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On the domination and signed domination numbers of zero-divisor graph

Ebrahim Vatandoost, Fatemeh Ramezani

Department of Basic Science,
Imam Khomeini International University,
Qazvin, Iran.

vatandoost@sci.ikiu.ac.ir, ramezani@ikiu.ac.ir

Abstract

Let R be a commutative ring (with 1) and let $Z(R)$ be its set of zero-divisors. The zero-divisor graph $\Gamma(R)$ has vertex set $Z^*(R) = Z(R) \setminus \{0\}$ and for distinct $x, y \in Z^*(R)$, the vertices x and y are adjacent if and only if $xy = 0$. In this paper, we consider the domination number and signed domination number on zero-divisor graph $\Gamma(R)$ of commutative ring R such that for every $0 \neq x \in Z^*(R)$, $x^2 \neq 0$. We characterize $\Gamma(R)$ whose $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) \in \{n+1, n, n-1\}$, where $|Z^*(R)| = n$.

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1. Introduction

The study on graphs from algebraic structures is an interesting subject for mathematician. In recent years, many algebraists as well as graph theorists have focused on the *zero-divisor* graph of rings. In [1], Anderson and Livingston introduced the zero-divisor graph of a commutative ring R with identity, denoted by $\Gamma(R)$, as the graph with vertices $Z^*(R) = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct vertices x and y are adjacent if and only if $xy = 0$.
A *dominating set* for Γ is a subset D of V such that every vertex not in D is adjacent to at least

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one member of D. The *domination number* is the number of vertices in a smallest dominating set for Γ and denoted by $\gamma(\Gamma)$. Oystein Ore introduced the terms "dominating set" and " domination number" in [10] and has proved if Γ has n vertices and no isolated vertices, then $\gamma(\Gamma) \leq \frac{n}{2}$.
For a vertex $v \in V(\Gamma)$, the closed neighborhood $N[v]$ of v is the set consisting of v and all of its neighbors. For a function $g : V(\Gamma) \longrightarrow \{-1,1\}$ and a vertex $v \in V$ we define $g[v] = \sum_{u\in N[v]} g(u)$. A *signed dominating function* of Γ is a function $g : V(\Gamma) \longrightarrow \{-1,1\}$ such that $g[v] > 0$ for all $v \in V(\Gamma)$. The *weight* of a function g is $\omega(g) = \sum_{v\in V(\Gamma)} g(v)$. The *signed domination number* $\gamma_s(\Gamma)$ is the minimum weight of a signed dominating function on Γ. A signed dominating function of weight $\gamma_s(\Gamma)$ is called a $\gamma_s(\Gamma)-$function. This concept was defined in [3] and has been studied by several authors (see for instance [4, 7, 8, 13, 14]). For a graph Γ the set of all vertices of Γ is denoted by $V(\Gamma)$. If Γ is a graph, then the *complement* of Γ, denoted by $\overline{\Gamma}$ is a graph with vertex set $V(\Gamma)$ in which two vertices are adjacent if and only if they are not adjacent in Γ. A graph is said to be *connected* if each pair of vertices are joined by a walk. The number of edges in a shortest walk joining v_i and v_j is called the *distance* between v_i and v_j and denoted by $d(v_i, v_j)$. The maximum value of the distance function in a connected graph Γ is called the *diameter* of Γ and denoted by $diam(\Gamma)$. The *complete graph* K_n is the graph with n vertices in which each pair of vertices are adjacent. The *corona* $\Gamma_1 \circ \Gamma_2$ is the graph formed by one copy of Γ_1 and $|V(\Gamma_1)|$ copies of Γ_2 where the ith vertex of Γ_1 is adjacent to every vertex in the ith copy of Γ_2.

In this work, we consider the domination and signed domination number on zero-divisor graph $\Gamma(R)$ for commutative ring R. The main results are in the following.

Theorem 1.1. $\gamma_s(\Gamma(R)) = n$ *if and only if* $\Gamma(R)$ *is isomorphic to* $K_{1,n-1}$ *or* $K_3 \circ K_1$.

Theorem 1.2. *Let* $|R|$ *be odd. Then* $\gamma_s(\Gamma(R)) = n - 2$ *if and only if* $\Gamma(R)$ *is a cycle* C_4.

Theorem 1.3. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n$ *if and only if* $\Gamma(R)$ *is a cycle* C_4 *or a path* P_3.

Theorem 1.4. $\gamma(\Gamma(R))+\gamma(\overline{\Gamma(R)}) = n-1$ *if and only if* $\Gamma(R)$ *is isomorphic to a* $K_{1,3}$ *or a* $K_3 \circ K_1$.

2. Preliminaries

First we give some facts that are needed in the next sections.

Theorem 2.1. *[1] Let* R *be a commutative ring. Then* $\Gamma(R)$ *is connected and* $diam(\Gamma(R)) \leq 3$. *Moreover, if* $\Gamma(R)$ *contains a cycle, then* $girth(\Gamma(R)) \leq 7$.

Theorem 2.2. *[1] Let* R *be a finite commutative ring with* $|\Gamma(R)| \geq 4$. *Then* $\Gamma(R)$ *is a star graph if and only if* $R = Z_2 \times F$ *where* F *is a finite field. In particular, if* $\Gamma(R)$ *is a star graph, then* $|\Gamma(R)| = p^n$ *for some prime* p *and* $n \geq 0$. *Conversely, each star graph of order* p *can be realized as* $\Gamma(R)$.

Theorem 2.3. *[10] If a graph* Γ *has* n *vertices and no isolated vertices, then* $\gamma(\Gamma) \leq \frac{n}{2}$.

Theorem 2.4. *[9] For any graph Γ with n vertices:*

i. $\gamma(\Gamma) + \gamma(\overline{\Gamma}) \leq n + 1$.

ii. $\gamma(\Gamma)\gamma(\overline{\Gamma}) \leq n$.

Theorem 2.5. *[11][5] For a graph Γ with even order n and no isolated vertices, $\gamma(\Gamma) = \frac{n}{2}$ if and only if the components of Γ are the cycle C_4 or the corona $H \circ K_1$ where H is a connected graph.*

Lemma 2.1. *[8] Let Γ be a complete graph of order n, then*

$$\gamma_s(\Gamma) = \begin{cases} 1 & n \text{ is odd.} \\ 2 & n \text{ is even.} \end{cases}$$

Theorem 2.6. *[8] Let Γ be a graph with n vertices, then*

i. $\gamma_s(\Gamma) + \gamma_s(\overline{\Gamma}) = 2n$ *and* $\gamma_s(\Gamma)\gamma_s(\overline{\Gamma}) = n^2$ *if and only if* $\Gamma \in \{P_1, P_2, \overline{P}_2, P_3, \overline{P}_3, P_4\}$, *where P_i is a path on i vertices.*

ii. $\gamma_s(\Gamma) + \gamma_s(\overline{\Gamma}) = 2n - 2$ *and* $\gamma_s(\Gamma)\gamma_s(\overline{\Gamma}) = n^2 - 2n$ *for exactly* 12 *graph in Figure 1.*

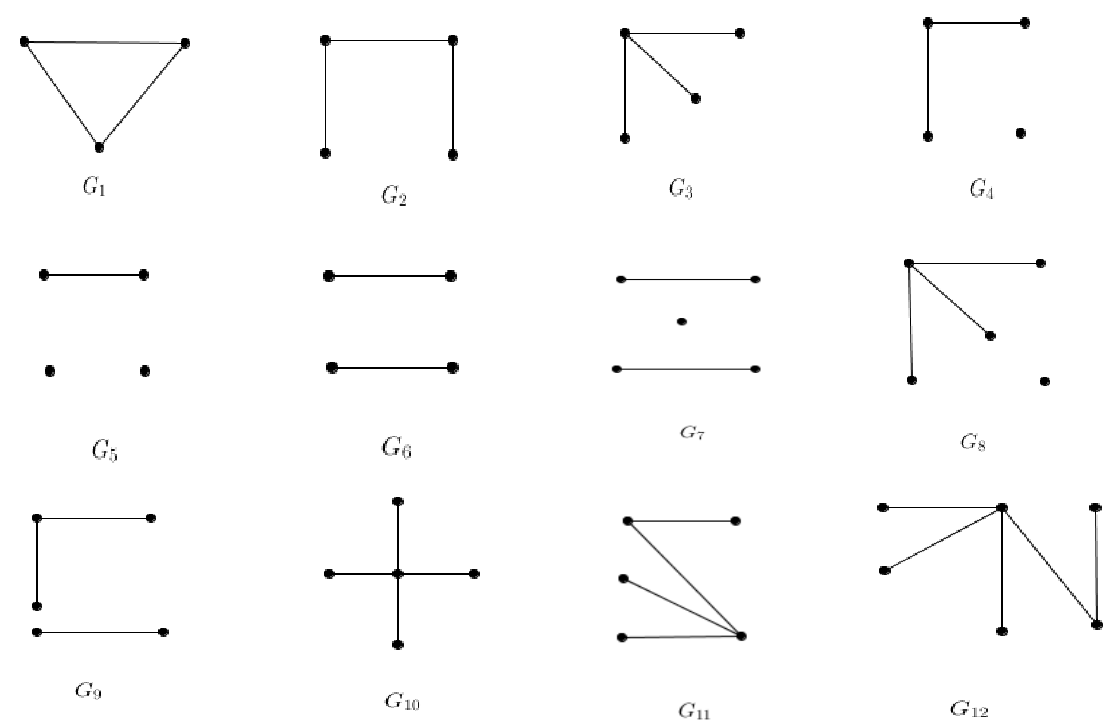

Figure 1. $\gamma_s(\Gamma) + \gamma_s(\overline{\Gamma}) = 2n - 2$ and $\gamma_s(\Gamma)\gamma_s(\overline{\Gamma}) = n^2 - 2n$.

Lemma 2.2. *[8] A graph Γ has $\gamma_s(\Gamma) = n$ if and only if every $v \in \Gamma$ is either isolated, an endvertex or adjacent to an endvertex.*

3. Signed domination number on zero-divisor graph

Throughout this paper, R is a commutative ring such that $|Z^*(R)| = n$ and for every non-zero element x, $x^2 \neq 0$. Also $\overline{\Gamma(R)}$ denotes the complement graph of the zero-divisor graph on R.

Lemma 3.1. *The cycle C_n is a zero-divisor graph of a ring if and only if $n = 4$.*

Proof. Let $\Gamma(R)$ be the zero-divisor graph of a commutative ring R. Since $girth(\Gamma(R)) \leq 7$, then $n \leq 7$. On the contrary, let $\Gamma(R) \simeq C_n$ and $n \geq 5$ or $n = 3$. If $n \geq 5$, then $a_1 - a_2 - \ldots - a_n - a_1$. So $a_1 + a_3 \in ann(a_2) = \{0, a_1, a_3\}$ and so $a_1 + a_3 = 0$. Thus $a_4 a_1 = 0$. This is impossible. Let $\Gamma(R)$ be K_3. Then $Z(R) = \{0, a, b, c\}$. So $ann(a) = \{0, b, c\}$ and $ann(b) = \{0, a, c\}$. Thus $b = -c = a$. This is a contradiction. Conversely, the zero divisor graph of ring $Z_3 \times Z_3$ is a cycle C_4. □

Proof of Theorem 1.1. Let $\gamma_s(\Gamma(R)) = n$. Since $\Gamma(R)$ is a connected graph, by Lemma 2.2, every vertex is an endvertex or adjacent to an end-vertex. If $x \in Z^*(R)$ and $deg(x) = 1$, then $ann(x) = \{0, y\}$ where $xy = 0$. So $O(y) = 2$ in group $(R, +)$. Hence $|R|$ has even order. Let $A = \{a\,;\, deg(a) > 1\}$. Since $diam(\Gamma(R)) \leq 3$, the induced subgraph on A is a complete graph. Consider four cases:

Case 1. If $|A| = 1$, then $\Gamma(R)$ is $K_{1,n-1}$.

Case 2. Let $A = \{a, b\}$. Then $ann(a) \cap ann(b) = \{0\}$. Suppose that $u \in ann(a)$ and $v \in ann(b)$. Since $deg(a), deg(b) > 1$, then $deg(u) = deg(v) = 1$ and also $uva = uvb = 0$. Hence, $uv \in ann(a) \cap ann(b)$ and so $uv = 0$. This is a contradiction by $deg(u) = deg(v) = 1$.

Case 3. Let $A = \{a, b, c\}$. Let $E(a)$ be the set of endvertex adjacent to a. Since $b, c \in ann(a)$ and $O(a) = O(b) = 2$, $ann(a)$ is a subgroup of $(R, +)$ of even order. Hence $|E(a)|$ is odd. The same conclusion can be drawn for b, c. We claim that $|E(a)| = 1$. On the contrary, suppose that $|E(a)| \geq 3$. There is no loos of generality in assuming $E(a) = \{x_1, x_2, x_3\}$. So $ann(a) = \{0, b, c, x_1, x_2, x_3\}$. Hence $x_1 = -x_3$ and $O(x_2) = 2$ or $O(x_i) = 2$ for $i \in \{1, 2, 3\}$. In the both cases, $x_1 + x_2, x_2 + x_3 \neq 0$. Let $y \in E(b)$. Then $x_1 y a = x_1 y b = 0$. So $x_1 y \in ann(a) \cap ann(b) = \{0, c\}$. Since $deg(y) = 1$, $x_1 y = c$. In the same manner we can see that $x_2 y = x_3 y = c$. Hence $y(x_1 + x_2) = y(x_2 + x_3) = 2c = 0$. Thus $x_1 + x_2, x_2 + x_3 \in ann(y) = \{0, b\}$. So $x_1 + x_2 = x_2 + x_3 = b$ and so $x_1 = x_3$. This is a contradiction. Therefore $|E(a)| = |E(b)| = |E(c)| = 1$ and $\Gamma(R)$ is $K_3 \circ K_1$.

Case 4. Let $A = \{a_1, \ldots, a_t\}$ and $t > 3$. Then $ann(a_i) = \{0, a_1, \ldots, \hat{a}_i, \ldots a_t\} \cup E(a_i)$ for $i \in \{1, \ldots, t\}$. So $\bigcap_{i=1}^{t-2} ann(a_i) = \{0, a_{t-1}, a_t\}$. Hence $a_{t-1} = -a_t$. Since $N(a_{t-1}) \neq N(a_t)$, this is impossible. □

Corollary 3.1. *If $\gamma_s(\Gamma(R)) = n$, then $\gamma_s(\overline{\Gamma(R)}) \in \{0, 3\}$.*

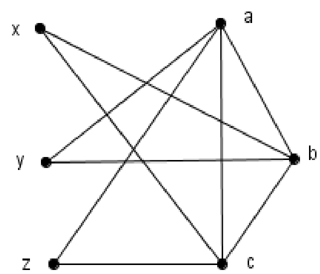

Figure 2. $\overline{K_3 \circ K_1}$.

Proof. By Theorem 1.1, $\Gamma(R) \simeq K_{1,n-1}$ or $K_3 \circ K_1$. If $\Gamma(R) \simeq K_{1,n-1}$, then $\overline{\Gamma(R)}$ is $K_1 \cup K_{n-1}$. Since $|Z(R)|$ is even, then n is odd and so $\gamma_s(K_{n-1}) = 2$ and $\gamma_s(\overline{\Gamma(R)}) = 3$. If $\Gamma(R) \simeq K_3 \circ K_1$, then $\overline{\Gamma(R)}$ is the graph in Figure 2. Let $V_1 = \{x, y, z\}$ and $V_2 = \{a, b, c\}$. Define $f : V(\overline{\Gamma(R)}) \longrightarrow \{-1, +1\}$ such that

$$f(u) = \begin{cases} -1 & u \in V_1; \\ +1 & u \in V_2. \end{cases}$$

It is clear that f is a signed dominating function and $\omega(f) = 0$. If g is a function such that $\omega(g) < 0$, then g is not a signed dominating function. Therefore $\gamma_s(\overline{\Gamma(R)}) = 0$. □

Corollary 3.2. *If $\gamma_s(\Gamma(R)) = n$, then $|R| \in \{2^k, 2p^k\}$ where p is prime.*

Proof. By Theorem 1.1, $\Gamma(R) \simeq K_{1,n-1}$ or $K_3 \circ K_1$. If $\Gamma(R) \simeq K_{1,n-1}$, then by Theorem 2.2, $R \simeq Z_2 \times F$ where F is a finite field. So $|R| = 2p^k$. Let $\Gamma(R) \simeq K_3 \circ K_1$. Let $V(\Gamma(R)) = \{a_i, x_i\,;\, deg(x_i) = 1\,,\, deg(a_i) = 3\,,\, 1 \leq i \leq 3\}$. So $|R|$ is even. If $p \mid |R|$ (p is odd prime number), then there is $0 \neq r \in R$ such that $O(r) = p$. Hence $pr = 0$. Also $(p-1)a_i = 0$. Thus $ra_i = r(pa_i) = 0$. So $r \in ann(a_i)$ for every $1 \leq i \leq 3$. Hence $r = 0$. This is a contradiction. Therefore $|R| = 2^k$. □

The Proof of Theorem 1.2 Since $|R|$ is odd, $\delta \geq 2$. Let $x \in R$ and $deg(x) = 2k+1$. Then $|ann(x)| = 2k+2$. This is a contradiction by $|R|$ is odd. So all vertices have even degree. Since $diam(\Gamma(R)) \leq 3$, there are three cases:

Case 1. If $diam(\Gamma(R)) = 1$, then $\Gamma(R)$ is complete graph K_n. Since all vertices have even degree, n is odd and so $\gamma_s(\Gamma(R)) = 1$. Hence $n = 3$ and $\Gamma(R)$ is K_3. This is impossible by Lemma 3.1.

Case 2. If $diam(\gamma(R)) = 3$, then there are $a, b \in Z^*(R)$ such that $d(a, b) = 3$. Define signed dominating function $f : V(\Gamma(R)) \longrightarrow \{-1, +1\}$ such that $f(a) = f(b) = -1$ and $f(x) = 1$ for $x \in Z^*(R) \setminus \{a, b\}$. Thus $\gamma_s(\Gamma(R)) < n - 2$. This is impossible.

Case 3. Let $diam(\Gamma(R)) = 2$. If $\Delta = 2$, then $\Gamma(R)$ is a cycle. So $\Gamma(R) \simeq C_4$, by Theorem 3.1. Let $deg(y) = \Delta \geq 4$. Let $ann(y) = \{0, a_1, \ldots, a_t\}$ where t is even and $t \geq 4$. So $O(a_i) \neq 2$. Hence, $-a_i \in ann(y)$. Thus $ann(y) = \{0, a_1, -a_1, \ldots, a_{\frac{t}{2}}, -a_{\frac{t}{2}}\}$. Let $x \in N(a_1)$. If there is $2 \leq j \leq \frac{t}{2}$ such that $\{a_1\ , a_j\} \notin E(\Gamma(R))$, then $d(x, a_j) > 2$. Otherwise, there is $z \in N(a_j) \setminus ann(y)$ and so $d(x, z) = 3$. This is not true. So for every $x \in N(a_1)$, $deg(x) \geq 4$. Define $f : V(\Gamma(R)) \longrightarrow \{-1, +1\}$ such that $f(a_1) = f(-a_1) = -1$ and $f(v) = 1$ for every $v \in V(\Gamma(R)) \setminus \{a_1, -a_1\}$. So f is a signed dominating function and so $\gamma(\Gamma(R)) < n - 2$. This is a contradiction. □

Theorem 3.1. *If $\gamma_s(\Gamma(R)) + \gamma_s(\overline{\Gamma(R)}) = 2n$, then $|R| \in \{2^k, 2 \times 3^k\}$.*

Proof. Since $\Gamma(R)$ is a connected graph, by Theorem 2.6, $\Gamma(R)$ is one of the paths in $\{P_1, P_2, P_3, P_4\}$. It is known P_4 is not a zero-divisor graph.
If $\Gamma(R)$ is P_1, then $Z(R) = \{0, x\}$. So $x^2 = 0$. This is impossible.
Let $\Gamma(R)$ be P_2. Then $Z(R) = \{0, a, b\}$ and $O(a) = O(b) = 2$. So $|R|$ is even. If $p \mid |R|$ where p is an odd prime number, then there is $r \in R$ such that $O(r) = p$. Hence $(p-1)a = 0$. Thus $ra = r(pa) = 0$. So $r \in ann(a)$ and so $r = b$. This is a contradiction. If $\Gamma(R)$ is $a - c - b$, then $ann(c) = \{0, a, b\}$. So $b = -a$ and so $O(a) = 3$. Also $O(c) = 2$. Also by Theorem 2.2, $R \simeq Z_2 \times F$. So $|R| = 2 \times 3^k$. □

Theorem 3.2. *If $\gamma_s(\Gamma(R)) + \gamma_s(\overline{\Gamma(R)}) = 2n - 2$, then $|R| = 2p^k$ where p is an odd prime.*

Proof. By Theorem 2.6 and Lemma 3.1 and since $\Gamma(R)$ is a connected graph, $\Gamma(R) \in \{K_{1,3}, K_{1,4}, G_1, G_2\}$ where G_1, G_2 are two graphs in Figure 3. We show that G_1 and G_2 are not a zero-divisor graph. If G_1 is a zero-divisor graph, then $b(a+e) = 0$. So $a + e \in ann(b) = \{0, a, e\}$. Hence $e = -a$. This is contradiction by $c, d \notin ann(a)$. Similar argument applies for G_2.
If $\Gamma(R)$ is $K_{1,3}$ or $K_{1,4}$, then likewise Corollary 3.2, $|R| = 2p^k$. □

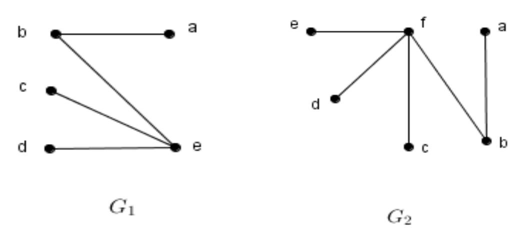

Figure 3. G_1 and G_2 in Theorem 3.2.

4. Domination number on zero-divisor graph

Theorem 4.1. $\gamma(\Gamma(R)) = \frac{n}{2}$ *if and only if $\Gamma(R)$ is a cycle C_4 or a $K_3 \circ K_1$.*

Proof. Let $\gamma(\Gamma(R)) = \frac{n}{2}$. By Theorem 2.5, $\Gamma(R)$ is the a cycle C_4 or the corona $H \circ K_1$ where H is a connected graph. If $\Gamma(R)$ is not C_4, then $\Gamma(R) \simeq H \circ K_1$. Let $A = \{a_i\,;\, deg(a_i) > 1\}$. Since $diam(\Gamma(R)) \leq 3$, the induced subgraph on A is complete. If $|A| = 2$, then $\Gamma(R)$ is a path P_4. This is impossible. If $|A| > 3$, then $\bigcap_{i=1}^{t-2} ann(a_i) = \{0, a_{t-1}, a_t\}$. Hence $a_t = -a_{t-1}$. This is a contradiction. So $|A| = 3$ and so $\Gamma(R) \simeq K_3 \circ K_1$. The converse is clear. □

Theorem 4.2. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n + 1$ *if and only if* $\Gamma(R)$ *is complete graph* K_n*.*

Proof. Let $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n + 1$. By Theorem 2.3, $\gamma(\Gamma(R)) \leq \frac{n}{2}$. So $\gamma(\overline{\Gamma(R)}) > \frac{n}{2}$ and so $\overline{\Gamma(R)}$ has isolated vertex. Hence $\gamma(\Gamma(R)) = 1$ and $\gamma(\overline{\Gamma(R)}) = n$. Thus all vertices of $\overline{\Gamma(R)}$ are isolated. Therefore $\Gamma(R) \simeq K_n$. □

Proof of Theorem 1.3. Let $\gamma(\Gamma(R))+\gamma(\overline{\Gamma(R)}) = n$. Since $\Gamma(R)$ is a connected graph, $\gamma(\Gamma(R)) \leq \frac{n}{2}$. We consider following cases:

Case 1. Let $\gamma(\Gamma(R)) = \frac{n}{2}$. By Theorem 4.1 and above equality, $\Gamma(R)$ is a C_4.

Case 2. If $\gamma(\Gamma(R)) < \frac{n}{2}$, then $\gamma(\overline{\Gamma(R)}) > \frac{n}{2}$. So $\overline{\Gamma(R)}$ has an isolated vertex and so $\gamma(\Gamma(R)) = 1$. Also $\gamma(\overline{\Gamma(R)}) = n - 1$. Thus $\overline{\Gamma(R)}$ is $P_2 \cup (n-2)K_1$. It is clear that $n \geq 3$.

Sub case I. If $n > 3$, then likewise the proof of Theorem 4.1, the contradiction reaches.

Sub case II. If $n = 3$, then $\overline{\Gamma(R)} \simeq P_2 \cup K_1$. So $\Gamma(R)$ is the path P_3.

The converse is easy. □

Proof of Theorem 1.4. Let $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$. Since $\Gamma(R)$ has no isolated vertices, $\gamma(\Gamma(R)) \leq \frac{n}{2}$. There are three cases:

Case 1. If $\gamma(\Gamma(R)) = \frac{n}{2}$, then $\Gamma(R)$ is $K_3 \circ K_1$ or C_4 by Theorem 4.1. But $K_3 \circ K_1$ is not satisfied in $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$.

Case 2. Let $\gamma(\Gamma(R)) = \frac{n}{2} - 1$. Then $\gamma(\overline{\Gamma(R)}) = \frac{n}{2}$. By Theorem 2.4, $0 \leq n \leq 6$. So $n \in \{4, 6\}$.

Sub case I. Let $n = 4$. Then $\gamma(\Gamma(R)) = 1$ and $\gamma(\overline{\Gamma(R)}) = 2$. So $\Gamma(R)$ is $K_{1,3}$ or G in Figure 4. Let G be a zero-divisor graph. Since $deg(a) = 1$, $O(b) = 2$. On the other hand, $ann(c) = \{0, b, d\}$. So $d = -b$. This is not true.

Sub case II. If $n = 6$, then $\gamma(\Gamma(R)) = 2$ and $\gamma(\overline{\Gamma(R)}) = 3$. So $\overline{\Gamma(R)}$ is a graph without isolated vertex. Hence by Theorem 2.5, $\overline{\Gamma(R)}$ is $C_4 \cup P_2, 3P_2$ or $K_3 \circ K_1$. So $\Gamma(R)$ is G_1, G_2 and G_3 in Figure 4, respectively. In graph G_1, $c(d + e) = 0$ and so $d + e \in ann(c)$. Hence $d + e = 0$ or f. Thus $ad = 0$ or $bd = 0$. This is a contradiction. In graph G_2, $d + f \in ann(a)$. But all cases are impossible. In graph G_3, Since $b(d + f) = 0$, $d = -f$. So $cf = 0$. This is not true.

Case 3. If $\gamma(\Gamma(R)) < \frac{n}{2} - 1$, then $\overline{\Gamma(R)}$ has an isolated vertex. So $\gamma(\Gamma(R)) = 1$ and so $\gamma(\overline{\Gamma(R)}) = n - 2$. Hence $\overline{\Gamma(R)}$ is $P_3 \cup (n-3)K_1$ or $K_3 \cup (n-3)K_1$. If $n = 4$, then $\Gamma(R)$ is G in Figure 4 or $K_{1,3}$ respectively. But G is not a zero-divisor graph of a ring. For $n > 4$, the contradiction reached by the same method in Theorem 4.1. □

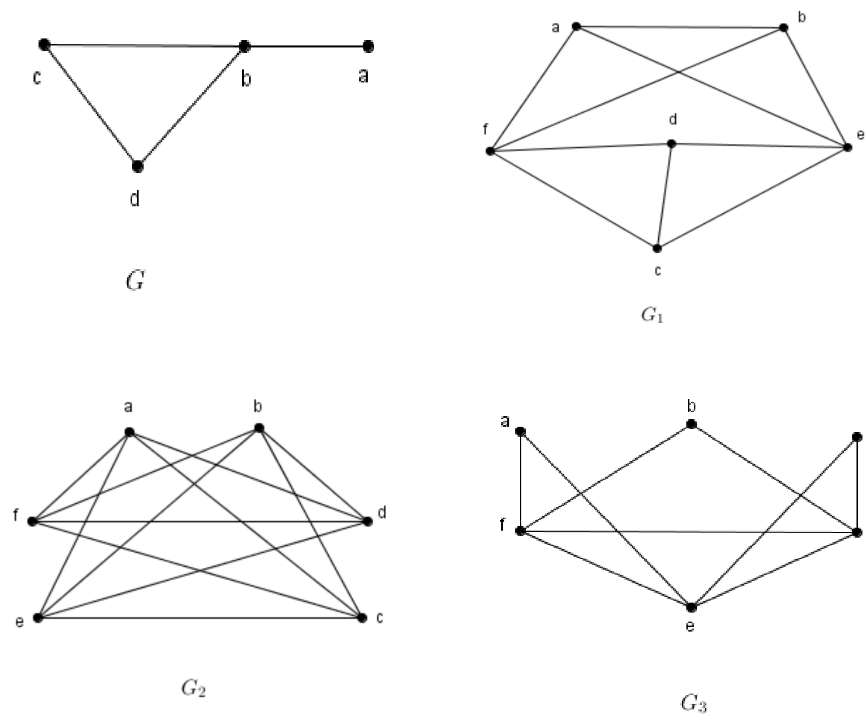

Figure 4. $\Gamma(R)$ in the proof of Theorem 1.4, Cases 2 and 3.

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