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On (super) edge-magic deficiency of some classes of graphs

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Abstract

A graph G of order p and size q is called *edge-magic total* if there exists a bijection ϕ from $V(G) \cup E(G)$ to the set $\{1, 2, \ldots, p+q\}$ such that $\phi(s) + \phi(st) + \phi(t)$ is a constant for every edge st in $E(G)$. An edge-magic total graph with $\phi(V(G)) = \{1, 2, \ldots, p\}$ is called *super edge-magic total*. Furthermore, the *edge-magic deficiency* of a graph G is the smallest integer $n \geq 0$ such that $G \cup nK_1$ is edge-magic total. The *super edge-magic deficiency* of a graph G is either the smallest integer $n \geq 0$ such that $G \cup nK_1$ is super edge-magic total or $+\infty$ if there exists no such integer n. In this paper, we study the (super) edge-magic deficiency of join product graphs and 2-regular graphs.

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1. Introduction

Let G be a finite and simple graph with vertex set $V(G)$ and edge set $E(G)$ such that $p = |V(G)|$ and $q = |E(G)|$. An *edge-magic total labeling* (EMTL) of a graph G is a bijection $\phi : V(G) \cup E(G) \to \{1, 2, \cdots, p+q\}$ such that $\phi(s) + \phi(st) + \phi(t)$ is a constant k, called the *magic*

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constant of ϕ, for every $st \in E(G)$. An EMTL ϕ of G is called a *super edge-magic total labeling* (SEMTL) if $\phi(V(G)) = \{1, 2, \cdots, p\}$. A graph that admits a (S)EMTL is called (*super*) *edge-magic total* ((S)EMT). The notions of an EMTL and an EMT graph were introduced in [12] while the concepts of a SEMTL and a SEMT graph were introduced in [3].

The next lemma provides necessary and sufficient conditions for a graph to be a SEMT graph.

Lemma 1.1. [4] *A graph G is* SEMT *if and only if there exists a bijection $\phi : V(G) \rightarrow \{1, 2, \cdots, p\}$ such that the set $S = \{\phi(s) + \phi(t) : st \in E(G)\}$ is a set of q consecutive integers.*

In [12], Kotzig and Rosa also introduced the notion of edge-magic deficiency (EMD) of a graph. The *EMD* of a graph G, $\mu(G)$, is the minimum integer $n \geq 0$ such that $G \cup nK_1$ is an EMT graph. Kotzig and Rosa proved that every graph has finite EMD. Figueroa-Centeno *et al.* [5] introduced the notion of super edge-magic deficiency (SEMD) of a graph. The *SEMD* of a graph G, $\mu_s(G)$, is defined as either the minimum integer $n \geq 0$ such that $G \cup nK_1$ is a SEMT graph or $+\infty$ if there exists no such integer n. Thus, a (S)EMT graph is a graph with zero (S)EMD. Unlike the EMD, not all graphs have finite SEMD. Lemma 1.2 provides necessary conditions for a graph to have infinite SEMD.

Lemma 1.2. [5] *If G is a graph with $q \equiv 2$ (mod 4) edges and every vertex of G has even degrees, then $\mu_s(G) = +\infty$.*

Several papers dealing with (S)EMD of graphs for instants see [1, 14, 15, 16, 17, 18]. The latest developments in these labelings can be found in [7]. Here, we study the (S)EMD of join product graphs and 2-regular graphs.

2. SEMD of Join Product Graphs

The *join product* of two graphs G and H, $G + H$, is a graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{st : s \in V(G), t \in V(H)\}$. If H is an isolated vertex, then it is denoted by $G + K_1$. To present the results on SEMD of join product graphs, we need the following lemma. First, we define a star S_n as a graph with $n + 1$ vertices where one vertex has degree n and n vertices have degree one.

Lemma 2.1. *Let G be a graph with $p \geq 8$ vertices and $q = 2p - 3$ edges. If $\mu_s(G) = 0$, then $2C_3$, $C_3 \cup S_3$ or $2S_3$ are subgraphs of G.*

Proof. Let ϕ be a SEMTL of G. Since $q = 2p - 3$, then $\{\phi(s) + \phi(t) : st \in E(G)\}$ is a unique set $S = \{3, 4, 5, \ldots 2p - 1\}$. To get 3, 4, and 5 in S, vertices with labels 1, 2, and 3 should form a cycle C_3 or vertices with labels 1, 2, 3, and 4 should form a star S_3, where vertices with labels 2, 3, and 4 are adjacent to the vertex with label 1. In a similar way, vertices with labels p, $p - 1$, and $p - 2$ should form a cycle C_3 or vertices with labels p, $p - 1$, $p - 2$ and $p - 3$ should form a star S_3. Since $p \geq 8$, so 1, 2, 3, 4, $p - 3$, $p - 2$, $p - 1$ and p are distinct integers. Thus, we have desire results. □

Lemma 2 is not true in reverse. Let's look at graph G in Figure 1 as an example. This graph has $p = 8$ vertices, $q = 2p-3 = 13$ edges, and having subgraphs isomorphic to $2C_3$, $C_3 \cup S_3$, and $2S_3$. Suppose $\mu_s(G) = 0$ with a SEMTL ϕ. Then, $\{\phi(u)+\phi(v) : uv \in E(G)\} = \{3, 4, 5, \ldots, 15\}$. So,

$$5\phi(z) + \phi(v_1) + \phi(v_2) + \phi(v_3) + 2\phi(u) = \sum_{i=3}^{15} i - 2\sum_{i=1}^{8} i = 45.$$

It can be verified that we do not get a SEMTL of G for any solutions of this equation. Thus, $\mu_s(G) \geq 1$. By this fact and the labeling of $G \cup K_1$ in Figure 1, we conclude that $\mu_s(G) = 1$.

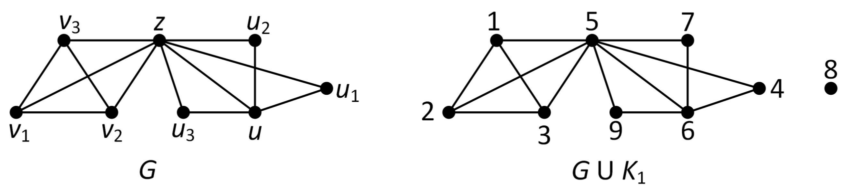

Figure 1. A grap G with 8 vertices and 13 edges having subgraphs $2C_3$, $C_3 \cup S_3$, and $2S_3$ and a vertex labeling of $G \cup K_1$.

In [19], Ngurah and Simanjuntak proved the following lemma. They also showed that the lemma is attainable by some classes of trees and forests.

Lemma 2.2. [19] *Let G be a graph without cycle and isolated vertices. If the SEMD of $G + K_1$ is zero, then G is a tree or a forest.*

We now relax the condition of the Lemma 2.2 as in Lemma 2.3. The proof of Lemma 2.3 is identical to the proof of Lemma 2.2.

Lemma 2.3. *Let G be a graph without isolated vertices. If $\mu_s(G + K_1) = 0$, then G is a tree, a forest, a union of cycles and trees, or a union of unicyclic graphs and trees.*

The *corona product*, $G \odot H$, of two graphs G and H is defined as the graph formed by taking one copy of G and $|V(G)|$ copies of H, then connecting the ith vertex of G to every vertex in the ith copy of H. If H is an isolated vertex, then it is denoted by $G \odot K_1$. The next results show that Lemma 2.3 is attainable.

Theorem 2.1. a) $\mu_s([C_n \cup P_m] + K_1) = 0$ *if and only if* $n = 3$ *and* $2 \leq m \leq 4$.
b) $\mu_s([(C_3 \odot K_1) \cup P_m] + K_1) = 0$ *if and only if* $2 \leq m \leq 4$.

Proof. a) First, let $F_{n,m} = [C_n \cup P_m] + K_1$ be a graph with $V(F_{n,m}) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq m\} \cup \{z\}$ and $E(F_{n,m}) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{y_j y_{j+1} : 1 \leq j \leq m-1\} \cup \{z x_i, z y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Thus, $F_{n,m}$ has $p = n+m+1$ vertices and $q = 2n+2m-1 = 2(n+m+1)-3$ edges.

Next, we show that, for $m \geq 5$, $\mu_s(F_{3,m}) > 0$. For $m \geq 5$, suppose that $\mu_s(F_{3,m}) = 0$ with a SEMTL ϕ. Then, $S = \{\phi(u)+\phi(v) : uv \in E(F_{3,m})\} = \{3, 4, 5, \ldots, 2m+7\}$. Notice that $F_{3,m}$ has no subgraphs isomorphic to $2S_3$. By Lemma 2.1, we should consider the following cases.

Case 1. Vertices with labels 1, 2, 3 and vertices with label $p = m+4$, $p-1$, $p-2$ form a $2C_3$. If $\{\phi(x_1), \phi(x_2), \phi(x_3)\} = \{1, 2, 3\}$ and $\phi(z) \in \{p, p-1, p-2\}$ then, we fail to get 6 in S. If $\{\phi(x_1), \phi(x_2), \phi(x_3)\} = \{p, p-1, p-2\}$ and $\phi(z) \in \{1, 2, 3\}$ then, we fail to get $2p-4$ in S.

Case 2. Vertices with labels 1, 2, 3 form a C_3 and vertices with labels p, $p-1$, $p-2$, $p-3$ form a S_3.
In this case, it is easy to check that we fail to get 6 in S.

Case 3. Vertices with labels p, $p-1$, $p-2$, form a C_3 and vertices with label 1, 2, 3, 4 form a S_3.
In this case, we fail to get $2p-4$ in S. Thus, for $m \geq 5$, $\mu_s(F_{3,m}) > 0$.

Now, we show that, for any $n \geq 4$ and $m \geq 2$, $\mu_s(F_{n,m}) > 0$. Since, for $n \geq 4$ and $m > 2$, $F_{n,m}$ is a graph with $|V(F_{n,m})| \geq 8$ and has no subgraphs isomorphic to $2C_3$, $C_3 \cup S_3$ and $2S_3$, by Lemma 2.1, $\mu_s(F_{n,m}) > 0$. Thus, the remaining case is to show that $\mu_s(F_{4,2}) > 0$. Let $\mu_s(F_{4,2}) = 0$ and let ϕ be a SEMTL of $F_{4,2}$. Then, $4\phi(z) + \sum_{i=1}^{4} \phi(x_i) = 32$. It is simple to confirm that none of the equation's solutions result in a SEMTL of $F_{4,2}$. Thus, $\mu_s(F_{4,2}) > 0$.

Finaly, we show that, for $2 \leq m \leq 4$, $\mu_s(F_{3,m}) = 0$. Define a vertex labeling ϕ as follows. $\{\phi(x_1), \phi(x_2), \phi(x_3)\} = \{1, 2, 3\}$, $\phi(z) = 5$, for $m = 2$, set $\{\phi(y_1), \phi(y_2)\}$ to $\{4, 6\}$, for $m = 3$, set $(\phi(y_1), \phi(y_2), \phi(y_3))$ to $(4, 6, 7)$, and for $m = 4$, set $(\phi(y_1), \phi(y_2), \phi(y_3), \phi(y_4))$ to $(4, 6, 8, 7)$.

b) Let $H_m = [(C_3 \odot K_1) \cup P_m] + K_1$ be a graph with $V(H_m) = \{u_i, v_i : 1 \leq i \leq 3\} \cup \{w_j : 1 \leq j \leq m\} \cup \{z\}$ and $E(H_m) = \{u_iu_{i+1} : 1 \leq i \leq 2\} \cup \{u_3u_1\} \cup \{u_iv_i : 1 \leq i \leq 3\} \cup \{w_jw_{j+1} : 1 \leq j \leq m-1\} \cup \{zu_i, zv_i, zw_j : 1 \leq i \leq 3, 1 \leq j \leq m\}$. Thus H_m has $p = m+7$ vertices and $q = 2m+11 = 2(m+7)-3$ edges.

First, we prove that, for $2 \leq m \leq 4$, $\mu_s([(C_3 \odot K_1) \cup P_m] + K_1) = 0$. Define a vertex labeling ϕ as follows. $(\phi(u_1), \phi(u_2), \phi(u_3)) = (1, 2, 3)$, $(\phi(v_1), \phi(v_2), \phi(v_3)) = (5, 6, 4)$, $\phi(z) = 8$, for $m = 2$, set $\{\phi(w_1), \phi(w_2)\}$ to $\{7, 9\}$, for $m = 3$, set $(\phi(w_1), \phi(w_2), \phi(w_3))$ to $(7, 9, 10)$, and for $m = 4$, set $(\phi(w_1), \phi(w_2), \phi(w_3), \phi(w_4))$ to $(7, 9, 11, 10)$. It is simple to confirm that ϕ extends to a SEMTL of $[(C_3 \odot K_1) \cup P_m] + K_1$ for $2 \leq m \leq 4$.

Next, suppose that, for any $m \geq 5$, $\mu_s(H_m) = 0$. Then, there exists a SEMTL ϕ of H_m such that $S = \{\phi(u) + \phi(v) : uv \in E(H_m)\} = \{3, 4, 5, \ldots, 2p-1\}$, in which we note that $2p-1 = 2m+13$. By Lemma 2.1, we should consider three cases.

Case 1. Vertices with labels 1, 2, 3 and vertices with labels p, $p-1$, $p-2$ form a $2C_3$.
If $(\phi(u_1), \phi(u_2), \phi(u_3)) = (1, 2, 3)$ and $\phi(z) \in \{p, p-1, p-2\}$ then, to get 6, 7, and 8 in S, set $(\phi(v_1), \phi(v_2), \phi(v_3))$ to $(5, 6, 4)$ or $(6, 4, 5)$. Thus, we fail to get 9 in S. If $(\phi(u_1), \phi(u_2), \phi(u_3)) = (p, p-1, p-2)$ and $\phi(z) \in \{1, 2, 3\}$ then, to get $2p-4$, $2p-5$, and $2p-6$, set $(\phi(v_1), \phi(v_2), \phi(v_3))$ to $(p-5, p-3, p-4)$ or $(p-4, p-5, p-3)$. In this case, it is not possible to get $2p-7$ in S.

Case 2. Vertices with labels 1, 2, 3 form a C_3 and vertices with labels p, $p-1$, $p-2$, $p-3$ form a S_3.
If $(\phi(u_1), \phi(u_2), \phi(u_3)) = (1, 2, 3)$ and $\phi(z) \in \{p, p-1, p-2, p-3\}$ then, by a similar argument as in the Case 1, we fail to get 9 in S. If $\phi(z) \in \{1, 2, 3\}$ and $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(v_1)) = (p, p-2, p-3, p-1)$, then we fail to get $2p-4$ in S. If $\phi(z) \in \{1, 2, 3\}$ and $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(v_1)) = (p, p-1, p-3, p-2)$, then we fail to get either $2p-5$ or $2p-6$ in S.

Case 3. Vertices with labels p, $p-1$, $p-2$ form a C_3 and vertices with labels 1, 2, 3, 4 form a S_3.
By a similar argument as in the Case 2, this case also do not lead to a SEMTL of H_m.

Case 4. Vertices with labels 1, 2, 3, 4 and vertices with labels $p, p-1, p-2, p-3$ form a $2S_3$. If $\phi(z) \in \{1, 2, 3, 4\}$ and $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(v_1)) = (p, p-2, p-3, p-1)$, then we fail to get $2p-4$ in S. If $\phi(z) \in \{1, 2, 3, 4\}$ and $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(v_1)) = (p, p-1, p-3, p-2)$, then we fail to get either $2p-5$ or $2p-6$ in S. If $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(v_1)) = (1, 3, 4, 2)$ and $\phi(z) \in \{p, p-1, p-2, p-3\}$ then, it is not possible to get 6 in S. If $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(v_1)) = (1, 2, 4, 3)$ and $\phi(z) \in \{p, p-1, p-2, p-3\}$ then, it is not possible to get either 7 or 8 in S. Thus, $\mu_s(H_m) > 0$, if $m \geq 5$. This complete the proof of part b). □

3. (S)EMD of 2-Regular Graphs

In [8], Holden et al. proved that $C_5 \cup (2t)C_3$, $C_4 \cup (2t-1)C_3$, and $C_7 \cup (2t)C_3$ are strong vertex-magic total (SVMT) graphs. It is easy to verify that a (S)VMT 2-regular graph is equivalent to a (S)EMT 2-regular graph. Based on these results, they posed the Conjecture 1.

Conjecture 1. [8]. *Let G be a 2-regular graph of odd order. The graph G is* SVMT *if and only if G is not one of $C_4 \cup C_3, C_4 \cup 3C_3$ or $C_5 \cup 2C_3$.*

Cichacz et al. [2] provide a partial solution to Conjecture 1, introducing a method for generating (S)VMT labelings of 2-regular graphs.

Theorem 3.1. [2]. *Let $k \geq 1$ be an integer and $F = \cup_{i=1}^{k} C_{n_i}$ be a 2-regular graph. If F is a* (S)VMT *graph, then $G = \cup_{i=1}^{k} C_{mn_i}$ is a* (S)VMT *graph for every odd $m \geq 3$.*

In [6], Figueroa-Centeno et al. provided the following result.

Theorem 3.2. [6]. *If a 3-colorable graph G is* (S)EMT*, then mG is* (S)EMT *for any odd integer m.*

As mentioning in [7], Ichishima and Oshima [11] investigate the SEMD of 2-regular graphs $C_m \cup C_n$ for $m = 3, 4, 5, 7$ and any n. Krisnawati et al. [13] investigate the SEMD of a 2-regular graph with three components, namely $2C_3 \cup C_n$. In the next theorem, we study the SEMD of a 2-regular graph $2C_4 \cup C_n$.

Theorem 3.3. *Let $n \geq 4$ be a positive integer. Then*

$$\mu_s(2C_4 \cup C_n) = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2, \\ 1, & \text{if } n = 4 \text{ and } n \equiv 8, 12 \pmod{16}, \\ +\infty, & \text{if } n \equiv 2 \pmod 4, \end{cases}$$

and $\mu_s(2C_4 \cup C_n) \leq 2$, if $4 < n \equiv 0, 4 \pmod{16}$.

Proof. First, let $H = 2C_4 \cup C_n$ be a graph with $V(H) = \{u_i, v_i : 1 \leq i \leq 4\} \cup \{w_i : 1 \leq i \leq n\}$ and $E(H) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq 3\} \cup \{u_4 u_1, v_4 v_1\} \cup \{w_i w_{i+1} : 1 \leq i \leq n-1\} \cup \{w_n w_1\}$.

Next, we show that the graph H is SEMT if and only if n is odd. If H is a SEMT graph then its magic constant is $\frac{1}{2}(5n+43)$. So, n should be an odd integer. Next, for odd $n \geq 5$ define $\phi : V(H) \longrightarrow \{1, 2, \ldots, n+8\}$ as follows. $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(u_4)) = (1, \frac{1}{2}(n+$

$9), 2, \frac{1}{2}(n+13))$, $(\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)) = (3, \frac{1}{2}(n+15), 5, \frac{1}{2}(n+17))$. When $n = 7$, label $(\phi(w_1), \phi(w_2), \phi(w_3), \phi(w_4), \phi(w_5), \phi(w_6), \phi(w_7)$ with $(6, 7, 13, 9, 14, 4, 15)$. When $n \geq 9$, define $\phi(w_1) = 4, \phi(w_2) = \frac{1}{2}(n+11), \phi(w_3) = n+7, \phi(w_4) = \frac{1}{2}(n+7)$, and label the remaining vertices as follows.

Case 1. $n \equiv 1 \pmod 4$.

$$\phi(w_i) = \begin{cases} \frac{1}{2}(2n+21-i), & \text{if } i = 5, 9, 13, \ldots, n, \\ \frac{1}{2}(n+9-i), & \text{if } i = 6, 10, 14, \ldots, n-3, \\ \frac{1}{2}(2n+17-i), & \text{if } i = 7, 11, 15, \ldots, n-2, \\ \frac{1}{2}(n+13-i), & \text{if } i = 8, 12, 16, \ldots, n-1. \end{cases}$$

Case 2. $7 < n \equiv 3 \pmod 4$.

$$\phi(w_i) = \begin{cases} n+8, & \text{if } i = 5, \\ n+6, & \text{if } i = 7, \\ 6, & \text{if } i = n-1, \\ \frac{1}{2}(n+9-i), & \text{if } i = 6, 10, 14, \ldots, n-5, \\ \frac{1}{2}(n+13-i), & \text{if } i = 8, 12, 16, \ldots, n-3, \\ \frac{1}{2}(2n+17-i), & \text{if } i = 9, 13, 17, \ldots, n-2, \\ \frac{1}{4}(2n+21-i), & \text{if } i = 11, 15, 19, \ldots, n. \end{cases}$$

It can be checked that $\{\phi(x) + \phi(y) : xy \in E(H)\} = \{\frac{1}{2}(n+11), \frac{1}{2}(n+13), \ldots, \frac{1}{2}(3n+25)\}$. By Lemma 1.1, H is a SEMT graph.

Now, we prove that, for $n = 4$ and $n \equiv 8, 12 \pmod{16}$, $\mu_s(H) = 1$. If $n = 4$, $\mu_s(3C_4) = 1$ have been proved in [10] (see Corollary 3.6). Next, for $n \equiv 8, 12 \pmod{16}$, consider the graph $H \cup K_1$ and define $\phi : V(H \cup K_1) \longrightarrow \{1, 2, \ldots, n+9\}$ as follows. $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(u_4)) = (1, \frac{1}{2}(n+10), 2, \frac{1}{2}(n+14))$, $(\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)) = (4, \frac{1}{2}(n+12), 5, \frac{1}{2}(n+16))$, $\phi(w_1) = 3, \phi(w_2) = \frac{1}{2}(n+24)$, $\phi(w_i) = \frac{1}{2}(i+9)$ for $i = 3, 5, 7, \ldots, n-1$, $\phi(w_n) = \frac{1}{2}(n+22)$, and $\phi(K_1) = \frac{1}{4}(3n+28)$.

Case 3. $n \equiv 8 \pmod{16}$.

$$\phi(w_i) = \begin{cases} \frac{1}{2}(n+20), & \text{if } i = 4, \\ \frac{1}{2}(n+22+i), & \text{if } i = 6, 14, 22, \ldots, \frac{n}{2}-6, \\ & \cup\ i = 8, 16, 24, \ldots, \frac{n}{2}-4, \\ & \cup\ i = 10, 18, 26, \ldots, \frac{n}{2}-2, \\ \frac{1}{2}(n+6+i), & \text{if } i = 12, 20, 28, \ldots, \frac{n}{2}, \\ \frac{1}{2}(n+20+i), & \text{if } i = \frac{n}{2}+2, \frac{n}{2}+4, \frac{n}{2}+6, \ldots, n-2. \end{cases}$$

Case 4. $n \equiv 12 \pmod{16}$.

$$\phi(w_i) = \begin{cases} \frac{1}{2}(n+26), & \text{if } i = 4, \\ \frac{1}{2}(n+18), & \text{if } i = 6, \\ \frac{1}{2}(n+22+i), & \text{if } i = 8, 16, 24, \ldots, \frac{n}{2}-6, \\ & \cup\ i = 10, 18, 26, \ldots, \frac{n}{2}-4, \\ & \cup\ i = 12, 20, 28, \ldots, \frac{n}{2}-2, \\ \frac{1}{2}(n+6+i), & \text{if } i = 14, 22, 30, \ldots, \frac{n}{2}, \\ \frac{1}{2}(n+20+i), & \text{if } i = \frac{n}{2}+2, \frac{n}{2}+4, \frac{n}{2}+6, \ldots, n-2. \end{cases}$$

It is simple to confirm that, by Lemma 1.1, $H \cup K_1$ a SEMT graph. Hence, $\mu_s(H) = 1$ for $n = 4$ and $n \equiv 8, 12$ (mod 16).

Next, we show that, or $4 < n \equiv 0, 4$ (mod 16), $\mu_s(H) \leq 2$. Let us consider the graph $H \cup 2K_1$ and define $\phi : V(H \cup 2K_1) \longrightarrow \{1, 2, \ldots, n+10\}$ as follows. $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(u_4)) = (1, \frac{1}{2}(n+10), 2, \frac{1}{2}(n+14)$, $(\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)) = (4, \frac{1}{2}(n+12), 5, \frac{1}{2}(n+16))$, $\phi(w_1) = 3, \phi(w_2) = \frac{1}{2}(n+24), \phi(w_4) = \frac{1}{2}(n+26), \phi(w_6) = \frac{1}{2}(n+18), \phi(w_i) = \frac{1}{2}(i+9), i = 3, 5, 7, \ldots, n-1,$ and $\phi(w_n) = \frac{1}{2}(n+22)$.

Case 1. $n \equiv 0$ (mod 16).

$$\phi(\{2K_1\}) = \{\frac{1}{4}(3n+40), n+7\}$$

and

$$\phi(w_i) = \begin{cases} \frac{1}{2}(n+22+i), & \text{if } i = 8, 16, 24, \ldots, \frac{n}{2}-8, \\ & \cup\, i = 10, 18, 26, \ldots, \frac{n}{2}-6, \\ & \cup\, i = 12, 20, 28, \ldots, \frac{n}{2}-4, \\ \frac{1}{4}(3n+48), & \text{if } i = \frac{n}{2}, \\ \frac{1}{4}(3n+52), & \text{if } i = \frac{n}{2}+2, \\ \frac{1}{4}(3n+56), & \text{if } i = \frac{n}{2}+4, \\ \frac{1}{2}(n+6+i), & \text{if } i = 14, 22, 30, \ldots, \frac{n}{2}+6, \\ \frac{1}{2}(n+24+i), & \text{if } i = \frac{n}{2}+8, \frac{n}{2}+16, \frac{n}{2}+24, \ldots, n-8, \\ & \cup\, i = \frac{n}{2}+10, \frac{n}{2}+18, \frac{n}{2}+26, \ldots, n-6, \\ & \cup\, i = \frac{n}{2}+12, \frac{n}{2}+20, \frac{n}{2}+28, \ldots, n-4, \\ \frac{1}{2}(n+8+i), & \text{if } i = \frac{n}{2}+14, \frac{n}{2}+22, \frac{n}{2}+30, \ldots, n-2. \end{cases}$$

Case 2. $4 < n \equiv 4$ (mod 16).

$$\phi(\{2K_1\}) = \{\frac{1}{4}(3n+36), n+8\}$$

and

$$f(w_i) = \begin{cases} \frac{1}{2}(n+22+i), & \text{if } i = 8, 16, 24, \ldots, \frac{n}{2}-2, \\ & \cup\, i = 10, 18, 26, \ldots, \frac{n}{2}-8, \\ & \cup\, i = 12, 20, 28, \ldots, \frac{n}{2}-6, \\ \frac{1}{4}(3n+48), & \text{if } i = \frac{n}{2}, \\ \frac{1}{4}(3n+52), & \text{if } i = \frac{n}{2}+2, \\ \frac{1}{2}(n+6+i), & \text{if } i = 14, 22, 30, \ldots, \frac{n}{2}+4, \end{cases}$$

The labeling of w_i when i is even and $i \geq \frac{n}{2}+6$ is determined by the following subcases.

Subcase 2.1. $4 < n \equiv 4$ (mod 48).

$$\phi(w_i) = \begin{cases} \frac{1}{2}(n+24+i), & \text{if } i = \frac{n}{2}+6, \frac{n}{2}+12, \frac{n}{2}+18, \ldots, n-14, \\ & \cup\, i = \frac{n}{2}+8, \frac{n}{2}+14, \frac{n}{2}+20, \ldots, n-12, \\ n+10, & \text{if } i = n-8, \\ n+7, & \text{if } i = n-6, \\ n+9, & \text{if } i = n-2, \\ \frac{1}{2}(n+12+i), & \text{if } i = \frac{n}{2}+10, \frac{n}{2}+16, \frac{n}{2}+22, \ldots, n-4. \end{cases}$$

Subcase 2.2. $n \equiv 20 \pmod{48}$.

When $n = 20$, define $\phi(w_{16}) = 26$ and $\phi(w_{18}) = 29$. When $n \geq 68$, label the remaining vertices by the following formula.

$$\phi(w_i) = \begin{cases} \frac{1}{2}(n+24+i), & \text{if } i = \frac{n}{2}+6, \frac{n}{2}+12, \frac{n}{2}+18, \ldots, n-16, \\ & \cup\ i = \frac{n}{2}+8, \frac{n}{2}+14, \frac{n}{2}+20, \ldots, n-14, \\ n+9, & \text{if } i = n-10, \\ n+6, & \text{if } i = n-8, \\ n+10, & \text{if } i = n-4, \\ n+7, & \text{if } i = n-2, \\ \frac{1}{2}(n+12+i), & \text{if } i = \frac{n}{2}+10, \frac{n}{2}+16, \frac{n}{2}+22, \ldots, n-6. \end{cases}$$

Subcase 2.3. Subcase $n \equiv 36 \pmod{48}$.

$$\phi(w_i) = \begin{cases} \frac{1}{2}(n+24+i), & \text{if } i = \frac{n}{2}+6, \frac{n}{2}+12, \frac{n}{2}+18, \ldots, n-6, \\ & \cup\ i = \frac{n}{2}+8, \frac{n}{2}+14, \frac{n}{2}+20, \ldots, n-4, \\ \frac{1}{2}(n+12+i), & \text{if } i = \frac{n}{2}+10, \frac{n}{2}+16, \frac{n}{2}+22, \ldots, n-2. \end{cases}$$

By Lemma 1.1, $H \cup 2K_1$ is a SEMT graph. Thus, $\mu_s(H) \leq 2$ for $n \equiv 0, 4 \pmod{16}$.

Finally, as a direct consequence of Lemma 1.2, $\mu_s(H) = +\infty$, if $n \equiv 2 \pmod 4$. □

The following corollary is a direct consequence of applying Theorem 3.2 to the Theorem 3.3.

Corollary 3.1. *Let $l \equiv 2(mod\ 4)$, $m \equiv 1(mod\ 2)$ and $n \geq 4$ be positive integers. Then*

$$\mu_s(lC_4 \cup mC_n) = \begin{cases} 0, & \textit{if } n \equiv 1 \textit{ (mod 2)}, \\ +\infty, & \textit{if } n \equiv 2 \textit{ (mod 4)}, \end{cases}$$

$\mu_s(lC_4 \cup mC_n) \leq m$, if $n = 4$ and $n \equiv 8, 12$ (mod 16), and $\mu_s(lC_4 \cup mC_n) \leq 2m$, if $4 < n \equiv 0, 4$ (mod 16).

The following result is obtained by applying Theorems 3.1 and 3.2 to the facts that $\mu_s(2C_4 \cup C_n) = 0$ for $n \equiv 1 \pmod 2$ and $\mu_s(2C_4 \cup C_n) = +\infty$ for $n \equiv 2 \pmod 4$.

Corollary 3.2. *Letl $\equiv 2(mod\ 4)$, $m \equiv 1(mod\ 2)$, $n \geq 4$ and r be positive integers. Then*

$$\mu_s(lC_{4(2r+1)} \cup mC_{n(2r+1)}) = \begin{cases} 0, & \textit{if } n \equiv 1 \textit{ (mod 2)}, \\ +\infty, & \textit{if } n \equiv 2 \textit{ (mod 4)}, \end{cases}$$

Unresolved issues related to these results are detailed as follows.

Open Problem 1. *Determine the exact value of $\mu_s(2C_4 \cup C_n)$ when $n \equiv 0, 4$ (mod 16). Further, for $l \equiv 2$ (mod 4) and $m \equiv 1$ (mod 2), find the upper bound of the $\mu_s(lC_{4(2r+1)} \cup mC_{n(2r+1)})$ when $n \equiv 0$ (mod 4).*

In [2], Chicacz et al. prove that $2C_n$ is VMT for $n \in \{4, 6, 8, 10\}$ and posed a conjecture that, for even value of n, the graph $2C_n$ is VMT. We able to give support of the validity of this conjecture by proving $2C_n$ is EMT for $n \in \{5, 7, 9, 11, 12, 13, 14\}$, see Figure 2. Theorem 3.4 is a direct consequence of applying Theorems 3.1 and 3.2 to these results.

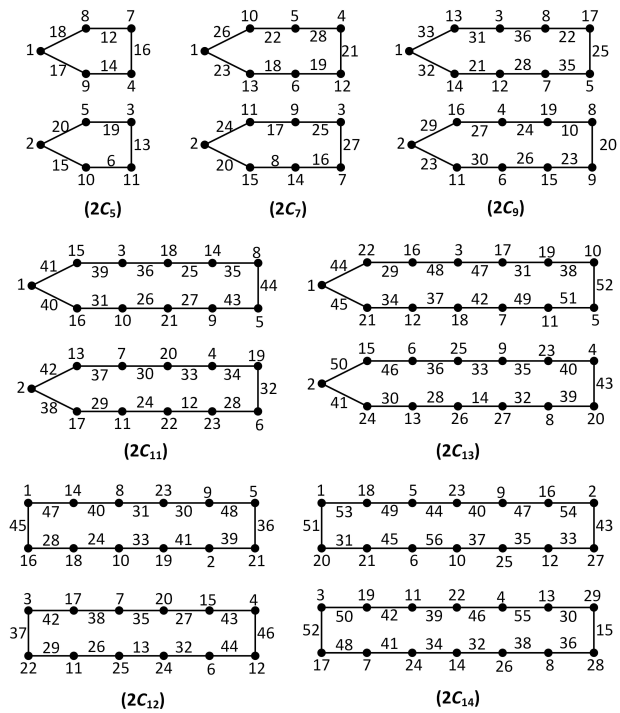

Figure 2. An EMTL of $2C_n$ for $n \in \{5, 7, 9, 11, 12, 13, 14\}$.

Theorem 3.4. *Let $m \equiv 2(mod\ 4)$, r be positive integers and $n \in \{5, 7, 9, 11, 12, 13, 14\}$. Then $mC_{n(2r+1)}$ has zero* EMD.

Based on Theorem 3.4, the following conjecture is proposed.

Conjecture 2. *For any integers $m \equiv 2$ (mod 4), $n \geq 3$, and $r \geq 1$, the 2-regular graph $mC_{n(2r+1)}$ has zero* EMD.

A graph $H \cup kK_1$ is called *pseudo super edge-magic* (PSEM) if there exists a bijection $\phi : V(H \cup kK_1) \to \{1, 2, \ldots, |V(H)| + k\}$ such that $\{\phi(s) + \phi(t) : st \in E(H)\} \cup \{2\phi(u) : u \in kK_1\}$ is a set of $|E(H)| + k$ consecutive integers. In such a case ϕ is called a *pseudo super edge-magic labeling* (PSEML) of $H \cup kK_1$. These concepts was introduced in [9]. It is easy to verify that if H is a 2-regular graph and $H \cup kK_1$ is a PSEM graph, then $|V(H)| + k$ should be an odd integer. We discover that Theorem 3.1 also valid for a PSEM graph $H \cup kK_1$. As an example, see Figure 3. This fact is not mentioned by the authors in [2].

Theorem 3.5. *For any integer $n \geq 4$, the following graphs are* PSEM.
a) $C_n \cup K_1$, *if* $n \equiv 0\ (mod\ 2)$.
b) $C_n \cup 2K_1$, *if* $n \equiv 1\ (mod\ 2)$.
c) $C_4 \cup C_n \cup K_1$, *if* $n \equiv 0\ (mod\ 4)$.
d) $C_4 \cup C_n \cup K_1$, *if* $n \in \{6, 10, 14, 18, 22, 26\}$.
e) $2C_4 \cup C_n \cup K_1$, *if* $n \equiv 8, 12\ (mod\ 16)$.

Proof. a) In [5], Ichishima et al. proved that $\mu_s(C_n) = 1$, if $n \equiv 0 \pmod 4$. The labeling defined to prove this result actually is a PSEML of $C_n \cup K_1$. Next, for $n \equiv 2 \pmod 4$, let $V(C_n \cup K_1) = \{u_i : 1 \leq i \leq n\} \cup \{v\}$ and $E(C_n \cup K_1) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\}$. Define a vertex labeling $\phi : V(C_n \cup K_1) \rightarrow \{1, 2, \ldots, n+1\}$ as follows. When $n = 6, 10$ and 14, set $(\phi(u_1), \phi(u_2), \ldots, \phi(u_6); \phi(v))$ to $(1, 7, 2, 5, 6, 4; 3)$, $(\phi(u_1), \phi(u_2), \ldots, \phi(u_{10}); \phi(v))$ to $(1, 8, 2, 9, 3, 11, 5, 10, 7, 6; 4)$, and $(\phi(u_1), \phi(u_2), \ldots, \phi(u_{14}); \phi(v))$ to $(1, 10, 2, 12, 3, 13, 4, 9, 14, 6, 15, 7, 11, 8, 1; 5)$, respectively. When $n \geq 18$, define

$$\phi(v) = \frac{1}{4}(n+6)$$

and

$$\phi(u_i) = \begin{cases} \frac{1}{2}(1+i), & \text{if } i = 1, 3, 5, \ldots, \frac{n}{2}, \\ \frac{1}{4}(n+10), & \text{if } i = \frac{2}{2}+2, \\ \frac{1}{2}(n+4+i), & \text{if } i = 2, 4, 6, \ldots, \frac{1}{2}(n-2). \end{cases}$$

To label the remaining vertices, we consider the following two cases.

Case 1. $n \equiv 2 \pmod 8$.

$$\phi(u_i) = \begin{cases} \frac{1}{2}(5+i), & \text{if } i = \frac{n}{2}+4, \frac{n}{2}+8, \frac{n}{2}+12, \ldots, n-1, \\ \frac{1}{2}(1+i), & \text{if } i = \frac{n}{2}+6, \frac{n}{2}+10, \frac{n}{2}+14, \ldots, n-3, \\ \frac{1}{2}(n+6+i), & \text{if } i = \frac{n}{2}+1, \frac{n}{2}+5, \frac{n}{2}+9, \ldots, n-4, \\ \frac{1}{2}(n+2+i), & \text{if } i = \frac{n}{2}+3, \frac{n}{2}+7, \frac{n}{2}+11, \ldots, n-2, \\ \frac{1}{2}(n+2), & \text{if } i = n. \end{cases}$$

Case 2. $n \equiv 6 \pmod 8$.

$$\phi(u_i) = \begin{cases} \frac{1}{2}(5+i), & \text{if } i = \frac{n}{2}+4, \frac{n}{2}+8, \frac{n}{2}+12, \ldots, n-7, \\ \frac{1}{2}(1+i), & \text{if } i = \frac{n}{2}+6, \frac{n}{2}+10, \frac{n}{2}+14, \ldots, n-5, \\ \frac{n}{2}, & \text{if } i = n-3, \\ \frac{1}{2}(n+4), & \text{if } i = n-1, \\ \frac{1}{2}(n+6+i), & \text{if } i = \frac{n}{2}+1, \frac{n}{2}+5, \frac{n}{2}+9, \ldots, n-10, \\ \frac{1}{2}(n+2+i), & \text{if } i = \frac{n}{2}+3, \frac{n}{2}+7, \frac{n}{2}+11 \ldots, n-8, \\ n-1, & \text{if } i = n-6, \\ n+1, & \text{if } i = n-4, \\ n, & \text{if } i = n-2, \\ \frac{1}{2}(n+2), & \text{if } i = n. \end{cases}$$

It can be checked that $\{\phi(u_i) + \phi(u_{i+1}) : 1 \leq i \leq n-1\} \cup \{\phi(u_n) + \phi(u_1)\} \cup \{2\phi(v)\}$ is $\{\frac{1}{2}(n+4), \frac{1}{2}(n+6), \ldots \frac{1}{2}(3n+4)\}$. Hence, $C_n \cup K_1$ is a PSEM graph.

b) For odd n, let $V(C_n \cup 2K_1) = \{u_i : 1 \leq i \leq n\} \cup \{v, w\}$ and $E(C_n \cup 2K_1) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\}$. Define a vertex labeling ϕ of $C_n \cup 2K_1$ as follows.

Case 1. $n \equiv 1 \pmod 4$.

$$f(\{v, w\}) = \{\frac{1}{2}(n+1), \frac{1}{4}(3n+5)\}$$

and

$$\phi(u_i) = \begin{cases} \frac{1}{2}(1+i), & \text{if } i = 1, 3, 5, \ldots, n-2, \\ \frac{1}{2}(n+3), & \text{if } i = n, \\ \frac{1}{2}(n+3+i), & \text{if } i = 2, 4, 6, \ldots, \frac{1}{2}(n-5), \\ \frac{1}{2}(n+5+i), & \text{if } i = \frac{1}{2}(n-1), \frac{1}{2}(n+3), \frac{1}{2}(n+7), \ldots, n-1. \end{cases}$$

Case 2. $n \equiv 3 \pmod 4$.

When $n = 7$ and 11, define $(\phi(u_1), \phi(u_2), \ldots, \phi(u_7); \phi(v), \phi(w)) = (1, 7, 2, 9, 4, 8, 6; 3, 5)$ and $(\phi(u_1), \phi(u_2), \ldots, \phi(u_{11}); \phi(v), \phi(w)) = (1, 9, 2, 11, 3, 13, 5, 10, 7, 12, 8; 4, 6)$, respectively. When $n \geq 15$, define

$$\phi(\{v, w\}) = \{\frac{1}{2}(n+3), \frac{1}{4}(n+5)\}$$

and

$$\phi(u_i) = \begin{cases} \frac{1}{2}(1+i), & \text{if } i = 1, 3, 5, \ldots, \frac{1}{2}(n-1), \\ \frac{1}{2}(n+5+i), & \text{if } i = 2, 4, 6, \ldots, \frac{1}{2}(n-3), \\ \frac{1}{4}(n+9), & \text{if } i = \frac{1}{2}(n+3), \end{cases}$$

Subcase 2.1. $n \equiv 3 \pmod 8$.

$$\phi(u_i) = \begin{cases} \frac{1}{2}(n+7+i), & \text{if } i = \frac{1}{2}(n+1), \frac{1}{2}(n+9), \ldots, n-9, \\ \frac{1}{2}(n+3+i), & \text{if } i = \frac{1}{2}(n+5), \frac{1}{2}(n+13), \ldots, n-7, \\ n, & \text{if } i = n-5, \\ n+2, & \text{if } i = n-3, \\ n+1, & \text{if } i = n-1, \\ \frac{1}{2}(5+i), & \text{if } i = \frac{1}{2}(n+7), \frac{1}{2}(n+15), \ldots, n-6, \\ \frac{1}{2}(1+i), & \text{if } i = \frac{1}{2}(n+11), \frac{1}{2}(n+19), \ldots, n-4, \\ \frac{1}{2}(n+1), & \text{if } i = n-2, \\ \frac{1}{2}(n+5), & \text{if } i = n. \end{cases}$$

Subcase 2.1. $n \equiv 7 \pmod 8$.

$$\phi(x) = \begin{cases} \frac{1}{2}(n+7+i), & \text{if } i = \frac{1}{2}(n+1), \frac{1}{2}(n+9), \ldots, n-3, \\ \frac{1}{2}(n+3+i), & \text{if } i = \frac{1}{2}(n+5), \frac{1}{2}(n+13), \ldots, n-1, \\ \frac{1}{2}(5+i), & \text{if } i = \frac{1}{2}(n+7), \frac{1}{2}(n+15), \ldots, n, \\ \frac{1}{2}(1+i), & \text{if } i = \frac{1}{2}(n+11), \frac{1}{2}(n+19), \ldots, n-2. \end{cases}$$

It is a routine procedure to verify that ϕ is a PSEML of G.

c) Let us define $G_n \cong C_4 \cup C_n \cup K_1$, $n \equiv 0 \pmod 4$, as a graph having

$$V(G_n) = \{u_i : 1 \leq i \leq 4\} \cup \{v_i : 1 \leq i \leq n\} \cup \{K_1\}$$

and

$$E(G_n) = \{u_i u_{i+1} : 1 \leq i \leq 3\} \cup \{u_4 u_1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}.$$

Next, define $\phi : V(G_n) \longrightarrow \{1, 2, \ldots, n+5\}$ as follows.

Case 1. $n = 4$.

$$(\phi(u_1), \phi(u_2), \phi(u_3), \phi(u_4)) = (1, 5, 3, 6), \ (\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)) = (2, 8, 4, 9), \ \phi(K_1) = 7.$$

Case 2. $n \geq 8$.

$$\phi(x) = \begin{cases} (1, \frac{1}{2}(n+6), 2, \frac{1}{2}(n+10)), & \text{if } x = (u_1, u_2, u_3, u_4), \\ \frac{1}{2}(i+7), & \text{if } x = v_i \text{ for } i = 1, 5, 9, \ldots, n-3, \\ \frac{1}{2}(i+3), & \text{if } x = v_i \text{ for } i = 3, 7, 11, \ldots, n-1, \\ \frac{1}{2}(n+8), & \text{if } x = v_n, \\ \frac{1}{4}(3n+16), & \text{if } x = K_1. \end{cases}$$

To label the remaining vertices, we consider the following two subcases.

Case 2.1 $n \equiv 0 \pmod 8$.

$$\phi(x) = \begin{cases} \frac{1}{2}(n+12+i), & \text{if } x = v_i \text{ for } i = 2, 6, 10, \ldots, \frac{n}{2}-6, \\ \frac{1}{2}(n+8+i), & \text{if } x = v_i \text{ for } i = 4, 8, 12, \ldots, \frac{n}{2}-4, \\ \frac{1}{2}(n+14+i), & \text{if } x = v_i \text{ for } i = \frac{n}{2}-2, \frac{n}{2}+2, \frac{n}{2}+6, \ldots, n-6, \\ \frac{1}{2}(n+10+i), & \text{if } x = v_i \text{ for } i = \frac{n}{2}, \frac{n}{2}+4, \frac{n}{2}+8, \ldots, n-4, \\ n+5, & \text{if } x = v_{n-2}. \end{cases}$$

Case 2.2 $n \equiv 4 \pmod 8$.

$$\phi(x) = \begin{cases} \frac{1}{2}(n+12), & \text{if } x = v_i \text{ for } i = 2, \\ \frac{1}{2}(n+12+i), & \text{if } x = v_i \text{ for } i = 4, 8, 12, \ldots, \frac{n}{2}-6, \\ \frac{1}{2}(n+8+i), & \text{if } x = v_i \text{ for } i = 6, 10, 14, \ldots, \frac{n}{2}-4, \\ \frac{1}{2}(n+14+i), & \text{if } x = v_i \text{ for } i = \frac{n}{2}-2, \frac{n}{2}+2, \frac{n}{2}+6, \ldots, n-4, \\ \frac{1}{2}(n+10+i), & \text{if } x = v_i \text{ for } i = \frac{n}{2}, \frac{n}{2}+4, \frac{n}{2}+8, \ldots, n-2. \end{cases}$$

It can be checked that ϕ is a PSEML of G.

d) For $n \in \{6, 10, 14, 18, 22, 26\}$, let $H_n = C_4 \cup C_n \cup K_1$, $V(H_n) = \{s_i : 1 \leq i \leq 4\} \cup \{t_i : 1 \leq i \leq n\} \cup \{w\}$ and $E(H_n) = \{s_i s_{i+1} : 1 \leq i \leq 3\} \cup \{s_4 s_1\} \cup \{t_i t_{i+1} : 1 \leq i \leq n-1\} \cup \{t_n t_1\} \cup \{ww\}$. It is simple to confirm that $\phi : V(H_n) \to \{1, 2, \ldots, n+5\}$ is a PSEML of H_n.

Case 1. $n = 6$. $(\phi(s_1), \phi(s_2), \phi(s_3), \phi(s_4)) = (1, 11, 2, 7)$, $(\phi(t_1), \phi(t_2), \ldots, \phi(t_6)) = (3, 8, 9, 6, 10, 4)$, and $\phi(w) = 5$.

Case 2. $n = 10$. $(\phi(s_1), \phi(s_2), \phi(s_3), \phi(s_4)) = (1, 15, 2, 9)$, $(\phi(t_1), \phi(t_2), \ldots, \phi(t_{10})) = (4, 10, 3, 12, 11, 8, 14, 7, 13, 5)$, and $\phi(w) = 6$.

Case 3. $n = 14$. $(\phi(s_1), \phi(s_2), \phi(s_3), \phi(s_4)) = (1, 19, 2, 11)$, $(\phi(t_1), \phi(t_2), \ldots, \phi(t_{14})) = (4, 12, 3, 15, 14, 9, 18, 10, 16, 8, 17, 5, 6, 13)$, and $\phi(w) = 7$.

Case 4. $n = 18$. $(\phi(s_1), \phi(s_2), \phi(s_3), \phi(s_4)) = (1, 23, 2, 13)$, $(\phi(t_1), \phi(t_2), \ldots, \phi(t_{18})) = (4, 14, 3, 20, 10, 17, 18, 11, 21, 12, 22, 9, 19, 7, 6, 16, 5, 15)$, and $\phi(w) = 8$.

Case 5. $n = 22$. $(\phi(s_1), \phi(s_2), \phi(s_3), \phi(s_4)) = (1, 27, 2, 15)$, $(\phi(t_1), \phi(t_2), \ldots, \phi(t_{22})) = (4, 16, 3, 24, 12, 21, 20, 11, 23, 14, 26, 13, 25, 10, 22, 8, 7, 19, 6, 18, 5, 17)$, and $f(w) = 9$.

Case 6. $n = 26$. $(\phi(s_1), \phi(s_2), \phi(s_3), \phi(s_4)) = (1, 31, 2, 17)$, $(\phi(t_1), \phi(t_2), \ldots, \phi(t_{26})) = (4, 18, 3, 28, 13, 30, 16, 29, 15, 27, 12, 26, 14, 23, 24, 11, 25, 9, 8, 22, 7, 21, 6, 20, 5, 19)$, and $\phi(w) = 10$.

e) The labeling ϕ in the proof of Theorem 3.3 is a PSEML of $2C_4 \cup C_n \cup K_1$ for $n \equiv 8, 12 \pmod{16}$. □

Corollary 3.3 is a direct consequence of applying first Theorem 3.1 and then Theorem 3.2 to the graphs in Theorem 3.5.

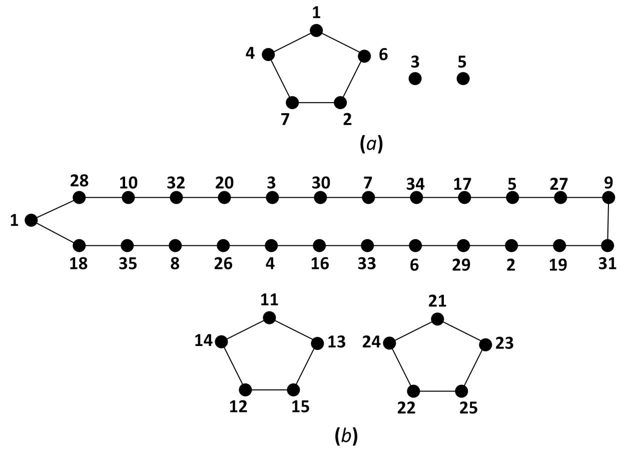

Figure 3. (a). A PSEML of $C_5 \cup 2K_1$. (b). A SEMTL of $C_{25} \cup 2C_5$ which is obtained by applying Theorem 3.1 to $C_5 \cup 2K_1$.

Corollary 3.3. *Let $m, r \geq 1$, $l \geq 2$, and $n \geq 4$ be positive integers such that m is odd and $l \equiv 2$ (mod 4). Then all the following graphs have zero* SEMD*.*
a) $m[C_{n(2r+1)} \cup C_{2r+1}]$*, if* $n \equiv 0\ (mod\ 2)$.
b) $m[C_{n(2r+1)} \cup 2C_{2r+1}]$*, if* $n \equiv 1\ (mod\ 2)$.
c) $m[C_{4(2r+1)} \cup C_{n(2r+1)} \cup C_{2r+1}]$*, if* $n \equiv 0\ (mod\ 4)$.
d) $m[C_{4(2r+1)} \cup C_{n(2r+1)} \cup C_{2r+1}]$*, if* $n \in \{6, 10, 14, 18, 22, 26\}$.
e) $lC_{4(2r+1)} \cup m[C_{n(2r+1)} \cup C_{2r+1}]$*, if* $n \equiv 8, 12\ (mod\ 16)$.

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