Electronic Journal of Graph Theory and Applications **8** (2) (2020), 365–372

Electronic Journal of
Graph Theory and Applications

Restricted size Ramsey number for P_3 versus cycle

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Abstract

Let F, G and H be simple graphs. We say $F \to (G, H)$ if for every red-blue coloring of the edges of F there exists a red copy of G or a blue copy of H in F. The Ramsey number $r(G, H)$ is defined as $r(G, H) = min\{|V(F)| : F \to (G, H)\}$, while the restricted size Ramsey number $r^*(G, H)$ is defined as $r^*(G, H) = min\{|E(F)| : F \to (G, H), |V(F)| = r(G, H)\}$. In this paper we determine previously unknown restricted size Ramsey numbers $r^*(P_3, C_n)$ for $7 \le n \le 12$. We also give new upper bound $r^*(P_3, C_n) \le 2n - 2$ for even $n \ge 10$.

Keywords: size Ramsey number, restricted, path, cycle.
Mathematics Subject Classification: 05C55, 05D10
DOI: 10.5614/ejgta.2020.8.2.12

1. Introduction

Paul Erdős had a tremendous impact on many areas of mathematics, one of these areas is Ramsey theory. His contributions started with the classical Ramsey numbers $r(G, H)$. In 1978, Erdős *et al.* in [3] defined the *size Ramsey number* $\hat{r}(G, H)$ as the smallest size of a graph F such that, under any red-blue coloring of its edges, the graph F contains a red copy of G or a blue copy of H. In [6] one can find a survey of results along with the influence of Paul Erdős on the development of size Ramsey theory.

The *restricted size Ramsey number* $r^*(G, H)$ is a problem connecting Ramsey number and size Ramsey number. For the restricted size Ramsey number, if r is the Ramsey number of G and

Received: 2 April 2019, Revised: 19 June 2020, Accepted: 2 August 2020.

H then F must be a spanning subgraph of K_r with the smallest size such that for any red-blue coloring of edges of F we have a red copy of G or a blue copy of H in F. Therefore, the size of K_r is the upper bound for the restricted size Ramsey number of G and H and the restricted size Ramsey number must be greater or equal to the size Ramsey number for a given pair of graphs. In addition, we have $\tilde{r}(G, H) \leq \hat{r}(G, H)$, where $\tilde{r}(G, H)$ is the on-line Ramsey number (the definition and properties of these numbers can be found in [2]). If both G and H are complete graphs then $F = K_r$ (see [3]). The case of complete graph is one of a few cases for which that upper bound is reached. In general, the more sparse both graphs G and H are, the problem of finding the restricted size Ramsey number for those pair of graphs is harder. Only two results for the exact value of restricted size Ramsey number involving a class of graph known so far, that are, for $K_{1,k}$ versus K_n ([7]) and G versus $K_{1,k}$, where G is K_3, $K_4 - e$, or C_5 ([4]). For other few classes of graphs, the problem is solved partially.

Some results for size Ramsey number was presented by Faudree and Schelp in 2002 ([6]). It had shown that $r^*(P_3, C_3) = 8$, $r^*(P_3, C_4) = 6$, $r^*(P_3, C_5) = 9$. In 2015, Silaban *et al.* proved the last known exact value, namely $r^*(P_3, C_6) = 9$ [9]. In addition, they give lower and upper bound for $r^*(P_3, C_n)$, where $n \geq 8$ is even (see Theorem 3 below). In this paper, we determine previously unknown restricted size Ramsey numbers, namely $r^*(P_3, C_n)$ for $7 \leq n \leq 12$, and we improve the upper bound for $r^*(P_3, C_n)$, that is we prove that $r^*(P_3, C_n) \leq 2n - 2$ for even $n \geq 10$.

In general, we follow graph theory terminology and notation of [9].

2. Known results

In this section, we list a few known definitions and theorems that we will need in proving our results.

The *Turán number* $ex(n, G)$ is the maximum number of edges in any n-vertex graph which does not contain a subgraph isomorphic to G. A graph on n vertices is said to be *extremal with respect to* G if it does not contain a subgraph isomorphic to G and has exactly $ex(n, G)$ edges.

In 1989, Clapham *et al.* [1] determined all values of $ex(n, C_4)$ for $n \leq 21$. They also characterized all the corresponding extremal graphs. In Theorem 1 we quote value for $ex(7, C_4)$ and we show (Figure 1) five corresponding extremal graphs. We will use this in the proof of Theorem 5.

Theorem 1 ([1]).

$$ex(7, C_4) = 9$$

and there are 5 extremal graphs for this number illustrated in Figure 1.

In our work we will also use a well known the Ramsey number for paths and cycles that was calculated by Faudree *et al.* in [5].

Theorem 2 ([5]). *For all integers* $n \geq 4$*,*

$$r(P_3, C_n) = n.$$

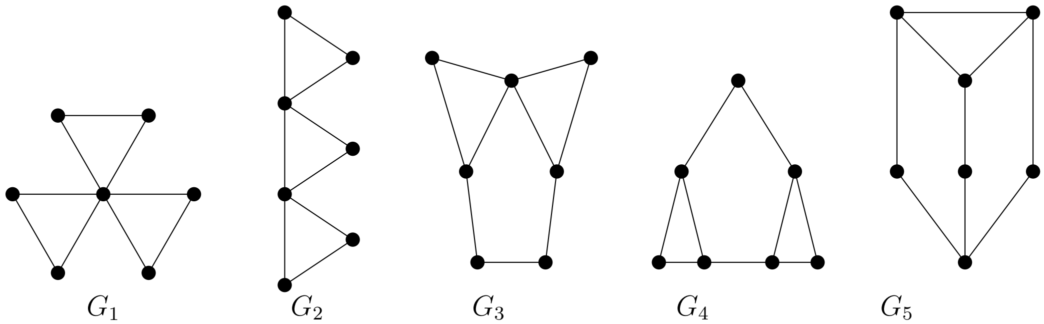

Figure 1. All extremal graphs for $ex(7, C_4)$.

In 2015, Silaban *et al.* [9] proved the lower and the upper bound for the restricted size Ramsey number for P_3 and cycles. At the end of our article we improve the upper bound for this number.

Theorem 3 ([9])**.** *For even $n \geq 8$,*

$$\frac{3}{2}n + 2 \leq r^*(P_3, C_n) \leq 2n - 1.$$

3. New results

In order to find the value of $r^*(P_3, C_n)$, we find a graph F with the smallest possible size such that $F \to (P_3, C_n)$. According to Theorem 2 the graph F must have n vertices.

3.1. Determining the value of $r^*(P_3, C_7)$

First, we give the following condition for graph F satisfying $F \to (P_3, C_7)$.

Lemma 4. *Let F be a graph with $|V(F)| = 7$ and $C_4 \subseteq \overline{F}$, then $F \nrightarrow (P_3, C_7)$.*

Proof. Suppose there is F with $|V(F)| = 7$ such that $\overline{F}$ contains cycle C_4, say v_1, v_2, v_3, v_4, v_1. By coloring possible edges v_1v_3, $v_2v_4 \in E(F)$ by red and the remaining edges of F by blue, we obtain a 2-coloring of F which contains neither a red P_3 nor a blue C_7. □

Theorem 5. $r^*(P_3, C_7) = 13$.

Proof. First, we will prove that $r^*(P_3, C_7) \geq 13$. From Lemma 4 and Theorem 1 we imply that $r^*(P_3, C_7) \geq 12$. Suppose that $r^*(P_3, C_7) = 12$. Let F be a graph on 7 vertices and 12 edges. By Lemma 4, if $F \to (P_3, C_7)$, then $C_4 \nsubseteq \overline{F}$ and therefore $\overline{F}$ is one of the five graphs G_i, $1 \leq i \leq 5$ from Figure 1. Furthermore, since $\Delta(G_i) \geq 4$ for $i \in \{1, 2, 3\}$, and by coloring u_2u_7, u_3u_5, v_1v_6, v_3v_7, v_4v_5 in red (see Figure 2) we obtain, for all $\overline{G_i}$, a 2-coloring of edges which contains neither a red P_3 nor a blue C_7. In fact, if $\Delta(G_i) \geq 4$, then there is a vertex of degree at most 2 in $\overline{G_i}$. To avoid a blue C_7 we color in red one edge coming out of this vertex (if any). Hence, $F \nrightarrow (P_3, C_7)$ and consequently we have $r^*(P_3, C_7) \geq 13$.

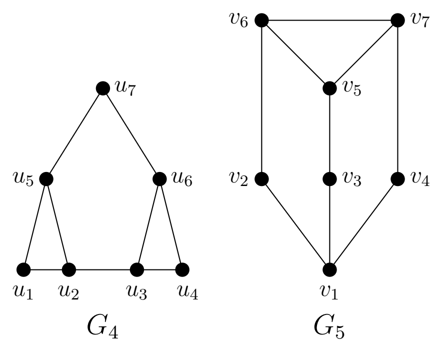

Figure 2. Two extremal graphs G_4 and G_5 for $ex(7, C_4)$.

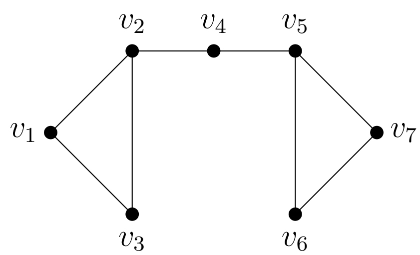

Figure 3. The complement of the graph F_7.

Next, we will show that $r^*(P_3, C_7) \leq 13$. Let F_7 be the complement of the graph shown in Figure 3. To prove that $F_7 \to (P_3, C_7)$, let χ be any red-blue coloring of edges of F_7 such that there is no red P_3 in F_7. We will show that the coloring χ will imply a blue C_7 in F_7. To do so, consider vertex v_4. There are 4 edges incidence to this vertex, at most one of them can be colored by red. Up to the symmetry of F_7, without loss of generality, we can assume that v_1v_4 is red or all edges v_iv_4, $i \in \{1, 3, 6, 7\}$ are blue. Nonexistence a red P_3 forces the red edges to be a matching and that it suffices to consider maximum matchings. Then, using symmetries, there are only five subcases to discuss.

1. Edge v_1v_4 is red.
   - 1.1 if v_2v_5 and v_3v_6 is red, then $v_1, v_5, v_3, v_4, v_6, v_2, v_7, v_1$ is the blue cycle,
   - 1.2 if v_2v_6 and v_3v_5 are red, then $v_1, v_5, v_2, v_7, v_4, v_3, v_6, v_1$ is the blue cycle,
   - 1.3 if v_2v_6 and v_3v_7 are red, then the cycle $v_1, v_6, v_4, v_3, v_5, v_2, v_7, v_1$ is blue.
2. All edges v_iv_4, $i \in \{1, 3, 6, 7\}$ are blue. Then we have two subcases:
   - 2.1 if v_2v_5, v_1v_6, v_3v_7 are red, then we obtain the following blue cycle: $v_1, v_5, v_3, v_4, v_6, v_2, v_7, v_1$,
   - 2.2 if v_2v_6, v_1v_5, v_3v_7 are red, then the cycle: $v_1, v_4, v_6, v_3, v_5, v_2, v_7, v_1$ is blue.

For all cases, there is always a blue C_7, so $F_7 \to (P_3, C_7)$ and the proof is complete. □

3.2. Upper bounds for $r^*(P_3, C_n)$

In [9] Silaban *et al.* proved that $r^*(P_3, C_n) \leq 2n - 1$. In this section we will show that this upper bound can be improved and we prove the following theorem.

Theorem 6. *For even $n \geq 12$,*

$$r^*(P_3, C_n) \leq 2n - 2.$$

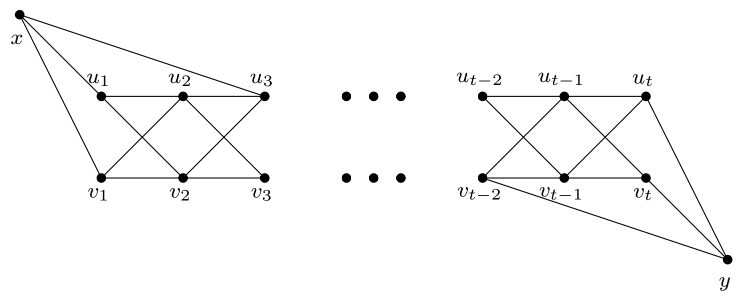

Figure 4. The graph $F_n \rightarrow (P_3, C_n)$ for $n \geq 12$ and even n, $t = \frac{n-2}{2}$.

Proof. Let $t = \frac{n-2}{2}$ and let F_n be a graph with

$$V(F_n) = \{x, y\} \cup \{u_i, v_i | i = 1, \ldots, t\}$$

and

$$E(F_n) = \{xu_1, xv_1, xu_3, u_t y, v_t y, v_{t-2} y\} \cup S,$$

where

$$S = \{u_i u_{i+1}, v_i v_{i+1}, v_i u_{i+1}, u_i v_{i+1} | i = 1, \ldots, t-1\}$$

(see Fig. 4). In order to prove that $F_n \rightarrow (P_3, C_n)$, let χ be any red-blue coloring of edges of F_n such that there is no red P_3 in F_n. We will show that the coloring F_n will imply a blue C_n in F_n.

FACT 1. Observe that if we have any two independent blue paths to u_i and v_i, then we can extend these paths step by step to vertices u_j and to v_j for $1 \leq i < j \leq t$. To do so, let us consider the vertex u_i. Since under the coloring χ there is no red P_3, at most one of edges $\{u_i u_{i+1}, u_i v_{i+1}\}$ can be red. If $u_i u_{i+1}$ is red, then $\{u_i v_{i+1}, v_i u_{i+1}\}$ must be blue. Using these 2 blue edges, we can extend our blue paths to u_{i+1} and v_{i+1}, independently. If $u_i v_{i+1}$ is red, then $\{u_i u_{i+1}, v_i v_{i+1}\}$ must be blue. Using these 2 blue edges, we also can extend our blue paths to u_{i+1} and v_{i+1}, independently. We can do the same process to extend our blue paths until reaching u_j and v_j.

FACT 2. There are always two independent blue paths from x to u_i and from x to v_i for $i = 1$ or $i = 3$. To prove this fact, let us consider the the vertex x. There are 3 incident edges to this vertex, at most one of them can be colored by red. Up to the symmetry of F_n, we can assume that at most one edge of set $\{xu_1, xu_3\}$ is red.

If xu_3 is red, then xu_1 and xv_1 must be blue, therefore we have two blue paths from x to u_1 and from x to v_1. Note that a similar situation occurs if none of edges incidence to x is red.

Now we can assume that xu_1 is red. In this case xv_1 and xv_3 are blue so we have one path from x to u_3. We will construct a path of size 6 with the set $\{u_1, u_2, v_1, v_2\}$) as inner vertices, namely the path from x to v_3. To do this consider the vertex u_2. Under the coloring χ, at most one of edges $\{u_2u_3, u_2v_3, u_2v_1\}$ can be red. In all cases we obtain one among two possible blue paths from x to v_3, namely $xv_1u_2u_1v_2v_3$ or $xv_1v_2u_1u_2v_3$.

Similarly, using the symmetry of F_n, we get two independent blue paths from y to u_j and from y to v_j for $j = t$ or $j = t - 2$.

By using Fact 1 and 2, we obtain a blue cycle C_n in F. Observe that the theorem holds for $3 \leq t - 2$ and $n \geq 12$. □

Silaban *et al.* ([9]) gave the upper bound for the restricted size Ramsey number of P_3 versus P_n. They proved that for even $n > 8$, $r^*(P_3, P_n) \leq 2n - 1$. From the proof of Theorem 6 we see that if we delete edge xu_3 then for any 2-coloring of edges of $F_n \backslash \{xu_3\}$ that avoid red P_3, it must imply a blue P_n in F_n. It means we get a better upper bound of the restricted size Ramsey number for P_3 versus P_n, even $n \geq 12$, as given in the following corollary.

Corollary 7. *For $n \geq 12$ and even n, $r^*(P_3, P_n) \leq 2n - 3$.*

3.3. Computational Approach

In this subsection we use a computational approach to determine the exact values of $r^*(P_3, C_n)$, $8 \leq n \leq 12$. We use the following Algorithm 1 to find such numbers.

Algorithm 1 Deciding whether graph $F \to (P_3, C_n)$ or not

Require: Adjacency matrix of biconnected graph F on n vertices.
Ensure: $F \to (P_3, C_n)$ or $F \nrightarrow (P_3, C_n)$.
for $m = \lfloor \frac{n}{2} \rfloor \to 0$ **do**
  for every subset S of m edges that compose independent edge set **do**
    $F' = F - S$
    find a Hamiltonian cycle in F'
    if no Hamiltonian cycle in F' **then return** $F \nrightarrow (P_3, C_n)$, Break.
    end if
  end for
  $m := m - 1$
end for
return $F \to (P_3, C_n)$

We generate all the adjacency matrices of biconnected graphs with n vertices ($8 \leq n \leq 12$) with minimum degree 3 by using a program called *geng* [8]. In order to find Hamiltonian cycle, algorithm uses recursive depth-first search. Note that the starting point should not matter as the cycle can be started from any point.

Table 1: Restricted size Ramsey numbers $r^*(P_3, C_n)$, $8 \le n \le 12$.

n	8	9	10	11	12
$r^*(P_3, C_n)$	15	17	18	20	22
$\#\{F \to (P_3, C_n), \lvert E(F)\rvert = r^*(P_3, C_n)\}$	10	16	2	4	8

From the above algorithm, we obtain the results which are presented in Table 1. This table provides the value of $r^*(P_3, C_n)$ and the number of non-isomorphic graphs F of order n and size $r^*(P_3, C_n)$ such that $F \to (P_3, C_n)$. Based on computer calculations, it turned out that in fact it was enough to consider cases where $m \in \{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor - 2\}$.

Examples of resulting graphs are presented in Fig. 5, 6, 7 and 4. For the number $r^*(P_3, C_8)$ an example is a graph $K_{4,4} - e$.

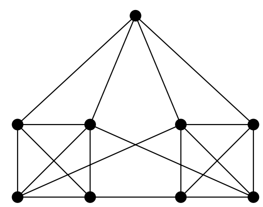

Figure 5. Complement of the graph $F_9 \to (P_3, C_9)$.

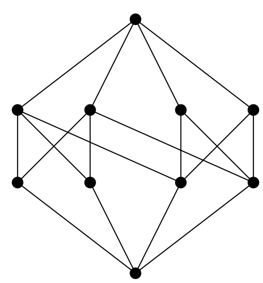

Figure 6. Graph $F_{10} \to (P_3, C_{10})$.

4. Conclusion

In this paper we established six new restricted size Ramsey numbers $r^*(P_3, C_n)$ for $7 \le n \le 12$. In addition, we gave the new upper bound for $n \ge 10$ and even n. It follows that the first open case of $r^*(P_3, C_n)$ is now $r^*(P_3, C_{13})$ and is certainly worth of further investigation. Based on results known earlier and described in this work as well as computer experiments for some bipartite graphs that are not presented here, let us formulate the following conjecture.

Conjecture 8. *For all $n \ge 10$, we have*

$$r^*(P_3, C_n) = 2n - 2.$$

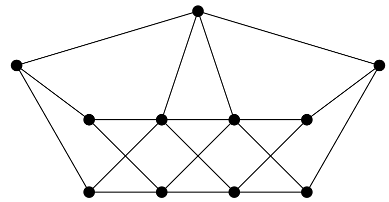

Figure 7. Graph $F_{11} \to (P_3, C_{11})$.

Acknowledgment

We would like to thank the student of the University of Gdańsk Maciej Godek for the independent performance of some computer experiments that confirmed the correctness of the results contained in the article.

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