{13}$ along with the H-blocks in B_2 and a leave consisting of the edges $\{0,1,2\}$ and $\{1,6,10\}$, which share a single vertex. Additionally, let

$$B_2' = \left(B_2 \setminus \{H[2,6,10,3,4,7,8,11,12]\}\right) \cup \{H[2,6,10,0,1,7,8,11,12]\}.$$

Then a maximum H-packing of $K_{13}^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $j \mapsto j+1 \pmod{13}$ along with the H-blocks in B_2' and a leave consisting of the edges $\{1,6,10\}$ and $\{2,3,4\}$, which are vertex-disjoint.

Example 19. Let $V\left(K_{15}^{(3)}\right) = \mathbb{Z}_{15}$ and let

$$B_1 = \{H[0,4,9,6,11,7,14,12,2],\ H[0,4,8,3,6,7,13,10,12],\ H[0,1,3,12,14,2,5,6,7],\ H[0,1,6,9,14,2,12,7,11],\ H[0,2,8,7,13,4,12,1,3],\ H[0,3,7,8,12,5,14,9,13],\ H[0,2,12,7,8,10,1,3,11]\},$$

$$B_2 = \{H[0,5,10,1,2,6,7,11,12],\ H[1,6,11,2,3,7,8,12,13],\ H[2,7,12,3,4,8,9,13,14],\ H[3,8,13,4,5,9,10,14,0],\ H[4,9,14,5,6,10,11,0,1],\ H[0,2,5,13,3,12,14,7,10],\ H[4,6,9,14,1,8,11,12,7],\ H[8,10,13,3,5,12,0,11,1]\}.$$

Then a maximum H-packing of $K_{15}^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $j \mapsto j+1 \pmod{15}$ along with the H-blocks in B_2 and a leave consisting of the edges $\{1,3,6\}$, $\{2,4,7\}$, and $\{9,11,14\}$, which are vertex-disjoint.