Electronic Journal of Graph Theory and Applications **8** (2) (2020), 391–400

Electronic Journal of Graph Theory and Applications

On a version of the spectral excess theorem

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Abstract

Given a regular (connected) graph $\Gamma = (X, E)$ with adjacency matrix A, $d+1$ distinct eigenvalues, and diameter D, we give a characterization of when its distance matrix A_D is a polynomial in A, in terms of the adjacency spectrum of Γ and the arithmetic (or harmonic) mean of the numbers of vertices at distance at most $D-1$ from every vertex. The same result is proved for any graph by using its Laplacian matrix L and corresponding spectrum. When $D = d$ we reobtain the spectral excess theorem characterizing distance-regular graphs.

Keywords: graph, adjacency algebra, spectrum, harmonic mean, distance-regular graph, Laplacian.
Mathematics Subject Classification: 05C50, 05E30
DOI: 10.5614/ejgta.2020.8.2.15

1. Preliminaries

Let $\Gamma = (X, E)$ be a (finite, simple, and connected) graph on $n = |X|$ vertices, with adjacency matrix $A \in \mathrm{Mat}_X(\mathbb{R})$, and spectrum $\mathrm{sp}\,\Gamma = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ are the distinct eigenvalues, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. If Γ has diameter D, we denote by $\Gamma_i(x)$ the set of vertices at distance $i = 1, \ldots, D$ from $x \in X$, and

Received: 6 August 2019, Revised: 6 July 2020, Accepted: 2 August 2020.

$k_i(x) = |\Gamma_i(x)|$. We abbreviate $k_1(x)$ by $k(x)$, the degree of vertex x. It is well known that $D \le d$, see for instance Biggs [1].

Given two symmetric square matrices $M, N \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, let $\mathrm{sum}(M)$ denote the sum of all entries of M, so that $\mathrm{tr}(MN) = \mathrm{sum}(M \circ N)$, where '$\circ$' stands for the Hadamard (or entrywise) product. The *predistance polynomials* $p_0, p_1, \dots, p_d$ of Γ, with $\deg(p_i) = i$ for $i = 0, \dots, d$, introduced by the first author and Garriga [11], are a sequence of orthogonal polynomials with respect to the scalar product

$$\langle f, g\rangle_A = \frac{1}{n}\mathrm{tr}(f(A)g(A)) = \frac{1}{n}\mathrm{sum}(f(A)\circ g(A)) = \frac{1}{n}\sum_{i=0}^{d} m(\lambda_i)f(\lambda_i)g(\lambda_i), \tag{1}$$

normalized in such a way that $\|p_i\|_A^2 = p_i(\lambda_0)$. For instance, since

$$\mathrm{tr}(A^h) = n\langle A^h, I\rangle_A = \sum_{i=0}^{d} m(\lambda_i)\lambda_i^h,$$

the two first predistance polynomials are $p_0(x) = 1$ and $p_1(x) = \frac{\lambda_0}{\overline{k}}x$, with $\overline{k} = \frac{1}{n}\sum_{x\in X} k(x)$ being the average degree of Γ, see also Lemma 1.1. (It is known that $\overline{k} \le \lambda_0$ with equality if and only if Γ is regular, see for instance Brouwer and Haemers [2].) Moreover, the value of the highest degree polynomial p_d at λ_0 can be computed from $\mathrm{sp}\,\Gamma$ as

$$p_d(\lambda_0) = n\left(\sum_{i=0}^{d}\frac{\phi_0^2}{m_i\phi_i^2}\right)^{-1}, \tag{2}$$

where $\phi_i = \prod_{j\ne i}(\lambda_i - \lambda_j)$, $i = 0, \dots, d$ (see [11]).

The *predistance matrices* $P_0, P_1, \dots, P_d$ are then defined by $P_i = p_i(A)$ for $i = 0, 1, \dots, d$. By [3, Prop. 2.2], there exist numbers α_i, β_i, and γ_i such that

$$PP_i (= P_iP) = \beta_{i-1}P_{i-1} + \alpha_i P_i + \gamma_{i+1}P_{i+1} \quad \text{for } i = 0, \dots, d,$$

where $P = P_1$, $P_{-1} = P_{d+1} = 0$, $\gamma_1 = 1$, and $\alpha_0 = 0$. Also, it can be shown that the polynomial $p_{d+1}(x) = (x - \alpha_d)p_d(x) - \beta_{d-1}p_{d-1}(x)$ is a scalar multiple of the minimal polynomial of A. Hence, $p_{d+1}(A) = 0$, and the distinct eigenvalues of Γ are precisely the zeros of p_{d+1}.

The above names come from the fact that, if Γ is a distance-regular graph, then the p_i's and P_i's correspond to the well-known distance polynomials and distance matrices A_i, respectively. In fact, a known characterization states that Γ is distance-regular if and only if such polynomials satisfy $p_i(A) = A_i$ for every $i = 1, \dots, D$; see, for instance, [7]. Moreover, in this case, $D = d$. If we do not impose that the degree of each polynomial coincide with its subindex, then it can be $D < d$ and the graph is called *distance-polynomial*, a concept introduced by Weichsel [13].

In fact, if $D = d$, the first author, Garriga, and Yebra [10] proved the following.

Proposition 1.1. *A regular graph Γ with diameter D and $d+1$ distinct eigenvalues is distance-regular if and only if $D = d$ and its highest degree predistance polynomial satisfies $p_d(A) = A_d$.*

From the predistance polynomials, we also consider their sums $q_i = p_0 + \cdots + p_i$ for $i = 0, \ldots, d$, which satisfy $1 = q_0(\lambda_0) < q_1(\lambda_0) < \cdots < q_d(\lambda_0) = |X|$, with $q_d = H$ being the Hoffman polynomial that characterizes the regularity of Γ by the equality $H(A) = J$, the all-1 matrix (see Hoffman [12]). Notice that $q_i(\lambda_0) = \|q_i(\lambda_0)\|_A^2$ for $i = 0, \ldots, d$.

We also recall that the Laplacian matrix of Γ is the matrix $L = K - A$, where $K = \operatorname{diag}(k(x_1), \ldots, k(x_n))$, where $x_i \in X$ for $i = 1, \ldots, n$. The Laplacian spectrum of Γ is $\operatorname{sp}_L \Gamma = \operatorname{sp} L = \{\theta_0^{m(\theta_0)}, \theta_1^{m(\theta_1)}, \ldots, \theta_d^{m(\theta_d)}\}$ with $\theta_0 = 0 < \theta_1 < \cdots < \theta_d$. In particular, since Γ is connected, $m_0 = 1$, and the eigenvalue 0 has eigenvector $\boldsymbol{j}$, the all-1 vector. As in the case of the adjacency spectrum, we can define the *Laplacian predistance polynomials* $r_0, r_1, \ldots, r_d$ as the sequence of orthogonal polynomials with respect to the scalar product

$$\langle f, g\rangle_L = \frac{1}{n}\operatorname{tr}(f(L)g(L)) = \frac{1}{n}\operatorname{sum}(f(L)\circ g(L)) = \frac{1}{n}\sum_{i=0}^{d} m(\theta_i) f(\theta_i) g(\theta_i),$$

normalized in such a way that $\|r_i\|_L^2 = r_i(0)$. The following result gives the first two Laplacian predistance polynomials.

Lemma 1.1. *Let Γ be a graph with Laplacian matrix $L = K - A$. Let $\overline{k^2}$ be the average of the square degrees of Γ. Then*

(i) $r_0(x) = 1$.

(ii) $r_1(x) = \frac{\overline{k}}{\overline{k}(\overline{k}-1)-\overline{k^2}}(x - \overline{k})$.

Proof. We only need to prove (ii). By using the method of Gram-Schmidt, we first find a polynomial $t(x)$ orthogonal to $r_0 = 1$. That is, $t(x) = x - \frac{\langle x, r_0\rangle_L}{\|r_0\|_L^2} r_0(x)$, where

$$\langle x, r_0\rangle_L = \frac{1}{n}\operatorname{tr}(L) = \frac{1}{n}\sum_{x\in X} k(x) = \overline{k} \qquad \text{and} \qquad \|r_0\|_L^2 = \frac{1}{n}\operatorname{tr}(I^2) = 1.$$

Now, $r_1(x) = \alpha t(x)$, where α is a constant to be determined by the normalization condition $\|r_1\|_L = r_1(0)$, which gives $\alpha = \frac{t(0)}{\|t\|_L^2}$. Moreover,

$$\begin{aligned}
\|t\|_L^2 &= \|x - \overline{k}\|_L^2 = \frac{1}{n}\operatorname{tr}([K - A - \overline{k}I]^2) = \frac{1}{n}\operatorname{tr}(K^2 + A^2 + (\overline{k})^2 I - 2KA - 2\overline{k}K + 2\overline{k}A)\\
&= \frac{1}{n}\sum_{x\in X} k(x)^2 + \frac{1}{n}\sum_{x\in X} k(x) + (\overline{k})^2 - 2\overline{k}\frac{1}{n}\sum_{x\in X} k(x) = \overline{k^2} + \overline{k} + (\overline{k})^2 - 2(\overline{k})^2\\
&= \overline{k^2} + \overline{k} - (\overline{k})^2.
\end{aligned}$$

Then, from $r_1(x) = \frac{t(0)}{\|t\|_L^2} t(x)$ we get the result. □

Also, as in the case of the predistance polynomials p_i's, we have

$$r_d(0) = n\left(\sum_{i=0}^{d} \frac{\psi_0^2}{m(\theta_i)\psi_i^2}\right)^{-1}, \tag{3}$$

with $\psi_i = \prod_{j\neq i}(\theta_i - \theta_j)$, $i = 0, \ldots, d$ (see [3]).

The analogue of Proposition 1.1, for not necessarily regular graphs, was proved by Van Dam and the first author in [5].

Proposition 1.2. *A graph Γ with Laplacian matrix L, $d+1$ distinct Laplacian eigenvalues, and diameter D is distance-regular if and only if $D = d$ and its highest degree Laplacian predistance polynomial satisfies $r_d(L) = A_d$.*

In fact, the regularity of Γ is already implied by the equation $r_1(L) = A$, as shown in the following lemma.

Lemma 1.2. *Let Γ be a graph with adjacency and Laplacian matrices A and L, respectively, and Laplacian predistance polynomial r_1. Then, Γ is k-regular if and only if $r_1(L) = A$.*

Proof. From the Cauchy-Schwartz inequality,

$$\overline{k^2} = \frac{1}{n}\sum_{x\in V} k(x)^2 \geq \left(\frac{1}{n}\sum_{x\in V} k(x)\right)^2 = (\overline{k})^2,$$

with equality if and only if Γ is k-regular. In this case, Lemma 1.1(ii) becomes $r_1(x) = k - x$ and, hence, $r_1(L) = kI - (kI - A) = A$. Conversely, if $r_1(L) = r_1(K - A) = A$, Lemma 1.1(ii) yields $A = \frac{\overline{k}}{\overline{k}(\overline{k}-1)-\overline{k^2}}(L - \overline{k}I)$ and, by equating coefficients, we get $\frac{\overline{k}}{\overline{k}(\overline{k}-1)-\overline{k^2}} = -1$, whence $\overline{k^2} = (\overline{k})^2$, Γ is k-regular, and $K = kI$. (Alternatively, note that $L - \overline{k}I$ can have zero diagonal if every entry on the diagonal of L equals $\overline{k}$.) □

In this context, we also consider the sum polynomials $s_i = r_0 + \cdots + r_i$ for $i = 0, \ldots, d$, with $H_L = s_d$ being a Hoffman-like polynomial satisfying $H(L) = J$ (independently of whether Γ is regular or not). For more details, see [5].

In our results we use the following simple result.

Lemma 1.3. *Let $\Gamma = (X, E)$ be a graph with adjacency matrix A and Laplacian matrix L. Given a vertex $x \in X$ and a polynomial $p \in \mathbb{R}_h[t]$,*

- (i) *If Γ is k-regular, then $\sum_{y\in X} p(A)_{xy} = p(k)$.*
- (ii) *For any Γ, we have $\sum_{y\in X} p(L)_{xy} = p(0)$.*

Proof. (i) Since Γ is k-regular, $(k, \boldsymbol{j})$ is an eigenpair of A and, hence, $p(A)\boldsymbol{j} = p(k)\boldsymbol{j}$. Then, the result follows by considering the x^{th} component of both vectors. Case (ii) is proved in the same way by considering that $(0, \boldsymbol{j})$ is an eigenpair of L. □

2. A version of the spectral excess theorem

The spectral excess theorem, due to Fiol and Garriga [11], states that a regular (connected) graph $\Gamma = (X, E)$ is distance-regular if and only if its spectral excess (a number which can be computed from the spectrum of Γ) equals its average excess $\overline{k}_d$ (the mean of the numbers of vertices at maximum distance from every vertex). More precisely, the spectral excess is the value of $p_d(\lambda_0)$ given in (2), and $\overline{k}_d = \frac{1}{|X|}\sum_{x\in X} k_d(x)$. Then, the theorem reads as follows:

Theorem 2.1 (The spectral excess theorem). *A connected regular graph on n vertices, with adjacency matrix A and $d+1$ distinct eigenvalues, is distance-regular if and only if*

$$\overline{k}_d = p_d(\lambda_0) = n\left(\sum_{i=0}^{d} \frac{\pi_0^2}{m(\lambda_i)\pi_i^2}\right)^{-1},$$

where $\pi_i = \prod_{j\neq i} |\lambda_i - \lambda_j|$ for $i = 0, 1, \ldots, d$.

For short proofs, see Van Dam [4], and Fiol, Gago, Garriga [9].

In this section we find a possible solution to the problem of deciding whether, from the adjacency spectrum of a (regular) graph $\Gamma = (X, E)$ and the harmonic (or arithmetic) mean of the numbers $(|X| - |\Gamma_D(x)|)_{x\in X}$, we can decide that A_D is a polynomial in A. To be more precise, we provide a characterization of when $A_D \in \text{span}\{p_0(A), \ldots, p_d(A)\}$, where the $p_i's$ are the predistance polynomials.

Before proving the main result, note that, for any $x \in X$ and any matrix indexed by the vertices of Γ, $C \in \text{Mat}_X(\mathbb{R})$, the Cauchy-Schwartz inequality yields

$$\left(\sum_{y\notin\Gamma_D(x)} (C_{xy})^2\right)\left(\sum_{y\notin\Gamma_D(x)} 1^2\right) \geq \left(\sum_{y\notin\Gamma_D(x)} C_{xy}\right)^2.$$

That is,

$$\sum_{y\notin\Gamma_D(x)} (C_{xy})^2 \geq \frac{1}{|X| - |\Gamma_D(x)|}\left(\sum_{y\notin\Gamma_D(x)} C_{xy}\right)^2, \tag{4}$$

and equality holds if and only if all the values of C_{xy} are the same for all $y \notin \Gamma_D(x)$.

Theorem 2.2. *Let $\Gamma = (X, E)$ be a connected k-regular graph, with adjacency matrix A having $d+1$ distinct eigenvalues, diameter D, and predistance polynomials $\{p_i\}_{i=0}^d$. Then,*

$$\frac{|X|}{\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}} \geq q_{D-1}(k) = |X| - \sum_{i=D}^{d} p_i(k), \tag{5}$$

with equality if and only if $A_D = \sum_{i=D}^{d} p_i(A)$.

Proof. We just adjust the proof of [4, Lemma 1], together with Lemma 1.3 and (4). Recalling that $q_{D-1} = \sum_{i=0}^{D-1} p_i$, we have that

$$\begin{aligned} q_{D-1}(k) &= \langle q_{D-1}, q_{D-1}\rangle_A = \frac{1}{|X|}\mathrm{tr}(q_{D-1}(A)^2) = \frac{1}{|X|}\sum_{x\in X}\left(q_{D-1}(A)^2\right)_{xx} \\ &= \frac{1}{|X|}\sum_{x\in X}\sum_{y\notin\Gamma_D(x)}(q_{D-1}(A)_{xy})^2 \geq \frac{1}{|X|}\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}\left[\sum_{y\notin\Gamma_D(x)} q_{D-1}(A)_{xy}\right]^2 \\ &= \frac{1}{|X|}\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}\left[q_{D-1}(k)\right]^2, \end{aligned}$$

(notice that, for the last equality, we used Lemma 1.3(i)), and this yields

$$\frac{|X|}{\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}} \geq q_{D-1}(k).$$

Since $|X| = \sum_{i=0}^{d} p_i(k) = q_{D-1}(k) + \sum_{i=D}^{d} p_i(k)$, the inequality follows.

If equality holds, then for each $x \in X$ the values of $q_{D-1}(A)_{xy}$ are the same, say α, for all $y \notin \Gamma_D(x)$. Moreover, since A is symmetric, $q_{D-1}(A)_{xy} = q_{D-1}(A)_{yx}$ for any $x \in X$ and $y \notin \Gamma_D(x)$, so that $q_{D-1}(A)_{xy} = \alpha$ for any $x, y \in X$ at distance less than D. Also, by Lemma 1.3(i), $\sum_{y\in X} q_{D-1}(A)_{xy} = q_{D-1}(k) = n_{D-1}\alpha$, where $n_{D-1} = \sum_{i=0}^{D-1}|\Gamma_i(x)| = |X| - |\Gamma_D(x)|$ for any $x \in X$. Finally, from $\|q_{D-1}\|^2 = q_{D-1}(k)$, we have

$$n_{D-1}\alpha = \|q_{D-1}\|^2 = \frac{1}{n}\mathrm{tr}(q_{D-1}(A)^2) = \frac{1}{n}\alpha^2 \mathrm{sum}\left(\sum_{i=0}^{D-1} A_i\right) = n_{D-1}\alpha^2,$$

where we have used (1). Then, $\alpha = 1$ and $q_{D-1}(A)_{xy} = 1$ for each pair of vertices x and y at distance less than $D-1$. Consequently, $q_{D-1}(A) = \sum_{i=0}^{D-1} A_i = J - A_D$, which yields $A_D = \sum_{i=D}^{d} p_i(A)$.

Conversely, assume that $\sum_{i=D}^{d} p_i(A) = A_D$. Then, $A_D\boldsymbol{j} = \sum_{i=D}^{d} p_i(k)\boldsymbol{j}$, and with $|\Gamma_D(x)| = \sum_{i=D}^{d} p_i(k)$, the equality follows. □

As a simple consequence, notice that, if Γ is a k-regular graph of diameter 2, then $|X| - |\Gamma_2(x)| = 1 + k$ for any $x \in X$. Besides, $q_1(k) = p_0(k) + p_1(k) = 1 + k$. Thus, equality in Theorem 2.2 holds, and Γ is distance polynomial, as already proved by Weichel in [13]. Another consequence of Theorem 2.2 is the following corollary.

Corollary 2.1. *Let $\Gamma = (X, E)$ be a connected k-regular graph on n vertices, with spectrum $\mathrm{sp}\,\Gamma = \{\lambda_0(= k)^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)} \ldots, \lambda_d^{m(\lambda_d)}\}$, diameter D, and predistance polynomials $\{p_i\}_{i=0}^d$. Then the following holds.*

(i) *In general,*

$$\frac{1}{|X|}\sum_{x\in X}(|X| - |\Gamma_D(x)|) \geq |X| - \sum_{i=D}^{d} p_i(k)$$

with equality if and only if $A_D = \sum_{i=D}^{d} p_i(A)$.

(ii) *If* $A_D \in \operatorname{span}\{I, A, \ldots, A^d\}$ *then*

$$|\Gamma_D(x)| \leq \sum_{i=D}^{d} p_i(k).$$

(iii) *If* $\frac{1}{|X|}\sum_{x\in X}(|X| - |\Gamma_D(x)|) = |X| - \sum_{i=D}^{d} p_i(k)$ *then* $A_D \in \operatorname{span}\{I, A, \ldots, A^d\}$.

(iv) *The graph* Γ *is distance-regular if and only if* $D = d$ *and*

$$\frac{|X|}{\sum_{x\in X}\frac{1}{|X|-|\Gamma_d(x)|}} = q_{d-1}(k) = |X| - p_d(k) = n\left[1 - \left(\sum_{i=0}^{d}\frac{\phi_0^2}{m_i\phi_i^2}\right)^{-1}\right] \qquad (6)$$

or, alternatively,

$$\frac{1}{|X|}\sum_{x\in X}(|X| - |\Gamma_D(x)|) = p_d(k) = n\left(\sum_{i=0}^{d}\frac{\phi_0^2}{m_i\phi_i^2}\right)^{-1}, \qquad (7)$$

where $\phi_i = \prod_{j\neq i}(\lambda_i - \lambda_j)$*, for* $i = 0, \ldots, d$.

Proof. (i) Let $a_1, a_2, \ldots, a_n$ be real numbers. Recall that the numbers

$$AM = \frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{and} \qquad HM = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

are the arithmetic and harmonic mean for the numbers $a_1, a_2, \ldots, a_n$, respectively, and we have $AM \geq HM$. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$. Now, the result of (i) follows from Theorem 2.2. The proofs of (ii) and (iii) are immediate from (i), or from Theorem 2.2. For instance, under the hypothesis of (ii), there exist constants $c_0, \ldots, c_d$ such that $A_D = \sum_{i=0}^{d} c_i A^i$. Thus, as Γ has the k-eigenvector $\boldsymbol{j}$, we have that $|\Gamma_D(x)| = (A_D\boldsymbol{j})_x = \sum_{i=0}^{d} c_i k^i$ for every $x \in X$ (a constant), and (i) gives the result. The results in (iv) correspond to different versions of the spectral excess theorem given in [4, 7] and [11], respectively. Thus, (6) is a consequence of Theorem 2.2 and Proposition 1.1, whereas (7) follows from Theorem 2.2 and (i). In these two cases, we also used $|X| = n$ and (2). □

3. The Laplacian approach

Theorem 3.1. *Let* $\Gamma = (X, E)$ *be a connected graph with* $d+1$ *distinct eigenvalues, diameter* D*, and Laplacian predistance polynomials* $\{r_i\}_{i=0}^{d}$*. Then,*

$$\frac{|X|}{\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}} \geq s_{D-1}(0) = |X| - \sum_{i=D}^{d} r_i(0), \qquad (8)$$

with equality if and only if $A_D = \sum_{i=D}^{d} r_i(L)$*. Moreover, in this case, if* $D = 2$*,* Γ *is regular.*

Proof. The proof follows the same line of reasoning as in Theorem 2.2 with the polynomial s_{D-1} instead of q_{D-1}. Thus, we have:

$$\begin{aligned} s_{D-1}(0) &= \|s_{D-1}\|_L^2 = \frac{1}{|X|}\mathrm{tr}(s_{D-1}(L)^2) = \frac{1}{|X|}\sum_{x\in X}(s_{D-1}(L)^2)_{xx} \\ &= \frac{1}{|X|}\sum_{x\in X}\sum_{y\notin\Gamma_D(x)}(s_{D-1}(L)_{xy})^2 \geq \frac{1}{|X|}\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}\left[\sum_{y\notin\Gamma_D(x)} s_{D-1}(L)_{xy}\right]^2 \\ &= \frac{1}{|X|}\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}\left[s_{D-1}(0)\right]^2, \end{aligned}$$

and this yields

$$\frac{|X|}{\sum_{x\in X}\frac{1}{|X|-|\Gamma_D(x)|}} \geq s_{D-1}(0) = |X| - \sum_{i=D}^{d} r_i(0).$$

If equality holds, then for each x the values of $s_{D-1}(L)_{xy}$ are the same, say α, for all $y \notin \Gamma_D(x)$. Moreover, since L is symmetric, $s_{D-1}(L)_{xy} = s_{D-1}(L)_{yx}$ for any x and $y \notin \Gamma_D(x)$. Also, by Lemma 1.3(ii), $\sum_{y\in X} s_{D-1}(L)_{xy} = s_{D-1}(0) = n_{D-1}\alpha$, where $n_{D-1} = \sum_{i=0}^{D-1}|\Gamma_i(x)|$ for every $x \in X$. Finally, from $\|s_{D-1}\|^2 = s_{D-1}(0)$, we have

$$n_{D-1}\alpha = \|s_{D-1}\|_L^2 = \frac{1}{n}\mathrm{tr}(s_{D-1}(L)^2) = \frac{1}{n}\alpha^2\mathrm{sum}\left(\sum_{i=0}^{D-1} A_i\right) = n_{D-1}\alpha^2,$$

so that $\alpha = 1$. That is, $s_{D-1}(L)_{xy} = 1$ for each pair of vertices x and y at distance less than $D-1$. Consequently, $s_{D-1}(L) = J - A_D$, which yields $A_D = \sum_{i=D}^{d} r_i(L)$. In particular, if equality holds and $D = 2$, we have $I + A = s_1(L) = r_0(L) + r_1(L) = I + r_1(L)$. Thus, $r_1(L) = A$ and, by Lemma 1.2, Γ is regular.

Conversely, assume that $\sum_{i=D}^{d} r_i(L) = A_D$. Then, $A_D\boldsymbol{j} = \sum_{i=D}^{d} r_i(0)\boldsymbol{j}$, and with $|\Gamma_D(x)| = \sum_{i=D}^{d} r_i(k)$, and the equality follows. □

From this theorem, we obtain the analogous results of Corollary 2.1(i)-(iv). In particular, the analogue of (iv) yields the following characterization of distance-regularity for a (not necessarily regular) graph.

Corollary 3.1. *Let $\Gamma = (X, E)$ be a graph on n vertices, with Laplacian matrix L, Laplacian spectrum $\mathrm{sp}\, L = \{\theta_0(=0)^{m(\theta_0)}, \theta_1^{m(\theta_1)}, \ldots, \theta_d^{m(\theta_d)}\}$, diameter D, and Laplacian predistance polynomials $\{r_i\}_{i=0}^{d}$. Then, Γ is distance-regular if and only if $D = d$ and*

$$\frac{|X|}{\sum_{x\in X}\frac{1}{|X|-|\Gamma_d(x)|}} = s_{d-1}(0) = |X| - r_d(0) = n\left[1 - \left(\sum_{i=0}^{d}\frac{\psi_0^2}{m(\theta_i)\psi_i^2}\right)^{-1}\right], \tag{9}$$

where $\psi_i = \prod_{j\neq i}(\theta_i - \theta_j)$ for $i = 0, \ldots, d$.

Proof. Use Theorem 2.2, Proposition 1.2, and (3). □

Also, as in Corollary 2.1(iv), the above result implies the characterization given in [5] by using the arithmetic mean of the numbers $|X| - |\Gamma_d(x)|$.)

4. Open problems

We finish the paper by formulating some open problems which could be of interest in further studies.

In the first problem we are interested in algebraic and combinatorial properties of graphs for which A_D can be written as a sum of predistance polynomials.

Research problem 1. *According to Corollary 2.1(iv), if $d = D$ and equality in* (5) *holds, then Γ is distance-regular. Classify all graphs for which equality in* (5) *holds.*

In the second problem we are interested in properties of walk-regular graphs for which $D < d$.

Research problem 2. *Let $\Gamma = (X, E)$ be a walk-regular graph (that is, for each $\ell \geq 0$, the number of closed walks of length ℓ from a vertex x to itself is the same for each x) with diameter D and $d + 1$ distinct eigenvalues. Assume that $D < d$. Prove or disprove that*

$$|\Gamma_D(x)| \leq p_d(k) \qquad \forall x \in X.$$

More generally, prove or disprove the same when Γ is a regular graph.

In the third problem we are interested in studying families of graphs for which the vector space generated by the adjacency matrix is closed with respect to ordinary and element-wise (Hadamard) multiplication.

Research problem 3. *Let $\Gamma = (X, E)$ be a graph with diameter D, adjacency matrix A, and $d+1$ distinct eigenvalues. Let $f_0, f_1, \ldots, f_d$ be linearly independent polynomials satisfying $\sum_{i=0}^{d} f_i(A) = J$, where f_i, for $i = 0, \ldots, d$, does not need to be of degree i. Find under what conditions on such polynomials we can obtain a version of the spectral excess theorem for quotient polynomial graphs. (For a definition of quotient polynomial graphs, see* [8]).

Acknowledgements

This research has been partially supported by AGAUR from the Catalan Government under project 2017SGR1087 and by MICINN from the Spanish Government under project PGC2018-095471-B-I00. The second author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0285 and Young Researchers Grant).

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