Restricted size Ramsey number for $P_3$ versus cycle

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Abstract

Let $F$, $G$ and $H$ be simple graphs. We say $F \rightarrow (G, H)$ if for every red-blue coloring of the edges of $F$ there exists a red copy of $G$ or a blue copy of $H$ in $F$. The Ramsey number $r(G, H)$ is defined as $r(G, H) = \min\{|V(F)| : F \rightarrow (G, H)\}$, while the restricted size Ramsey number $r^*(G, H)$ is defined as $r^*(G, H) = \min\{|E(F)| : F \rightarrow (G, H), |V(F)| = r(G, H)\}$. In this paper we determine previously unknown restricted size Ramsey numbers $r^*(P_3, C_n)$ for $7 \leq n \leq 12$. We also give new upper bound $r^*(P_3, C_n) \leq 2n - 2$ for even $n \geq 10$.

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1. Introduction

Paul Erdős had a tremendous impact on many areas of mathematics, one of these areas is Ramsey theory. His contributions started with the classical Ramsey numbers $r(G, H)$. In 1978, Erdős et al. in [3] defined the size Ramsey number $\hat{r}(G, H)$ as the smallest size of a graph $F$ such that, under any red-blue coloring of its edges, the graph $F$ contains a red copy of $G$ or a blue copy of $H$. In [6] one can find a survey of results along with the influence of Paul Erdős on the development of size Ramsey theory.

The restricted size Ramsey number $r^*(G, H)$ is a problem connecting Ramsey number and size Ramsey number. For the restricted size Ramsey number, if $r$ is the Ramsey number of $G$ and
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$H$ then $F$ must be a spanning subgraph of $K_r$ with the smallest size such that for any red-blue coloring of edges of $F$ we have a red copy of $G$ or a blue copy of $H$ in $F$. Therefore, the size of $K_r$ is the upper bound for the restricted size Ramsey number of $G$ and $H$ and the restricted size Ramsey number must be greater or equal to the size Ramsey number for a given pair of graphs. In addition, we have $\tilde{r}(G, H) \leq \hat{r}(G, H)$, where $\tilde{r}(G, H)$ is the on-line Ramsey number (the definition and properties of these numbers can be found in [2]). If both $G$ and $H$ are complete graphs then $F = K_r$ (see [3]). The case of complete graph is one of a few cases for which that upper bound is reached. In general, the more sparse both graphs $G$ and $H$ are, the problem of finding the restricted size Ramsey number for those pair of graphs is harder. Only two results for the exact value of restricted size Ramsey number involving a class of graph known so far, that are, for $K_{1,k}$ versus $K_n$ ([7]) and $G$ versus $K_{1,k}$, where $G$ is $K_3$, $K_4 - e$, or $C_5$ ([4]). For other few classes of graphs, the problem is solved partially.

Some results for size Ramsey number was presented by Faudree and Schelp in 2002 ([6]). It had shown that $r^*(P_3, C_4) = 8$, $r^*(P_3, C_5) = 6$, $r^*(P_3, C_6) = 9$. In 2015, Silaban et al. proved the last known exact value, namely $r^*(P_3, C_6) = 9$ [9]. In addition, they give lower and upper bound for $r^*(P_3, C_n)$, where $n \geq 8$ is even (see Theorem 3 below). In this paper, we determine previously unknown restricted size Ramsey numbers for $P_3$ versus $C_n$, namely $r^*(P_3, C_n)$ for $7 \leq n \leq 12$, and we improve the upper bound for $r^*(P_3, C_n)$, that is we prove that $r^*(P_3, C_n) \leq 2n - 2$ for even $n \geq 10$.

In general, we follow graph theory terminology and notation of [9].

2. Known results

In this section, we list a few known definitions and theorems that we will need in proving our results.

The Turán number $ex(n, G)$ is the maximum number of edges in any $n$-vertex graph which does not contain a subgraph isomorphic to $G$. A graph on $n$ vertices is said to be extremal with respect to $G$ if it does not contain a subgraph isomorphic to $G$ and has exactly $ex(n, G)$ edges.

In 1989, Clapham et al. [1] determined all values of $ex(n, C_4)$ for $n \leq 21$. They also characterized all the corresponding extremal graphs. In Theorem 1 we quote value for $ex(7, C_4)$ and we show (Figure 1) five corresponding extremal graphs. We will use this in the proof of Theorem 5.

**Theorem 1** ([1]).

$$ex(7, C_4) = 9$$

and there are 5 extremal graphs for this number illustrated in Figure 1.

In our work we will also use a well known the Ramsey number for paths and cycles that was calculated by Faudree et al. in [5].

**Theorem 2** ([5]). For all integers $n \geq 4$,

$$r(P_3, C_n) = n.$$
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In 2015, Silaban et al. [9] proved the lower and the upper bound for the restricted size Ramsey number for $P_3$ and cycles. At the end of our article we improve the upper bound for this number.

**Theorem 3** ([9]). For even $n \geq 8$,
\[
\frac{3}{2}n + 2 \leq r^*(P_3, C_n) \leq 2n - 1.
\]

3. New results

In order to find the value of $r^*(P_3, C_n)$, we find a graph $F$ with the smallest possible size such that $F \rightarrow (P_3, C_n)$. According to Theorem 2 the graph $F$ must have $n$ vertices.

3.1. Determining the value of $r^*(P_3, C_7)$

First, we give the following condition for graph $F$ satisfying $F \rightarrow (P_3, C_7)$.

**Lemma 4.** Let $F$ be a graph with $|V(F)| = 7$ and $C_4 \subseteq F$, then $F \not\rightarrow (P_3, C_7)$.

**Proof.** Suppose there is $F$ with $|V(F)| = 7$ such that $F$ contains cycle $C_4$, say $v_1, v_2, v_3, v_4, v_1$. By coloring possible edges $v_1v_3, v_2v_4 \in E(F)$ by red and the remaining edges of $F$ by blue, we obtain a 2-coloring of $F$ which contains neither a red $P_3$ nor a blue $C_7$. \qed

**Theorem 5.** $r^*(P_3, C_7) = 13$.

**Proof.** First, we will prove that $r^*(P_3, C_7) \geq 13$. From Lemma 4 and Theorem 1 we imply that $r^*(P_3, C_7) \geq 12$. Suppose that $r^*(P_3, C_7) = 12$. Let $F$ be a graph on 7 vertices and 12 edges. By Lemma 4, if $F \rightarrow (P_3, C_7)$, then $C_4 \not\subseteq F$ and therefore $F$ is one of the five graphs $G_i, 1 \leq i \leq 5$ from Figure 1. Furthermore, since $\Delta(G_i) \geq 4$ for $i \in \{1, 2, 3\}$, and by coloring $u_2u_7, u_3u_5, v_1v_6, v_3v_7, v_4v_5$ in red (see Figure 2) we obtain, for all $G_i$, a 2-coloring of edges which contains neither a red $P_3$ nor a blue $C_7$. In fact, if $\Delta(G_i) \geq 4$, then there is a vertex of degree at most 2 in $G_i$. To avoid a blue $C_7$ we color in red one edge coming out of this vertex (if any). Hence, $F \not\rightarrow (P_3, C_7)$ and consequently we have $r^*(P_3, C_7) \geq 13$. 

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![Figure 1. All extremal graphs for $ex(7, C_4)$](image-url)
Next, we will show that $r^*(P_3, C_7) \leq 13$. Let $F_7$ be the complement of the graph shown in Figure 3. To prove that $F_7 \rightarrow (P_3, C_7)$, let $\chi$ be any red-blue coloring of edges of $F_7$ such that there is no red $P_3$ in $F_7$. We will show that the coloring $\chi$ will imply a blue $C_7$ in $F_7$. To do so, consider vertex $v_4$. There are 4 edges incidence to this vertex, at most one of them can be colored by red. Up to the symmetry of $F_7$, without loss of generality, we can assume that $v_1v_4$ is red or all edges $v_iv_4$, $i \in \{1, 3, 6, 7\}$ are blue. Nonexistence a red $P_3$ forces the red edges to be a matching and that it suffices to consider maximum matchings. Then, using symmetries, there are only five subcases to discuss.

1. Edge $v_1v_4$ is red.
   1.1 if $v_2v_5$ and $v_3v_6$ is red, then $v_1, v_5, v_3, v_4, v_6, v_2, v_7, v_1$ is the blue cycle,
   1.2 if $v_2v_6$ and $v_3v_5$ are red, then $v_1, v_5, v_2, v_7, v_4, v_3, v_6, v_1$ is the blue cycle,
   1.3 if $v_2v_6$ and $v_3v_7$ are red, then the cycle $v_1, v_6, v_4, v_3, v_5, v_2, v_7, v_1$ is blue.

2. All edges $v_iv_4$, $i \in \{1, 3, 6, 7\}$ are blue. Then we have two subcases:
   2.1 if $v_2v_5$, $v_1v_6$, $v_3v_7$ are red, then we obtain the following blue cycle: $v_1, v_5, v_3, v_4, v_6, v_2, v_7, v_1$,
   2.2 if $v_2v_6$, $v_1v_5$, $v_3v_7$ are red, then the cycle: $v_1, v_4, v_6, v_3, v_5, v_2, v_7, v_1$ is blue.

For all cases, there is always a blue $C_7$, so $F_7 \rightarrow (P_3, C_7)$ and the proof is complete.
3.2. Upper bounds for $r^*(P_3, C_n)$

In [9] Silaban et al. proved that $r^*(P_3, C_n) \leq 2n - 1$. In this section we will show that this upper bound can be improved and we prove the following theorem.

**Theorem 6.** For even $n \geq 12$, $r^*(P_3, C_n) \leq 2n - 2$.

![Figure 4. The graph $F_n \rightarrow (P_3, C_n)$ for $n \geq 12$ and even $n, t = \frac{n-2}{2}$.](image)

**Proof.** Let $t = \frac{n-2}{2}$ and let $F_n$ be a graph with

$$V(F_n) = \{x, y\} \cup \{u_i, v_i| i = 1, \ldots, t\}$$

and

$$E(F_n) = \{xu_1, xv_1, xu_3, u_ty, v_ty, v_{t-2}y\} \cup S,$$

where

$$S = \{u_iu_{i+1}, v_iv_{i+1}, v_iu_{i+1}, u_iv_{i+1}| i = 1, \ldots, t - 1\}$$

(see Fig. 4). In order to prove that $F_n \rightarrow (P_3, C_n)$, let $\chi$ be any red-blue coloring of edges of $F_n$ such that there is no red $P_3$ in $F_n$. We will show that the coloring $F_n$ will imply a blue $C_n$ in $F_n$.

**FACT 1.** Observe that if we have any two independent blue paths to $u_i$ and $v_i$, then we can extend these paths step by step to vertices $u_j$ and to $v_j$ for $1 \leq i < j \leq t$. To do so, let us consider the vertex $u_i$. Since under the coloring $\chi$ there is no red $P_3$, at most one of edges $\{u_iu_{i+1}, u_iv_{i+1}\}$ can be red. If $u_iu_{i+1}$ is red, then $\{u_iu_{i+1}, u_iv_{i+1}\}$ must be blue. Using these 2 blue edges, we can extend our blue paths to $u_{i+1}$ and $v_{i+1}$, independently. If $u_iv_{i+1}$ is red, then $\{u_iu_{i+1}, v_iv_{i+1}\}$ must be blue. Using these 2 blue edges, we also can extend our blue paths to $u_{i+1}$ and $v_{i+1}$, independently. We can do the same process to extend our blue paths until reaching $u_j$ and $v_j$.

**FACT 2.** There are always two independent blue paths from $x$ to $u_i$ and from $x$ to $v_i$ for $i = 1$ or $i = 3$. To prove this fact, let us consider the vertex $x$. There are 3 incident edges to this vertex, at most one of them can be colored by red. Up to the symmetry of $F_n$, we can assume that at most one edge of set $\{xu_1, xu_3\}$ is red.
If \( xu_3 \) is red, then \( xu_1 \) and \( xv_1 \) must be blue, therefore we have two blue paths from \( x \) to \( u_1 \) and from \( x \) to \( v_1 \). Note that a similar situation occurs if none of edges incidence to \( x \) is red.

Now we can assume that \( xu_1 \) is red. In this case \( xv_1 \) and \( xv_3 \) are blue so we have one path from \( x \) to \( u_3 \). We will construct a path of size 6 with the set \( \{ u_1, u_2, v_1, v_2 \} \) as inner vertices, namely the path \( xv_3u_1v_2u_2v_3 \).

Similarly, using the symmetry of \( F_n \), we get two independent blue paths from \( y \) to \( u_j \) and from \( y \) to \( v_j \) for \( j = t \) or \( j = t - 2 \).

By using Fact 1 and 2, we obtain a blue cycle \( C_n \) in \( F_n \). Observe that the theorem holds for \( 3 \leq t - 2 \) and \( n \geq 12 \).

Silaban et al. ([9]) gave the upper bound for the restricted size Ramsey number of \( P_3 \) versus \( P_n \). They proved that for even \( n > 8 \), \( r^*(P_3, P_n) \leq 2n - 1 \). From the proof of Theorem 6 we see that if we delete edge \( xu_3 \) then for any 2-coloring of edges of \( F_n \setminus \{ xu_3 \} \) that avoid red \( P_3 \), it must imply a blue \( P_n \) in \( F_n \). It means we get a better upper bound of the restricted size Ramsey number for \( P_3 \) versus \( P_n \), even \( n \geq 12 \), as given in the following corollary.

**Corollary 7.** For \( n \geq 12 \) and even \( n \), \( r^*(P_3, P_n) \leq 2n - 3 \).

### 3.3. Computational Approach

In this subsection we use a computational approach to determine the exact values of \( r^*(P_3, C_n) \), \( 8 \leq n \leq 12 \). We use the following Algorithm 1 to find such numbers.

**Algorithm 1** Deciding whether graph \( F \rightarrow (P_3, C_n) \) or not

```plaintext
Require: Adjacency matrix of biconnected graph \( F \) on \( n \) vertices.
Ensure: \( F \rightarrow (P_3, C_n) \) or \( F \not\rightarrow (P_3, C_n) \).
1: \textbf{for} \( m = \lfloor \frac{n}{2} \rfloor \rightarrow 0 \) \textbf{do}
2: \quad \textbf{for} every subset \( S \) of \( m \) edges that compose independent edge set \textbf{do}
3: \quad \quad \( F' = F - S \)
4: \quad \quad \textbf{find a Hamiltonian cycle in} \( F' \)
5: \quad \quad \textbf{if} no Hamiltonian cycle in \( F' \) \textbf{then return} \( F \not\rightarrow (P_3, C_n) \), \textbf{Break}.
6: \quad \quad \textbf{end if}
7: \quad \textbf{end for}
8: \quad \( m := m - 1 \)
9: \textbf{end for}
10: \textbf{return} \( F \rightarrow (P_3, C_n) \)
```

We generate all the adjacency matrices of biconnected graphs with \( n \) vertices (\( 8 \leq n \leq 12 \)) with minimum degree 3 by using a program called *geng* [8]. In order to find Hamiltonian cycle, algorithm uses recursive depth-first search. Note that the starting point should not matter as the cycle can be started from any point.
Table 1: Restricted size Ramsey numbers $r^*(P_3, C_n)$, $8 \leq n \leq 12$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^*(P_3, C_n)$</td>
<td>15</td>
<td>17</td>
<td>18</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td>$#{F \rightarrow (P_3, C_n),</td>
<td>E(F)</td>
<td>= r^*(P_3, C_n)}$</td>
<td>10</td>
<td>16</td>
<td>2</td>
</tr>
</tbody>
</table>

From the above algorithm, we obtain the results which are presented in Table 1. This table provides the value of $r^*(P_3, C_n)$ and the number of non-isomorphic graphs $F$ of order $n$ and size $r^*(P_3, C_n)$ such that $F \rightarrow (P_3, C_n)$. Based on computer calculations, it turned out that in fact it was enough to consider cases where $m \in \{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor - 2\}$.

Examples of resulting graphs are presented in Fig. 5, 6, 7 and 4. For the number $r^*(P_3, C_8)$ an example is a graph $K_{4,4} - e$.

Figure 5. Complement of the graph $F_9 \rightarrow (P_3, C_9)$.

Figure 6. Graph $F_{10} \rightarrow (P_3, C_{10})$.

4. Conclusion

In this paper we established six new restricted size Ramsey numbers $r^*(P_3, C_n)$ for $7 \leq n \leq 12$. In addition, we gave the new upper bound for $n \geq 10$ and even $n$. It follows that the first open case of $r^*(P_3, C_n)$ is now $r^*(P_3, C_{13})$ and is certainly worth of further investigation. Based on results known earlier and described in this work as well as computer experiments for some bipartite graphs that are not presented here, let us formulate the following conjecture.

**Conjecture 8.** For all $n \geq 10$, we have

$$r^*(P_3, C_n) = 2n - 2.$$
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References


