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On distance labelings of 2-regular graphs

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Abstract

Let G be a graph with $|V(G)|$ vertices and $\psi : V(G) \longrightarrow \{1, 2, 3, \cdots, |V(G)|\}$ be a bijective function. The weight of a vertex $v \in V(G)$ under ψ is $w_\psi(v) = \sum_{u \in N(v)} \psi(u)$. The function ψ is called a distance magic labeling of G, if $w_\psi(v)$ is a constant for every $v \in V(G)$. The function ψ is called an (a, d)-distance antimagic labeling of G, if the set of vertex weights is $a, a + d, a + 2d, \ldots, a + (|V(G)| - 1)d$. A graph that admits a distance magic (resp. an (a, d)-distance antimagic) labeling is called distance magic (resp. (a, d)-distance antimagic). In this paper, we characterize distance magic 2-regular graphs and (a, d)-distance antimagic some classes of 2-regular graphs.

Keywords: distance magic labeling, (a, d)-distance antimagic labeling, 2-regular graph
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1. Introduction

Let $G = G(V, E)$ be a graph without isolated vertices. A *vertex labeling* of G is a one-to-one function with domain the set of all vertices and co-domain the set $\{1, 2, \cdots, |V(G)|\}$. A *vertex weight* of a vertex v under a vertex labeling is the sum of all vertex labels of the vertices adjacent to v. If every vertex has the same vertex weights, then it is called a *distance magic labeling* of G. If all vertices have distinct vertex weights, then it is called a *distance antimagic labeling* of G. In particular, if vertex weights of all vertices are an arithmetic sequence with the first term a and a

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common difference d then it is called an *(a, d)-distance antimagic labeling* of G. Formally, these concepts can be stated as the following definitions.

Definition 1. *A distance magic* (DM) *labeling of a graph G is a vertex labeling ψ of G such that $|\{w_\psi(v) = \sum_{u \in N(v)} \psi(u) : v \in V(G)\}| = 1$, where $N(v) = \{u : uv \in E(G)\}$. A graph that admits a DM labeling is called a DM graph.*

Definition 2. *Let $a > 0$ and $d \geq 0$ be fixed integers. An (a, d)-distance antimagic* (DA) *labeling of a graph G is a vertex labeling ψ of G such that $\{w_\psi(v) = \sum_{u \in N(v)} \psi(u) : v \in V(G)\}$ is the set $\{a, a+d, a+2d, \ldots, a+(|V(G)|-1)d$. Any graph which admits such a labeling is called an (a, d)-DA graph*

The concept of a DM labeling of a graph independently was introduced by Vilfred [12] and Miller et al. [8]. Vilfred [12] called this labeling as a sigma labeling and Miller et al. [8] called it as an 1-vertex magic vertex labeling. The term a DM labeling for this concept introduced by Sugeng et al. [11]. Meanwhile, the notion of an (a, d)-DA labeling was introduced by Arumugam and Kamatchi [1], in 2012.

Several papers on DM labelings have been published. Many classes of graphs have been shown to be DM, see for instance [2, 8, 11, 12]. Additionally, a generalization of DM labeling to an any set $D \subseteq \{1, 2, \ldots, diam(G)\}$ is introduced in [9, 10]. Another generalization of DM labeling can be seen in [3]. Meanwhile, some results on DA labelings can be seen in, for instance, [1, 4, 5, 7]. For more results in these subjects, we refer the readers to Gallian's paper, dynamic survey of graph labelings [6]. In this paper, we study these labelings for 2-regular graphs.

2. DM labeling of 2-regular graphs

In [8], Miler et al. provided some necessary conditions for graphs to have no DM labeling. One of them is given in Lemma 2.1.

Lemma 2.1. [8] *Let G be a graph and $x, y \in V(G)$. If $|N(x) \cap N(y)| = |N(x)| - 1 = |N(y)| - 1$, then G is not DM.*

They also gave the following result.

Theorem 2.1. [8] *The graph C_m has a DM labeling iff $m = 4$.*

We now generalize this result as follows:

Theorem 2.2. *The 2-regular graph G is a DM graph iff $G = tC_4$ for any positive integer t.*

Proof. Suppose G contains a component C_n where $n \neq 4$. Then by Lemma 2.1, G is not DM. Conversely, define $G = tC_4$ as a graph with $V(G) = \{x_{i,j} : 1 \leq i \leq t, 1 \leq j \leq 4\}$ and $E(G) = \{x_{i,j}x_{i,j+1} : 1 \leq i \leq t, 1 \leq j \leq 3\} \cup \{x_{i,4}x_{i,1} : 1 \leq i \leq t\}$. Next, Let $\mathfrak{A} = \{\{x_{i,1}, x_{i,3}\}, \{x_{i,2}, x_{i,4}\} : 1 \leq i \leq t\}$ and $\mathfrak{B} = \{\{i, 4t+1-i\} : 1 \leq i \leq 2t\}$. It is clear that $\cup_{i=1}^{2t}\{i, 4t+1-i\}$ is a partition of $\{1, 2, 3, \ldots, 4t\}$ and $|\mathfrak{A}| = |\mathfrak{B}|$. Also, it can be checked that any bijective function $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a DM labeling of G. $\square$

3. DA labeling of 2-regular graphs

In [1], Arumugam and Kamatchi proved the existence of an (a, d)-DA labeling for some graphs and they provided the following results.

Lemma 3.1. [1] *If G is an (a, d)-DA graph with n vertices, then*

$$d \leq \frac{2n\Delta - \Delta(\Delta - 1) - \delta(\delta + 1)}{2(n-1)}.$$

Corollary 3.1. [1] *Let G be a 2-regular graph with n vertices. If G is (a, d)-DA, then $a = \frac{n+3}{2}$ and $d = 1$.*

Theorem 3.1. [1] *Let $t \geq 1$ be an integer. The graph C_n is (a, d)-DA iff $n = 2t + 1$ and $d = 1$.*

They also posed the following problem.

Problem 1. [1] *Find the necessary and sufficient conditions such that disconnected 2-regular graphs are (a, d)-DA.*

In this section, we give a partial answer to Problem 1 by characterizing some classes of disconnected 2-regular (a, d)-DA graphs.

It is clear that if a graph has two vertices u and v such that $N(u) = N(v)$, then it is not a DA graph. Thus, a forbidden subgraph for 2-regular graphs to be DA is C_4. By this fact and Corollary 3.1, we have the following result.

Lemma 3.2. *Let $t > 1$ be an integer and $G = \cup_{i=1}^{t} C_{n_i}$. If G is (a, d)-DA, then $\sum_{i=1}^{t} n_i$ is odd, $n_i \neq 4$ for $1 \leq i \leq t$, and $d = 1$.*

To present the next three theorems, we use the following $3 \times (2t + 1)$ matrix $\mathbf{A}$,

$$\mathbf{A} = [a_{i,j}] = \begin{bmatrix} 1 & 2 & \dots & t & t+1 & \dots & 2t & 2t+1 \\ t+2 & t+3 & \dots & 2t+1 & 1 & \dots & t & t+1 \\ 2t & 2t-2 & \dots & 2 & 2t+1 & \dots & 3 & 1 \end{bmatrix}.$$

We can check that $\{a_{1,i} + a_{2,i} : 1 \leq i \leq 2t+1\} = \{a_{1,i} + a_{3,i} : 1 \leq i \leq 2t+1\} = \{a_{2,i} + a_{3,i} : 1 \leq i \leq 2t+1\} = \{t+2, t+3, \dots, 3t+1, 3t+2\}$. As we show later, this property of $\mathbf{A}$ preserves the antimagic properties of (a, d)-DA labelings of 2-regular graphs mC_n. The matrix $\mathbf{A}$ is obtained by increasing 1 each entry of the following $3 \times (2t + 1)$ Kotzig array.

$$\begin{bmatrix} 0 & 1 & \dots & t-1 & t & \dots & 2t-1 & 2t \\ t+1 & t+2 & \dots & 2t & 0 & \dots & t-1 & t \\ 2t-1 & 2t-3 & \dots & 1 & 2t & \dots & 2 & 0 \end{bmatrix}.$$

Additionally, we use the notation $b + \{a_i : 1 \leq i \leq t\} = \{b + a_i : 1 \leq i \leq t\}$. Also, in each figure, the numbers at the outside of the cycles are the labels of the vertices and the bold number inside of the cycles are the weights of the corresponding vertices.

The next theorem gives necessary and sufficient conditions of the disjoint union of odd numbers of C_n to be an (a, d)-DA graph.

Theorem 3.2. *Let m and n be positive integers. The graph mC_n is (a,d)-DA iff m,n are odd and $d=1$.*

Proof. Let mC_n be an (a,d)-DA graph. Then by Corollary 3.1, $a=\frac{1}{2}(mn+3)$ and $d=1$. Hence, m and n should be odd integers. Conversely, let m and n be odd integers. For $1\le i\le m$, define mC_n as a graph with

$$V(mC_n)=\{u_{i,j}:1\le j\le n\}$$

and

$$E(mC_n)=\{u_{i,j}u_{i,j+1}:1\le j\le n-1\}\cup\{u_{i,n}u_{i,1}\}.$$

Set $m=2t+1$ and, for $1\le i\le 2t+1$, define a vertex labeling f of $(2t+1)C_n$ as follows:

$$f(u_{i,j})=\begin{cases}\frac{1}{4}(j-1)(2t+1)+a_{1,i}, & \text{if } j\equiv 1 \text{ (mod 4)},\\ \frac{1}{4}(2n-1+j)(2t+1)+a_{2,i}, & \text{if } j\equiv 3 \text{ (mod 4)},\\ \frac{1}{2}(n-1)(2t+1)+a_{3,i}, & \text{if } j=n-1,\\ \frac{1}{4}(n-1+j)(2t+1)+a_{1,i}, & \text{if } n\equiv 1 \text{ (mod 4) and } n-1\ne j\equiv 0 \text{ (mod 4)},\\ \frac{1}{4}(3n-1+j)(2t+1)+a_{2,i}, & \text{if } n\equiv 1 \text{ (mod 4) and } j\equiv 2 \text{ (mod 4)},\\ \frac{1}{4}(3n-1+j)(2t+1)+a_{2,i}, & \text{if } n\equiv 3 \text{ (mod 4) and } j\equiv 0 \text{ (mod 4)},\\ \frac{1}{4}(n-1+j)(2t+1)+a_{1,i}, & \text{if } n\equiv 3 \text{ (mod 4) and } n-1\ne j\equiv 2 \text{ (mod 4)}.\end{cases}$$

The vertex weights of all vertices are as follows:
For $n-2, n\ne j\equiv 1,3$ (mod 4),

$$w_f(u_{i,j})=\frac{1}{2}(2n-1+j)(2t+1)+\mathfrak{F},$$

for $j\equiv 0,2$ (mod 4),

$$w_f(u_{i,j})=\frac{1}{2}(n-1+j)(2t+1)+\mathfrak{F},$$

for $j=n-2$,

$$w_f(u_{i,j})=\frac{1}{2}(3n-3)(2t+1)+\mathfrak{F}, \text{ and}$$

for $j=n$,

$$w_f(u_{i,j})=\frac{1}{2}(n-1)(2t+1)+\mathfrak{F},$$

where $\mathfrak{F}=\{t+2,t+3,\dots,3t+2\}$. Thus, for $1\le i\le 2t+1$, $\{w_f(u_{i,j}):1\le j\le n\}=\{\frac{(2t+1)n+3}{2},\frac{(2t+1)n+5}{2}\dots,\frac{3(2t+1)n+1}{2}\}$. So, f is an $(\frac{mn+3}{2},1)$-DA labeling of mC_n. □

Figure 1. shows the labeling in the proof of Theorem 3.2 for $m=3$ and $n=11$.

Theorem 3.3. *The graph $tC_6\cup C_3$ is a $(3t+3,1)$-DA graph for every positive integer t.*

Proof. First, define $tC_6\cup C_3$ as a graph with $V(tC_6\cup C_3)=\{x_{i,1},x_{i,2},x_{i,3},y_{i,1},y_{i,2},y_{i,3}:1\le i\le t\}\cup\{z_1,z_2,z_3\}$ and $E(tC_6\cup C_3)=\{x_{i,1}y_{i,1},x_{i,2}y_{i,2},x_{i,3j}y_{i,3}:1\le i\le t\}\cup\{y_{i,1}x_{i,2},x_{i,2}y_{i,3}:1\le i\le t\}\cup\{y_{i,3}x_{i,1}:1\le i\le t\}\cup\{z_1z_2,z_2z_3,z_3z_1\}$.

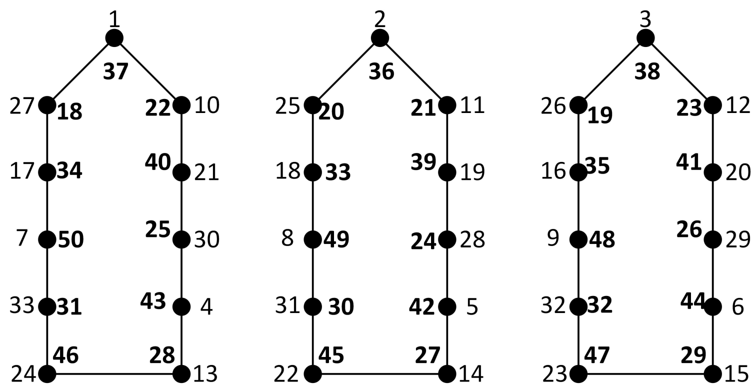

Figure 1. A (18, 1)-DA labeling of $3C_{11}$.

Next, for $1 \leq i \leq t$, consider $g : V(tC_6 \cup C_3) \longrightarrow \{1, 2, 3, \ldots, 6t + 3\}$ which is defined by

$$g(u) = \begin{cases} (j-1)(2t+1) + a_{j,i}, & \text{if } u = x_{i,j} \text{ and } 1 \leq j \leq 3, \\ (j-1)(2t+1) + a_{j,2t+2-i}, & \text{if } u = y_{i,j} \text{ and } 1 \leq j \leq 3, \\ (j-1)(2t+1) + a_{j,t+1}, & \text{if } u = z_j \text{ and } 1 \leq j \leq 3, \end{cases}$$

where $a_{i,j}$ is the entry of the matrix $\mathbf{A}$.

Clearly, f is a bijective function. We now can check that $\{w_g(x_{i,1}) : 1 \leq i \leq t\} = \{6t + 4, 6t + 5, 6t + 6, \ldots, 7t + 3\}$, $\{w_g(x_{i,2}) : 1 \leq i \leq t\} = \{3t + 5, 3t + 7, 3t + 9, \ldots, 5t + 3\}$, $\{w_g(x_{i,3}) : 1 \leq i \leq t\} = \{7t + 5, 7t + 6, 7t + 7, \ldots, 8t + 4\}$, $\{w_g(y_{i,1}) : 1 \leq i \leq t\} = \{3t + 4, 3t + 6, 3t + 8, \ldots, 5t + 2\}$, $\{w_g(y_{i,2}) : 1 \leq i \leq t\} = \{8t + 6, 8t + 7, 8t + 8, \ldots, 9t + 5\}$, $\{w_g(y_{i,3}) : 1 \leq i \leq t\} = \{5t + 4, 5t + 5, 5t + 6, \ldots, 6t + 3\}$, and $\{w_g(z_j) : 1 \leq j \leq 3\} = \{3t + 3, 7t + 4, 8t + 5\}$. Hence, g is a $(3t + 3, 1)$-DA labeling of $tC_6 \cup C_3$. □

Theorem 3.4. *The graph $tC_{10} \cup C_5$ is a $(5t + 4, 1)$-DA graph for every positive integer t.*

Proof. Let $V(tC_{10} \cup C_5) = \{x_{i,1}, x_{i,2}, x_{i,2}, x_{i,4}, x_{i,5}, y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}, y_{i,5} : 1 \leq i \leq t\} \cup \{z_1, z_2, z_3, z_4, z_5\}$ and $E(tC_{10} \cup C_5) = \{x_{i,1}y_{i,1}, x_{i,2}y_{i,2}, x_{i,3}y_{i,3}, x_{i,4}y_{i,4}, x_{i,5}y_{i,5} : 1 \leq i \leq t\} \cup \{y_{i,1}x_{i,2}, x_{i,2}y_{i,3}, x_{i,3}y_{i,4}, x_{i,4}y_{i,5} : 1 \leq i \leq t\} \cup \{y_{i,5}x_{i,1} : 1 \leq i \leq t\} \cup \{z_1z_2, z_2z_3, z_3z_4, z_4z_5, z_5z_1\}$.

For $1 \leq i \leq t$, define a bijective function $h : V(tC_{10} \cup C_5) \longrightarrow \{1, 2, 3, \ldots, 10t + 5\}$ as the following formula:

$$h(u) = \begin{cases} a_{1,i}, & \text{if } u = x_{i,1}, \\ 6t+3+a_{2,i}, & \text{if } u = x_{i,2}, \\ 2t+1+a_{1,i}, & \text{if } u = x_{i,3}, \\ 8t+4+a_{2,i}, & \text{if } u = x_{i,4}, \\ 4t+2+a_{3,i}, & \text{if } u = x_{i,5}, \\ a_{1,t+i}, & \text{if } u = y_{i,1}, \\ 6t+3+a_{2,t+i}, & \text{if } u = y_{i,2}, \\ 2t+1+a_{1,t+i}, & \text{if } u = y_{i,3}, \\ 8t+4+a_{2,t+i}, & \text{if } u = y_{i,4}, \\ 4t+2+a_{3,t+i}, & \text{if } u = y_{i,5}, \\ a_{1,2t+1}, & \text{if } u = z_1, \\ 8t+4+a_{2,2t+1}, & \text{if } u = z_2, \\ 6t+3+a_{2,2t+1}, & \text{if } u = z_3, \\ 4t+2+a_{3,2t+1}, & \text{if } u = z_4, \\ 2t+1+a_{1,2t+1}, & \text{if } u = z_5. \end{cases}$$

Notice that $a_{i,j}$ in this formula is the entry of the matrix $\mathbf{A}$. Under the labeling h, we can verify that $\{w_h(x_{i,1}) : 1 \le i \le t\} = \{6t+5, 6t+6, \ldots, 7t+3, 7t+4\}$, $\{w_h(x_{i,2}) : 1 \le i \le t\} = \{7t+5, 7t+7, \ldots, 9t+1, 9t+3\}$, $\{w_h(x_{i,3}) : 1 \le i \le t\} = \{9t+6, 9t+8, \ldots, 11t+2, 11t+4\}$, $\{w_h(x_{i,4}) : 1 \le i \le t\} = \{11t+7, 11t+9, \ldots, 13t+3, 13t+5\}$, and $\{w_h(x_{i,5}) : 1 \le i \le t\} = \{13t+9, 13t+10, \ldots, 14t+7, 14t+8\}$. Also, $\{w_h(y_{i,1}) : 1 \le i \le t\} = \{7t+6, 7t+8, \ldots, 9t+2, 9t+4\}$, $\{w_h(y_{i,2}) : 1 \le i \le t\} = \{9t+7, 9t+9, \ldots, 11t+3, 11t+5\}$, $\{w_h(y_{i,3}) : 1 \le i \le t\} = \{11t+8, 11t+10, \ldots, 13t+4, 13t+6\}$, $\{w_h(y_{i,4}) : 1 \le i \le t\} = \{14t+9, 14t+10, \ldots, 15t+7, 15t+8\}$, $\{w_h(y_{i,5}) : 1 \le i \le t\} = \{5t+4, 5t+5, \ldots, 6t+2, 6t+3\}$, and $\{w_h(z_j) : 1 \le j \le 5\} = \{6t+4, 9t+5, 11t+6, 13t+7, 13t+8\}$. Hence, h is a $(5t+4, 1)$-DA labeling of $tC_{10} \cup C_5$. □

In the next results, we characterize the graphs $tC_{2n} \cup C_n$ to be (a, d)-DA for $1 \le t \le 3$.

Theorem 3.5. *The graph $C_{2n} \cup C_n$ is (a, d)-DA iff $n \ge 3$ is odd and $d = 1$.*

Proof. By Corollary 3.1, if $C_{2n} \cup C_n$ is an (a, d)-DA graph then $a = \frac{3n+3}{2}$, $d = 1$ and thus n is odd. Conversely, we will show that $C_{2n} \cup C_n$ admits a $(\frac{3n+3}{2}, 1)$-DA labeling. The graphs $C_6 \cup C_3$ and $C_{10} \cup C_5$ admit the labeling by Theorems 3.3 and 3.4, respectively. For odd $n \ge 7$, let $V(C_{2n} \cup C_n) = \{x_i, y_i, z_i : 1 \le i \le n\}$ and $E(C_{2n} \cup C_n) = \{x_i y_i : 1 \le i \le n\} \cup \{y_i x_{i+1} : 1 \le i \le n-1\} \cup \{y_n x_1\} \cup \{z_i z_{i+1} : 1 \le i \le n-1\} \cup \{z_n z_1\}$. Next, define $\psi : V(C_{2n} \cup C_n) \longrightarrow \{1, 2, 3, \ldots, 3n\}$ as follows:

$$\psi(u)=\begin{cases}\frac{1}{2}(3i-1), & \text{if } u=x_i,\ n\neq i\equiv 1 \text{ (mod 2)},\\ \frac{1}{2}(3n+1), & \text{if } u=x_n,\\ \frac{3}{2}(n+1+i), & \text{if } u=x_i,\ i\equiv 0 \text{ (mod 2)},\\ \frac{1}{2}(3i+1), & \text{if } u=y_i,\ n\neq i\equiv 1 \text{ (mod 2)},\\ \frac{3}{2}(n+1), & \text{if } u=y_n,\\ \frac{1}{2}(3n-1+3i), & \text{if } u=y_i,\ i\equiv 0 \text{ (mod 2)},\\ \frac{3}{4}(i+3), & \text{if } u=z_i,\ i\equiv 1 \text{ (mod 4)},\\ \frac{1}{4}(6n+5+3i), & \text{if } u=z_i,\ i\equiv 3 \text{ (mod 4)},\\ \frac{1}{2}(3n-1), & \text{if } u=z_{n-1},\\ \frac{3}{4}(n+3+i), & \text{if } u=z_i,\ n\equiv 1 \text{ (mod 4) and } n-1\neq i\equiv 0 \text{ (mod 4)},\\ \frac{1}{4}(9n+5+3i), & \text{if } u=z_i,\ n\equiv 1 \text{ (mod 4) and } i\equiv 2 \text{ (mod 4)},\\ \frac{1}{4}(9n+5+3i), & \text{if } u=z_i,\ n\equiv 3 \text{ (mod 4) and } i\equiv 0 \text{ (mod 4)},\\ \frac{3}{4}(n+3+i), & \text{if } u=z_i,\ n\equiv 3 \text{ (mod 4) and } n-1\neq i\equiv 2 \text{ (mod 4)}.\end{cases}$$

The weights of all vertices are given by

$$w_\psi(u)=\begin{cases}\frac{1}{2}(3n+7), & \text{if } u=x_1,\\ \frac{1}{2}(3n-3+6i), & \text{if } u=x_i,\ 2\leq i\leq n-1,\\ \frac{1}{2}(9n-1), & \text{if } u=x_n,\\ \frac{1}{2}(3n+5+6i), & \text{if } u=y_i,\ 1\leq i\leq n-2,\\ \frac{1}{2}(9n+1), & \text{if } u=y_{n-1},\\ \frac{1}{2}(3n+3), & \text{if } u=y_n,\\ \frac{1}{2}(3n+7+3i), & \text{if } u=z_i,\ i \text{ is even},\\ \frac{1}{2}(6n+7+3i), & \text{if } u=z_i,\ n-2,n\neq i \text{ is odd},\\ \frac{1}{2}(9n-3), & \text{if } u=z_{n-2},\\ \frac{1}{2}(3n+5), & \text{if } u=z_n.\end{cases}$$

Thus, ψ is a $(\frac{3n+3}{2},1)$-DA of $C_{2n}\cup C_n$. □

Next, we consider the graph $2C_{2n}\cup C_n$ with $V(2C_{2n}\cup C_n)=\{x_{1,j},x_{2,j},y_{1,j},y_{2,j}:1\leq j\leq n\}\cup\{z_j:1\leq j\leq n\}$ and $E(2C_{2n}\cup C_n)=\{x_{1,j}y_{1,j},x_{2,j}y_{2,j}:1\leq j\leq n\}\cup\{y_{1,j}x_{1,j+1},y_{2,j}x_{2,j+1}:1\leq j\leq n-1\}\cup\{y_{1,n}x_{1,1},y_{2,n}x_{2,1}\}\cup\{z_jz_{j+1}:1\leq j\leq n-1\}\cup\{z_nz_1\}$. By Theorems 3.3 and 3.4, $2C_{2n}\cup C_n$ is $(a,1)$-DA for $n=3$ and 5. So, it is enough to consider $n\geq 7$.

Theorem 3.6. *For $n\geq 7$, the graph $2C_{2n}\cup C_n$ is (a,d)-DA iff n is odd and $d=1$.*

Proof. By Corollary 3.1, n is odd and $d=1$, if $2C_{2n}\cup C_n$ is an (a,d)-DA graph. Conversely, for odd $n\geq 7$ and $i=1,2$, define $f:V(2C_{2n}\cup C_n)\longrightarrow\{1,2,3,\ldots,5n\}$ as follows:

$$f(u)=\begin{cases}\frac{1}{2}(5j-7+4i), & \text{if } u=x_{i,j},\ n\neq j\equiv 1 \text{ (mod 2)},\\ \frac{1}{2}(5n+1+2i), & \text{if } u=x_{i,n},\\ \frac{1}{2}(5n+5j+9-6i), & \text{if } u=x_{i,j},\ j\equiv 0 \text{ (mod 2)},\\ \frac{1}{2}(5j-5+4i), & \text{if } u=y_{i,j},\ n\neq j\equiv 1 \text{ (mod 2)},\\ \frac{1}{2}(5n-3+2i), & \text{if } u=y_{i,n},\\ \frac{1}{2}(5n+5j+11-6i), & \text{if } u=y_{i,j},\ j\equiv 0 \text{ (mod 2)}.\end{cases}$$

$$f(z_j) = \begin{cases} \frac{1}{4}(5j+15), & \text{if } j \equiv 1 \text{ (mod 4)},\\ \frac{1}{4}(10n+7+5j), & \text{if } j \equiv 3 \text{ (mod 4)},\\ \frac{1}{2}(5n-3), & \text{if } j = n-1,\\ \frac{1}{4}(5n+15+5j), & \text{if } n \equiv 1 \text{ (mod 4) and } n-1 \neq j \equiv 0 \text{ (mod 4)},\\ \frac{1}{4}(15n+7+5j), & \text{if } n \equiv 1 \text{ (mod 4) and } j \equiv 2 \text{ (mod 4)},\\ \frac{1}{4}(15n+7+5j), & \text{if } n \equiv 3 \text{ (mod 4) and } j \equiv 0 \text{ (mod 4)},\\ \frac{1}{4}(5n+15+5j), & \text{if } n \equiv 3 \text{ (mod 4) and } n-1 \neq j \equiv 2 \text{ (mod 4)}. \end{cases}$$

Clearly, f is a bijective function. Also it is easy to verify that the weights of all vertices are as follows:

$$w_f(u) = \begin{cases} \frac{1}{2}(5n+3), & \text{if } u = x_{1,1},\\ \frac{1}{2}(5n-1+10j), & \text{if } u = x_{1,j},\ j \neq 1,\\ \frac{1}{2}(15n-1), & \text{if } u = y_{1,n-1},\\ \frac{1}{2}(5n+5), & \text{if } u = y_{1,n},\\ \frac{1}{2}(5n+5+10j), & \text{if } u = y_{1,j},\ j \neq n-1, n,\\ \frac{1}{2}(5n+9), & \text{if } u = x_{2,1},\\ \frac{1}{2}(15n-5), & \text{if } u = x_{2,n},\\ \frac{1}{2}(5n-3+10j), & \text{if } u = x_{2,j},\ j \neq 1, n,\\ \frac{1}{2}(15n-3), & \text{if } u = y_{2,n-1},\\ \frac{1}{2}(5n+11), & \text{if } u = y_{2,n},\\ \frac{1}{2}(5n+3+10j), & \text{if } u = y_{2,j},\ j \neq n-1, n,\\ \frac{1}{2}(15n-7), & \text{if } u = z_{n-2},\\ \frac{1}{2}(5n+7), & \text{if } u = z_n,\\ \frac{1}{2}(10n+11+5j), & \text{if } u = z_j,\ n-2, n \neq j \equiv 1\ (mod\ 2),\\ \frac{1}{2}(5n+11+5j), & \text{if } u = z_j,\ j \equiv 0\ (mod\ 2). \end{cases}$$

Hence, f is a $(\frac{5n+3}{2}, 1)$-DA labeling of $2C_{2n} \cup C_n$. □

Figure 2. shows the labeling defined in the proof of Theorem 3.6 for $n = 7$.

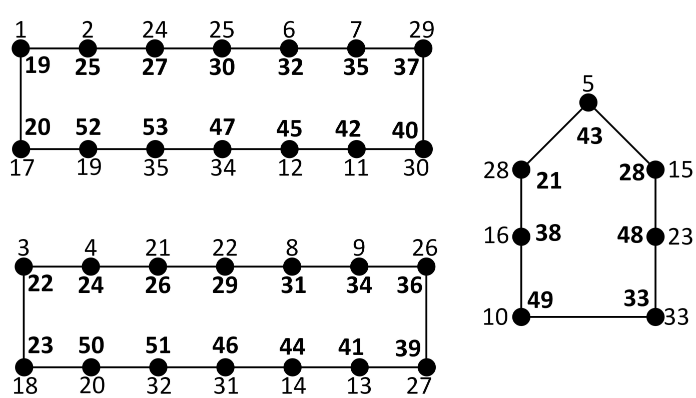

Figure 2. A (19, 1)-DA labeling of $2C_{14} \cup C_7$.

Now, we consider the graphs $3C_{2n} \cup C_n$, $n \geq 7$, where $V(3C_{2n} \cup C_n) = \{x_{1,j}, x_{2,j}, x_{3,j}, y_{1,j}, y_{2,j}, y_{3,j} : 1 \leq j \leq n\} \cup \{z_j : 1 \leq j \leq n\}$ and $E(2C_{2n} \cup C_n) = \{x_{1,j}y_{1,j}, x_{2,j}y_{2,j}, x_{3,j}y_{3,j} : 1 \leq j \leq n\} \cup \{y_{1,j}x_{1,j+1}, y_{2,j}x_{2,j+1}, y_{3,j}x_{3,j+1} : 1 \leq j \leq n-1\} \cup \{y_{1,n}x_{1,1}, y_{2,n}x_{2,1}, y_{3,n}x_{3,1}\} \cup \{z_j z_{j+1} : 1 \leq j \leq n-1\} \cup \{z_n z_1\}$.

Theorem 3.7. *For $n \geq 7$, the graph $3C_{2n} \cup C_n$ is (a,d)-DA iff n is odd and $d = 1$.*

Proof. From Corollary 3.1, if $3C_{2n} \cup C_n$ is an (a,d)-DA graph then n is odd and $d = 1$, Conversely, for odd $n \geq 7$ and $i = 1,2,3$, define $g : V(3C_{2n} \cup C_n) \longrightarrow \{1,2,3,\ldots,7n\}$ as follows:

$$g(u) = \begin{cases} \frac{1}{2}(7j-7+2i), & \text{if } u = x_{i,j},\ n \neq j \equiv 1 \text{ (mod 2)}, \\ \frac{1}{2}(7n+9-4i), & \text{if } u = x_{i,n}, \\ \frac{1}{2}(7n+7j+1+2i), & \text{if } u = x_{i,j},\ j \equiv 0 \text{ (mod 2)}, \\ \frac{1}{2}(7j+1+2i), & \text{if } u = y_{i,j},\ n \neq j \equiv 1 \text{ (mod 2)}, \\ \frac{1}{2}(7n+7-4i), & \text{if } u = y_{i,n}, \\ \frac{1}{2}(7n+7j-5+2i), & \text{if } u = y_{i,j},\ j \equiv 0 \text{ (mod 2)}. \end{cases}$$

$$g(z_j) = \begin{cases} \frac{1}{4}(7j+9), & \text{if } j \equiv 1 \text{ (mod 4)}, \\ \frac{1}{4}(14n-3+7j), & \text{if } j \equiv 3 \text{ (mod 4)}, \\ \frac{1}{2}(7n+7), & \text{if } j = n-1, \\ \frac{1}{4}(7n+9+7j), & \text{if } n \equiv 1 \text{ (mod 4) and } n-1 \neq j \equiv 0 \text{ (mod 4)}, \\ \frac{1}{4}(21n-3+7j), & \text{if } n \equiv 1 \text{ (mod 4) and } j \equiv 2 \text{ (mod 4)}, \\ \frac{1}{4}(21n-3+7j), & \text{if } n \equiv 3 \text{ (mod 4) and } j \equiv 0 \text{ (mod 4)}, \\ \frac{1}{4}(7n+9+7j), & \text{if } n \equiv 3 \text{ (mod 4) and } n-1 \neq j \equiv 2 \text{ (mod 4)}. \end{cases}$$

Clearly, g is a bijective function. Also it is easy to verify that, for $i = 1,2,3$,

$$w_g(u) = \begin{cases} \frac{1}{2}(7n+15-2i), & \text{if } u = x_{i,1}, \\ \frac{1}{2}(21n-5-2i), & \text{if } u = x_{i,n}, \\ \frac{1}{2}(7n+14j-11+4i), & \text{if } u = x_{i,j},\ j \neq 1, n, \\ \frac{1}{2}(7n+9-2i), & \text{if } u = y_{i,n}, \\ \frac{1}{2}(21n+3-2i), & \text{if } u = y_{i,n-1}, \\ \frac{1}{2}(7n+14j+1+4i), & \text{if } u = y_{i,j},\ j \neq n-1, n, \\ \frac{1}{2}(21n-5), & \text{if } u = z_{n-2}, \\ \frac{1}{2}(7n+15), & \text{if } u = z_n, \\ \frac{1}{2}(14n+3+7j), & \text{if } u = z_j,\ n-2, n \neq j \equiv 1\ (mod\ 2), \\ \frac{1}{2}(7n+3+7j), & \text{if } u = z_j,\ j \equiv 0\ (mod\ 2). \end{cases}$$

Hence, g is a $(\frac{7n+3}{2}, 1)$-DA labeling of $3C_{2n} \cup C_n$. □

Based on Theorems 3.3 – 3.7, we propose the following conjecture:

Conjecture 1. *For every positive integer t, the graph $tC_{2n} \cup C_n$ is (a,d)-DA if and only if n is odd and $d = 1$.*

Next, we consider two classes of 2-regular graphs with two components.

Theorem 3.8. *For $n \geq 7$, the graph $C_6 \cup C_n$ is (a,d)-DA iff n is odd and $d = 1$.*

Proof. As an immediate consequence of Corollary 3.1, if $C_6 \cup C_n$ is an (a,d)-DA graph then n is odd and $d = 1$. Next, for odd $n \geq 7$, let $V(C_6 \cup C_n) = \{u_1, u_2, u_3, u_4, u_5, u_6\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(C_6 \cup C_n) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_1\} \cup \{v_jv_{j+1} : 1 \leq j \leq n-1\} \cup \{v_nv_1\}$. Define $h : V(C_6 \cup C_n) \longrightarrow \{1, 2, 3, \ldots, n+6\}$ as follows:
Case $n \equiv 1 \pmod 4$

$h([u_1, u_2, u_3, u_4, u_5, u_6]) = [\frac{n-1}{2}, \frac{3n+17}{4}, \frac{n+3}{2}, \frac{3n+21}{4}, \frac{n+5}{2}, \frac{3n+13}{4}]$.

$$h(v_j) = \begin{cases} \frac{1}{4}(j+3), & \text{if } n-4, n \neq j \equiv 1 \pmod 4, \\ \frac{1}{4}(3n+9), & \text{if } j = n-4, \\ \frac{1}{4}(2n+11+j), & \text{if } n-2 \neq j \equiv 3 \pmod 4, \\ \frac{1}{4}(n+3), & \text{if } j = n-2. \end{cases}$$

Sub case $n \equiv 1 \pmod 8$

If $n = 9$, $h([v_1, v_2, v_3, \ldots, v_9]) = [1, 15, 13, 3, 2, 9, 14, 8, 5]$. If $n \geq 17$,

$$h(v_j) = \begin{cases} \frac{1}{4}(n-3+j), & \text{if } n-7 \neq j \equiv 2 \pmod 8, \\ \frac{1}{2}(n+1), & \text{if } j = n-7, \\ \frac{1}{4}(3n+21+j), & \text{if } n-5 \neq j \equiv 4 \pmod 8, \\ n+5, & \text{if } j = n-5, \\ \frac{1}{4}(n+5+j), & \text{if } n-3 \neq j \equiv 6 \pmod 8, \\ \frac{1}{2}(n-5), & \text{if } j = n-3, \\ \frac{1}{4}(3n+29+j), & \text{if } n-9, n-1 \neq j \equiv 0 \pmod 8, \\ n+6, & \text{if } j = n-9, \\ n+4, & \text{if } j = n-1, \\ \frac{1}{4}(3n+29), & \text{if } j = n. \end{cases}$$

Sub case $n \equiv 5 \pmod 8$

$$h(v_j) = \begin{cases} \frac{1}{4}(n-3+j), & \text{if } j \equiv 2 \pmod 8, \\ \frac{1}{4}(3n+29+j), & \text{if } n-1 \neq j \equiv 4 \pmod 8, \\ n+4, & \text{if } j = n-1, \\ \frac{1}{4}(n+5+j), & \text{if } n-7 \neq j \equiv 6 \pmod 8, \\ \frac{1}{2}(n+1), & \text{if } j = n-7, \\ \frac{1}{4}(3n+21+j), & \text{if } n-5 \neq j \equiv 0 \pmod 8, \\ n+6, & \text{if } j = n-5, \\ \frac{1}{4}(3n+25), & \text{if } j = n. \end{cases}$$

It can be checked that h is a bijective function. When $n = 9$, $\{w_h(u_i) : 1 \leq j \leq 6\} = \{10, 11, 13, 21, 22, 23\}$ and $\{w_h(v_j) : 1 \leq j \leq 9\} = \{9, 12, 14, 15, 16, 17, 18, 19, 20\}$. When $n \neq 9$, $w_h(v_2) = \frac{n+9}{2}$, $w_h(u_3) = \frac{3n+19}{2}$, and the weights of other vertices are consecutive integers from $\frac{n+11}{2}$ to $\frac{3n+17}{2}$.
Case $n \equiv 3 \pmod 4$

$h([u_1, u_2, u_3, u_4, u_5, u_6]) = [\frac{n+1}{2}, \frac{3n+11}{4}, \frac{n+3}{2}, \frac{3n+15}{4}, \frac{n+5}{2}, \frac{3n+19}{4}]$.

$$h(v_j) = \begin{cases} \frac{1}{4}(j+3), & \text{if } j \equiv 1 \text{ (mod 4)}, \\ \frac{1}{4}(2n+15+j), & \text{if } n-4, n \neq j \equiv 3 \text{ (mod 4)}, \\ \frac{1}{4}(3n+23), & \text{if } j = n-4, \\ \frac{1}{4}(3n+27), & \text{if } j = n, \\ \frac{1}{4}(n+3+j), & \text{if } n-1 \neq j \equiv 2 \text{ (mod 4)}, \\ \frac{1}{2}(n+7), & \text{if } j = n-1, \\ \frac{1}{4}(3n+27+j), & \text{if } j \equiv 0 \text{ (mod 4)}. \end{cases}$$

It can be checked that h is a bijective function, $w_h(v_n) = \frac{n+9}{2}$, $w_h(v_{n-2}) = \frac{3n+19}{2}$, and the weights of the remaining vertices are consecutive integers from $\frac{n+11}{2}$ to $\frac{3n+17}{2}$. □

Theorem 3.9. *For $n \geq 5$, the graph $C_8 \cup C_n$ is (a, d)-DA iff n is odd and $d = 1$.*

Proof. As a direct consequence of Corollary 3.1, if $C_8 \cup C_n$ is (a, d)-DA then n is odd and $d = 1$. Conversely, For odd $n \geq 5$, let $V(C_8 \cup C_n) = \{u_i : 1 \leq i \leq 8\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(C_8 \cup C_n) = \{u_i u_{i+1} : 1 \leq i \leq 7\} \cup \{u_8 u_1\} \cup \{v_j v_{j+1} : 1 \leq j \leq n-1\} \cup \{v_n v_1\}$. Define $f : V(C_8 \cup C_n) \longrightarrow \{1, 2, 3, \ldots, n+8\}$ as follows:

$$f([u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8]) = [1, 3, \frac{n+9}{2}, \frac{n+15}{2}, 2, 5, \frac{n+13}{2}, \frac{n+17}{2}].$$

Case $n \equiv 1$ (mod 4)

When $n = 5$, $f([v_1, v_2, v_3, v_4, v_5]) = [4, 6, 8, 13, 12]$. When $n \geq 9$ define f as the following formula:

$$f(v_j) = \begin{cases} 4, & \text{if } j = 1, \\ \frac{1}{4}(4n+41-j), & \text{if } 1 < j \equiv 1 \text{ (mod 8)}, \\ \frac{1}{2}(n+11), & \text{if } j = 3, \\ \frac{1}{4}(2n+17-j), & \text{if } 3 < j \equiv 3 \text{ (mod 8)}, \\ \frac{1}{4}(4n+33-j), & \text{if } j \equiv 5 \text{ (mod 8)}, \\ \frac{1}{2}(n+7), & \text{if } j = 7, \\ \frac{1}{4}(2n+25-j), & \text{if } 7 < j \equiv 7 \text{ (mod 8)}, \\ \frac{1}{4}(n+17-j), & \text{if } n \equiv 1 \text{ (mod 8) and } j \equiv 2 \text{ (mod 8)}, \\ \frac{1}{4}(3n+33-j), & \text{if } n \equiv 1 \text{ (mod 8) and } j \equiv 4 \text{ (mod 8)}, \\ \frac{1}{4}(n+25-j), & \text{if } n \equiv 1 \text{ (mod 8) and } j \equiv 6 \text{ (mod 8)}, \\ \frac{1}{4}(3n+41-j), & \text{if } n \equiv 1 \text{ (mod 8) and } j \equiv 0 \text{ (mod 8)}, \\ \frac{1}{4}(n+25-j), & \text{if } n \equiv 5 \text{ (mod 8) and } j \equiv 2 \text{ (mod 8)}, \\ \frac{1}{4}(3n+41-j), & \text{if } n \equiv 5 \text{ (mod 8) and } j \equiv 4 \text{ (mod 8)}, \\ \frac{1}{4}(n+17-j), & \text{if } n \equiv 5 \text{ (mod 8) and } j \equiv 6 \text{ (mod 8)}, \\ \frac{1}{4}(3n+33-j), & \text{if } n \equiv 5 \text{ (mod 8) and } j \equiv 0 \text{ (mod 8)}. \end{cases}$$

Case $n \equiv 3$ (mod 4)

When $n = 7$, $f([v_1, v_2, v_3, v_4, v_5, v_6, v_7]) = [4, 13, 15, 9, 6, 14, 7]$. When $n \geq 11$,

$$f(v_j)=\begin{cases}4, & \text{if } j=1,\\ \frac{1}{2}(n+14-j), & \text{if } j=3,7,11,,\\ n+7, & \text{if } j=5,\\ n+8, & \text{if } j=9,\\ n+6, & \text{if } j=13,\\ \frac{1}{4}(4n+33-j), & \text{if } 17\le j\equiv 1 \text{ (mod 8)},\\ \frac{1}{4}(4n+41-j), & \text{if } 21\le j\equiv 5 \text{ (mod 8)},\\ 6, & \text{if } j=n-3,\\ \frac{1}{4}(2n+17-j), & \text{if } n\equiv 3 \text{ (mod 8) and } 19\le j\equiv 3 \text{ (mod 8)},\\ \frac{1}{4}(n+20-j), & \text{if } n\equiv 3 \text{ (mod 8) and } 15\le j\equiv 7 \text{ (mod 8)},\\ \frac{1}{4}(3n+41-j), & \text{if } n\equiv 3 \text{ (mod 8) and } j\equiv 2 \text{ (mod 8)},\\ \frac{1}{4}(n+25-j), & \text{if } n\equiv 3 \text{ (mod 8) and } j\equiv 4 \text{ (mod 8)},\\ \frac{1}{4}(3n+33-j), & \text{if } n\equiv 3 \text{ (mod 8) and } j\equiv 6 \text{ (mod 8)},\\ \frac{1}{4}(n+17-j), & \text{if } n\equiv 3 \text{ (mod 8) and } n-3\ne j\equiv 0 \text{ (mod 8)},\\ \frac{1}{4}(2n+41-j), & \text{if } n\equiv 7 \text{ (mod 8) and } 19\le j\equiv 3 \text{ (mod 8)},\\ \frac{1}{4}(2n+25-j), & \text{if } n\equiv 7 \text{ (mod 8) and } 15\le j\equiv 7 \text{ (mod 8)},\\ \frac{1}{4}(3n+33-j), & \text{if } n\equiv 7 \text{ (mod 8) and } j\equiv 2 \text{ (mod 8)},\\ \frac{1}{4}(n+17-j), & \text{if } n\equiv 7 \text{ (mod 8) and } n-3\ne j\equiv 4 \text{ (mod 8)},\\ \frac{1}{4}(3n+41-j), & \text{if } n\equiv 7 \text{ (mod 8) and } j\equiv 6 \text{ (mod 8)},\\ \frac{1}{4}(n+25-j), & \text{if } n\equiv 7 \text{ (mod 8) and } j\equiv 0 \text{ (mod 8)}.\end{cases}$$

Clearly, f is a bijective function. When $n=5$ and 7, $\{w_f(x): x\in V(C_8\cup C_n)\}$ is $\{8,9,10,\ldots,20\}$ and $\{9,10,11,\ldots,23\}$, respectively. When $n\ge 9$, it can be checked that the weights of all vertices are consecutive integers with the first term $w_f(u_2)=\frac{n+11}{2}$ and the last term $w_f(v_4)=\frac{3n+25}{2}$. □

Figure 3. shows the labeling defined in the proof of Theorem 3.9 for $n=15$.

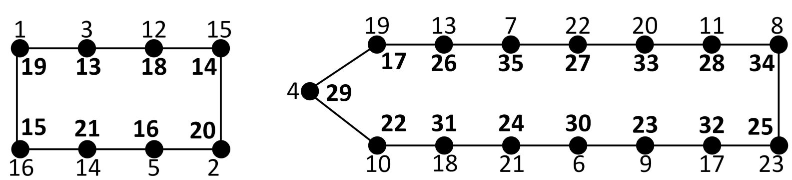

Figure 3. A (13, 1)-DA labeling of $C_8\cup C_{15}$.

Based on Theorems 3.8 and 3.9, we have the following conjecture:

Conjecture 2. *Let $k,n\ge 5$ be positive integers. The graph $C_{2k}\cup C_n$ is (a,d)-DA if and only if n is odd and $d=1$.*

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