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Non-isomorphic signatures on some generalised Petersen graph

Deepak Sehrawat, Bikash Bhattacharjya

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, India - 781039

deepakmath55555@iitg.ac.in, b.bikash@iitg.ac.in

Abstract

In this paper we find the number of different signatures of $P(3,1)$, $P(5,1)$ and $P(7,1)$ up to switching isomorphism, where $P(n,k)$ denotes the generalised Petersen graph, $2k < n$. We also count the number of non-isomorphic signatures on $P(2n+1,1)$ of size two for all $n \geq 1$, and we conjecture that any signature of $P(2n+1,1)$, up to switching, is of size at most $n+1$.

Keywords: signed graph, generalised Petersen graph, balance, switching, switching isomorphism
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1. Introduction

Throughout the paper we consider simple graphs. For all the graph-theoretic terms that have not been defined but are used in the paper, see Bondy [1]. Harary [4] firstly introduced the notion of signed graph and balance. Harary [2] used them to model social stress in small groups of people in social psychology. Subsequently, signed graphs have turned out to be valuable. The fundamental property of signed graphs is balance. A signed graph is balanced if all its cycles have positive sign product. The second basic property of signed graphs is switching equivalence. Switching is a way of turning one signature of a graph into another, without changing cycle signs. Many properties of signed graphs are unaltered by switching, the set of negative cycles is a notable example. In [6], the non-isomorphic signatures on the Heawood graph are studied. The author in [9] determined the non-isomorphic signed Petersen graph, using the fact that the minimum signature on a cubic

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graph is a matching. Using the same technique, we find the number of non-isomorphic signatures on $P(3,1), P(5,1)$ and $P(7,1)$. We also determine the number of non-isomorphic signatures of size two in $P(2n+1,1)$ for all $n \geq 1$.

2. Preliminaries

A *signified graph* is a graph G together with an assignment of $+$ or $-$ signs to its edges. If Σ is the set of negative edges, then we denote the signified graph by (G,Σ). The set Σ is called the *signature* of (G,Σ). Signature Σ can also be viewed as a function from $E(G)$ into $\{+1,-1\}$. A *resigning (switching)* of a signified graph at a vertex v is to change the sign of each edge incident to v. We say (G,Σ_2) is *switching equivalent* to (G,Σ_1) if it is obtained from (G,Σ_1) by a sequence of switchings. Equivalently, we say that (G,Σ_2) is switching equivalent to (G,Σ_1) if there exists a function $f : V \to \{+1,-1\}$ such that $\Sigma_2(e) = f(u)\Sigma_1(e)f(v)$ for each edge $e = uv$ of G. Resigning defines an equivalence relation on the set of all signified graphs over G (also on the set of signatures). Each such class is called a *signed graph* and is denoted by $[G,\Sigma]$, where (G,Σ) is any member of the class.

We say two signified graphs (G,Σ_1) and (H,Σ_2) to be *isomorphic* if there exists a graph isomorphism $\psi : V(G) \to V(H)$ which preserve the edge signs. We denote it by $\Sigma_1 \cong \Sigma_2$. They are said to be *switching isomorphic* if Σ_1 is isomorphic to a switching of Σ_2. That is, there exists a representation (H,Σ_2') which is equivalent to (H,Σ_2) such that $\Sigma_1 \cong \Sigma_2'$. We denote it by $\Sigma_1 \sim \Sigma_2$.

Proposition 2.1. *[5] If G has m edges, n vertices and c components, then there are $2^{(m-n+c)}$ distinct signed graphs of G.*

A cycle in a signified graph (G,Σ) is called *positive* if the product of its edge signs is positive and *negative*, otherwise. A signified graph (G,Σ) is called *balanced* if each cycle in (G,Σ) is positive and *unbalanced*, otherwise. One of the first theorems in the theory of signed graphs tells that the set of negative cycles uniquely determines the class of signed graphs to which a signified graph belongs. More precisely, we state the following theorem.

Theorem 2.1. *[8] Two signatures Σ_1 and Σ_2 of a graph G are equivalent if and only if they have the same set of negative cycles.*

3. Notations

The *distance* between two vertices x and y in a graph G, denoted by $d_G(x,y)$, is the length of a shortest path connecting x and y. The *distance* between two edges $e_1 = u_1u_2$ and $e_2 = v_1v_2$ in a graph G, denoted by $d_G(e_1,e_2)$, is $\min\{d_G(u_i,v_j) : i \in \{1,2\}, j \in \{1,2\}\}$. For example, $d_G(e_1,e_2) = 1$ for the edges $e_1 = u_0u_1$ and $e_2 = u_2v_2$ of the graph $P(3,1)$ in Figure 1. Throughout this paper, the solid lines and dotted lines in a graph represent positive and negative edges respectively.

In a signed graph $[G,\Sigma]$, a signature Σ' which is equivalent to Σ is said to be a *minimum signature* if the number of edges in Σ' is minimum among all equivalent signatures of Σ. We denote the number of edges in $\Sigma^{'}$ by $|\Sigma^{'}|$. For example, if $[G, \Sigma]$ is balanced then $\Sigma^{'} = \emptyset$ and thus $|\Sigma^{'}| = 0$. A signed graph may have more than one minimum signatures. To see this, take a signified complete graph (K_3, Σ), where $\Sigma = \{12, 23, 31\}$. Switching (K_3, Σ) by vertex 3 gives an equivalent signature $\Sigma_1 = \{12\}$ and by vertex 1 gives an equivalent signature $\Sigma_2 = \{23\}$. Clearly, Σ_1 and Σ_2 are minimum signatures of (K_3, Σ).

The following theorem tells about the maximum degree of a vertex in a minimum signature, when the signature is considered as a spanning subgraph of a given graph G.

Theorem 3.1. *Let $[G, \Sigma]$ be a signed graph on n vertices and let Σ' be an equivalent minimum signature of Σ. Then $d_{G_{\Sigma'}}(v) \leq \lfloor \frac{n-1}{2} \rfloor$ for each vertex $v \in V(G_{\Sigma'})$.*

Proof. Let, if possible, there exists a vertex $u \in V(G_\Sigma)$ such that $d_{G_\Sigma}(u) > \frac{n-1}{2}$. Resign at u to get an equivalent signature Σ_1. It is clear that $|\Sigma| > |\Sigma_1|$. We apply the same operation on Σ_1, if G_{Σ_1} has a vertex of degree greater than $\frac{n-1}{2}$. Repeated application, if needed, of this process will ultimately give us an equivalent signature $\tilde{\Sigma}$ of minimum number of edges such that degree of every vertex of $\tilde{\Sigma}$ is at most $\lfloor \frac{n-1}{2} \rfloor$. It is clear that $|\tilde{\Sigma}| = |\Sigma'|$, and every vertex of Σ' has degree at most $\lfloor \frac{n-1}{2} \rfloor$. □

The following theorem will remain our key result throughout this paper.

Theorem 3.2. *[9] Every minimum signature of a cubic graph is a matching.*

From now onward, matching of a graph G stands for a minimum signature. With a few exceptions, most of the time switching transforms a matching (when considered as a signature) of $P(n, 1)$ to a new matching. The notation $\Sigma(e_1, e_2, \ldots, e_k)$ denotes a signature or a set of edges Σ which contains the edges $e_1, e_2, \ldots, e_k$ of a graph. For example, in the graph $P(3, 1)$ of Figure 1, $\Sigma(u_0u_1, v_1v_2)$ denotes a signature containing the edges u_0u_1 and v_1v_2.

Further, we say that two signatures Σ_1 and Σ_2 of a graph G are *automorphic* if there exists an automorphism f of G such that $uv \in \Sigma_1$ if and only if $f(u)f(v) \in \Sigma_2$. If two signatures are automorphic then they are said to be *automorphic type* signatures. If two signatures Σ_1 and Σ_2 of a graph G are not automorphic to each other, then we say that they are *distinct automorphic type* signatures. For example, in the signed graphs $[P(5, 1), \{u_1u_2\}]$ and $[P(5, 1), \{u_3u_4\}]$, the signatures $\{u_1u_2\}$ and $\{u_3u_4\}$ are automorphic type signatures.

4. Generalised Petersen Graph

Let n and k be positive integers such that $2 \leq 2k < n$. The *generalized Petersen graph*, denoted by $P(n, k)$, is defined to have the vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge set

$$\{u_iu_{i+1} : i = 0, 1, \ldots, n-1\} \cup \{v_iv_{i+k} : i = 0, 1, \ldots, n-1\} \cup \{u_iv_i : i = 0, 1, \ldots, n-1\},$$

where the subscripts are read modulo n. We call the cycle $u_0u_1 \ldots u_{n-1}u_0$ as *outer cycle* and the cycle $v_0v_kv_{2k} \ldots v_0$ as *inner cycle* of $P(n, k)$. The edges of the form u_iv_i are called the *spokes* of $P(n, k)$. It is clear that $P(2n+1, 1)$ has $4n+2$ vertices and $6n+3$ edges. Now we discuss certain structural facts of $P(2n+1, 1)$, where $n \geq 1$.

Theorem 4.1. *For $n \geq 1$ and $2 \leq l \leq 2n+1$, the number of $2l$-cycles and the number of $(2n+1)$-cycles of $P(2n+1,1)$ are $2n+1$ and 2, respectively.*

Proof. It is obvious that the cycles given by $\{u_0, u_1, \ldots, u_{2n}\}$ and $\{v_0, v_1, \ldots, v_{2n}\}$ are the only cycles of length $2n+1$. This proves the second part of the theorem.

We prove the first part of the theorem by counting the number of $2l$-cycles, where $2 \leq l \leq (2n+1)$. It is important to note that any even cycle in $P(2n+1,1)$ must contain as many u_i's as v_i's. It is clear that for each $i = 0, 1, \ldots, 2n$, the cycle $u_i v_i v_{i+1} \ldots v_{i+(l-1)} u_{i+(l-1)} u_{i+(l-2)} \ldots u_{i+1} u_i$ is of length $2l$, and any even cycle of $P(2n+1,1)$ is of this form. Hence there are $2n+1$ cycles of $P(2n+1,1)$ of length $2l$, where $2 \leq l \leq (2n+1)$. This proves the theorem. □

Theorem 4.2. *The distance between any two edges in $P(2n+1,1)$ is at most n for all $n \geq 1$.*

Proof. It is clear that the distance between any two spokes of $P(2n+1,1)$ is at most n. Further, the distance between any two edges of the outer, as well as of the inner cycle, is at most n. Without loss of generality, if we pick the edge u_0u_1 from the outer cycle, then the edges $v_n v_{n+1}$ and $v_{n+1}v_{n+2}$ are the only edges of the inner cycle which are at maximum distance of n from u_0u_1. Similarly, $u_{n+1}v_{n+1}$ is the only spoke which is at maximum distance of n from u_0u_1. This completes the proof of the theorem. □

For each $k = 0, 1, 2, \ldots, 2n$, we define the permutations γ, ρ_k, δ_k of $V(G)$ such that for all $i = 0, 1, 2, \ldots, 2n$, we have

$$\gamma(u_i) = v_i, \gamma(v_i) = u_i \text{ and } \rho_k(u_i) = u_{i+k}, \rho_k(v_i) = v_{i+k};$$

$$\delta_k(u_i) = \begin{cases} u_i, & \text{if } \ i = k, \\ u_l, & \text{if } d(u_i, u_k) = d(u_l, u_k) \text{ and } i \neq k, i \neq l; \end{cases}$$

$$\delta_k(v_i) = \begin{cases} v_i, & \text{if } \ i = k, \\ v_l, & \text{if } d(v_i, v_k) = d(v_l, v_k) \text{ and } i \neq k, i \neq l. \end{cases}$$

Note that each ρ_k represents a clockwise rotation of $P(2n+1,1)$. Also each δ_k represents a reflection of $P(2n+1,1)$ about a line induced by the edge $u_k v_k$. Further, γ just swaps the inner and outer cycles of $P(2n+1,1)$. Thus the automorphism group of $P(2n+1,1)$ is given by

$$\text{Aut}(P(2n+1,1)) = \langle \rho_k,\ \delta_k,\ \gamma \mid k = 0, 1, 2, \ldots, 2n \rangle .$$

Accordingly, the automorphisms of $P(2n+1,1)$ are some combination of rotations, reflections and interchanges of v_i's with u_i's. Using this fact, if H_1 and H_2 are two given subgraphs of $P(2n+1,1)$, it is easier to decide whether there is an automorphism of $P(2n+1,1)$ that maps H_1 onto H_2.

Example 1. The graph $P(3,1)$ is given in Figure 1. The automorphism ρ_1 rotates the graph $P(3,1)$ clockwise through the angle $\frac{2\pi}{3}$, the automorphism δ_1 flips $P(3,1)$ about the line containing the edge u_1v_1 as its segment, and γ switches the cycles $u_0u_1u_2$ and $v_0v_1v_2$ to each other.

For more on automorphism group of generalised Petersen graph, see [3]. Yegnanarayanan [7] studied various aspects of the generalised Petersen graph.

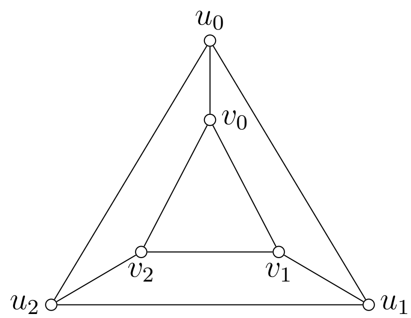


Figure 1: The graph $P(3,1)$.

5. Signings on $P(3,1)$

From Theorem 3.2, it is easy to see that finding non-isomorphic signatures on $P(3,1)$ is equivalent to determining the non-isomorphic matchings of $P(3,1)$ of size up to three. Let M_k denotes a matching of size k, where $k = 0, 1, 2, 3$. We classify all the automorphic type matchings of $P(3,1)$ of size up to three in the following lemmas. Let a matching of size zero be denoted by Σ_0.

Lemma 5.1. *The number of distinct automorphic type matchings of* $P(3,1)$ *of size one is two.*

Proof. A matching of size one that does not contain a spoke is $\Sigma_1(u_0u_1)$. A matching of size one containing a spoke is $\Sigma_2(u_0v_0)$. It is easy to see that any other matching of size one is automorphic to either Σ_1 or Σ_2, and that Σ_1 is not automorphic to Σ_2. This proves the lemma. □

Lemma 5.2. *The number of distinct automorphic type matchings of* $P(3,1)$ *of size two is four.*

Proof. We classify the matchings of size two by looking at the distance between their edges. Theorem 4.2 gives us that the distance between any two edges of $P(3,1)$ is at most one.

(i) Let M_2 have no spoke. We may assume that one edge is u_0u_1. There are two possibilities for such matchings of size two. One of such matchings is $\Sigma_3(u_0u_1, v_0v_1)$ and another is $\Sigma_4(u_0u_1, v_1v_2)$.

(ii) Let M_2 have one spoke and let it be u_0v_0. One of such matchings is $\Sigma_5(u_0v_o, u_1u_2)$.

(iii) Let M_2 have two spokes. One of such matchings is $\Sigma_6(u_0v_0, u_1v_1)$.

Any other matching of $P(3,1)$ of size two is automorphic to $\Sigma_3, \Sigma_4, \Sigma_5$ or Σ_6. Further, no two of these matchings are automorphic. This concludes the proof of the lemma. □

Lemma 5.3. *The number of distinct automorphic type matchings of* $P(3,1)$ *of size three is two.*

Proof. Any M_3 must contain at least one spoke, as at most one edge can be taken from the inner cycle as well as from the outer cycle. If a matching of size three contains two spokes of $P(3,1)$, then no other edge can be included in that matching. Thus, following are the possibilities for M_3.

(i) Let M_3 have one spoke and let it be u_0v_0. There is only one possibility for such a matching, and let it be $\Sigma_7(u_0v_0, u_1u_2, v_1v_2)$.

(ii) Let M_3 has three spokes and let that M_3 be $\Sigma_8(u_0v_0, u_1v_1, u_2v_2)$.

Any other matching of size three is automorphic to Σ_7 or Σ_8, and that Σ_7 is not automorphic to Σ_8. This completes the proof. □

The matchings obtained in the preceding lemmas along with Σ_0 give us nine different automorphic type matchings of $P(3,1)$ *viz.,* $\Sigma_0, \Sigma_1, \ldots, \Sigma_8$. However, some of these nine matchings may be switching isomorphic to each other. We have the following observations.

- In Σ_6, by resigning at u_0, u_1, u_2; we get a matching automorphic to Σ_2. Thus $\Sigma_6 \sim \Sigma_2$.
- In Σ_7, by resigning at u_1, v_1, v_0; we get a matching automorphic to Σ_4. Thus $\Sigma_7 \sim \Sigma_4$.
- In Σ_8, by resigning at u_0, u_1, u_2; we get a matching automorphic to Σ_0. Thus $\Sigma_8 \sim \Sigma_0$.

So we are left with the matchings $\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$, and their corresponding signed graphs are depicted in Figure 2, where the label of the vertices correspond to that of Figure 1. In the following theorem we show that these six matchings are not switching isomorphic to each other.

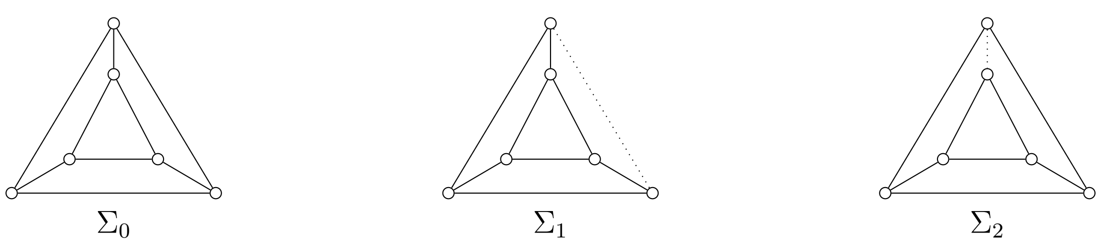
Theorem 5.1. *There are exactly six signed $P(3,1)$ up to switching isomorphisms.*

Proof. The number of negative 3-cycles and negative 4-cycles for the signed $P(3,1)$ shown in Figure 2 are given in Table 1. We see that the set of negative cycles are different for all these six

Table 1: Number of negative 3-cycles and negative 4-cycles of some signed $P(3,1)$.

	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5
Number of negative C_3	0	1	0	2	2	1
Number of negative C_4	0	1	2	0	2	3

signatures. So by Theorem 2.1, we conclude that all these six signatures are pairwise not switching isomorphic. This completes the proof. □



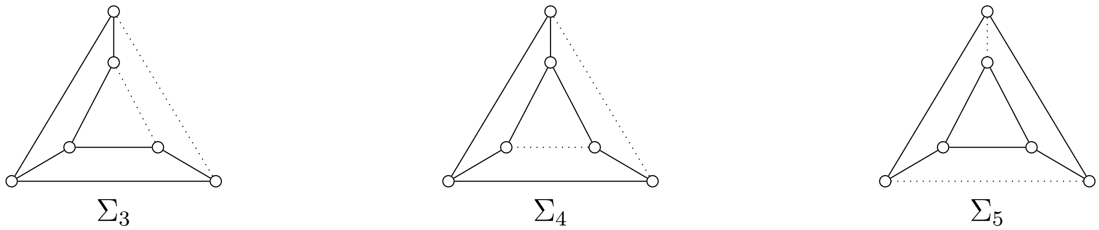


Figure 2: The six signed $P(3,1)$.

6. Signings on $P(5,1)$

The graph $P(5,1)$ is shown in Figure 3. Recall from Theorem 3.2 that finding non-isomorphic signatures of $P(5,1)$ is equivalent to finding matchings of $P(5,1)$ of sizes $0,1,2,3,4$ and 5, up to switching isomorphism. We now classify all the automorphic type matchings of $P(5,1)$ of sizes up to five. We denote a matching of size zero by Σ_0. We emphasize that at most two edges of a matching may lie on the outer cycle or on the inner cycle. We use this fact to get the possible automorphic type matchings of different sizes.

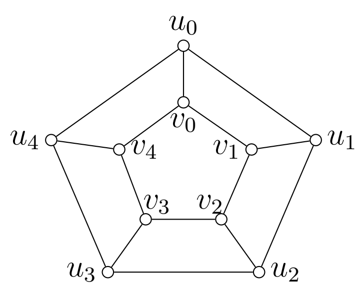


Figure 3: The graph $P(5,1)$.

Lemma 6.1. *The number of distinct automorphic type matchings of $P(5,1)$ of size one is two.*

Proof. We have only the following two cases.

(i) Let M_1 have no spoke. There is only one automorphic type matching of size one. One of such matchings is $\Sigma_1(u_0u_1)$.

(ii) Let M_1 have one spoke. There is also only one automorphic type matching of size one. One of such matchings is $\Sigma_2(u_0v_0)$.

Any other matching of $P(5,1)$ of size one is automorphic to Σ_1 or Σ_2, and that Σ_1 is not automorphic to Σ_2. This completes the proof. □

Lemma 6.2. *The number of distinct automorphic type matchings of $P(5,1)$ of size two is eight.*

Proof. We classify the matchings of size two by looking at the distance between the edges of the matching.

(i) Let the edges of the matching be at distance one. There are five different automorphic type matchings of size two and one of each such automorphic type matchings is $\Sigma_3(u_0u_1, v_0v_1)$, $\Sigma_4(u_0u_1, v_1v_2)$, $\Sigma_5(u_0u_1, u_2u_3), \Sigma_6(u_0u_1, v_2u_2)$ and $\Sigma_7(u_0v_0, u_1v_1)$. Let M_2 be a matching of size two other than $\Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6$ and Σ_7 whose edges are at distance one. Note that M_2 must contain either two spokes, or two edges from outer cycle, or two edges from inner cycle, or one edge from outer cycle and one from inner cycle, or one edge from outer/inner cycle and one spoke. In each of these cases, M_2 is automorphic to either $\Sigma_7, \Sigma_5, \Sigma_3, \Sigma_4$ or Σ_6. Thus $\Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6$ and Σ_7 are the only automorphic type matchings of size two whose edges are at distance one. It is clear that these matchings are pairwise non-automorphic.

(ii) Let edges of M_2 be at distance two. There are three automorphic type matchings of size two whose edges are at distance two. We denote them by $\Sigma_8(u_0u_1, v_2v_3), \Sigma_9(u_0u_1, v_3u_3)$ and $\Sigma_{10}(u_0v_0, u_2v_2)$. In a similar manner (as in case(i)), one can show that any other matching of size two whose edges are at distance two is automorphic to one of Σ_8, Σ_9 and Σ_{10}. The matchings Σ_8, Σ_9 and Σ_{10} are clearly pairwise non-automorphic.

This concludes the proof of the lemma. □

Lemma 6.3. *The number of distinct automorphic type matchings of* $P(5,1)$ *of size three is* 11.

Proof. We classify all the automorphic type matchings of size three by looking at the number of spokes contained in these matchings.

(i) Let M_3 have no spoke. Out of three edges of M_3, two edges lie on outer (inner) cycle and the remaining one edge lies on inner (outer) cycle. Because of the automorphism γ, we may assume that two edges are lying on the outer cycle, and let they be u_0u_1 and u_2u_3. Therefore, the possible automorphic type matchings for this case are $\Sigma_{11}(u_0u_1, v_0v_1, u_2u_3)$, $\Sigma_{12}(u_0u_1, v_1v_2, u_2u_3)$ and $\Sigma_{13}(u_0u_1, v_3v_4, u_2u_3)$. Any other matching of size three which does not contain a spoke is automorphic to one of Σ_{11}, Σ_{12} and Σ_{13}. Further, these matchings are not automorphic to each other.

(ii) Let M_3 have one spoke and let it be u_0v_0. If the other two edges of M_3 lie either on the outer cycle or on the inner cycle, then one of such matchings is $\Sigma_{14}(u_0v_0, u_1u_2, u_3u_4)$. If one edge of M_3 lies on the outer cycle and one lies on the inner cycle, then following are the only possibilities: $\Sigma_{15}(u_0v_0, u_1u_2, v_1v_2), \Sigma_{16}(u_0v_0, u_1u_2, v_2v_3)$ and $\Sigma_{17}(u_0v_0, u_2u_3, v_2v_3)$. Any other matching of size three containing only one spoke is automorphic to one of $\Sigma_{14}, \Sigma_{15}, \Sigma_{16}$ and Σ_{17}. Further, no two of these matchings are automorphic.

(iii) Let M_3 have two spokes. If the spokes are consecutive then there is only one possibility, *viz.,* $\Sigma_{18}(u_0v_0, u_1v_1, u_2u_3)$. If the spokes are not consecutive then there is also only one possibility, *viz.*, $\Sigma_{19}(u_0v_0, v_2u_2, u_3u_4)$. Any other matching of size three containing only two spokes is automorphic to Σ_{18} or Σ_{19}, and that Σ_{18} is not automorphic to Σ_{19}.

(iv) Let M_3 have three spokes. In this case, there are only two automorphic type matchings of size three and one of each such type of matchings is $\Sigma_{20}(u_0v_0, u_1v_1, u_2v_2)$ and $\Sigma_{21}(u_0v_0, u_1v_1, u_3v_3)$. Any other matching of size three containing only spokes is automorphic Σ_{20} or Σ_{21}.

This proves the lemma. □

Lemma 6.4. *The number of distinct automorphic type matchings of $P(5,1)$ of size four is 10.*

Proof. We classify the matchings of size four by considering the number of spokes contained in these matchings.

(i) Let M_4 have no spoke. Note that, at most two edges of M_4 may lie on the outer cycle and at most two edges may lie on the inner cycle. So, without loss of generality, let the edges u_0u_1 and u_2u_3 be lie on the outer cycle. Thus following are the only possibilities for the matchings of size four having no spoke: $\Sigma_{22}(u_0u_1, u_2u_3, v_0v_1, v_2v_3), \Sigma_{23}(u_0u_1, u_2u_3, v_0v_1, v_3v_4)$ and $\Sigma_{24}(u_0u_1, u_2u_3, v_1v_2, v_3v_4)$. Any other matching of size four which does not contain a spoke is automorphic to one of Σ_{22}, Σ_{23} and Σ_{24}. Further, these matchings are pairwise non-automorphic.

(ii) Let M_4 have one spoke and let it be u_0v_0. Out of the three remaining edges, two edges will lie on the outer (inner) cycle and one edge will lie on inner (outer) cycle. The two edges which lie on the outer cycle can be taken to be u_1u_2 and u_3u_4. Thus the possible automorphic type matchings of size four are $\Sigma_{25}(u_0v_0, u_1u_2, u_3u_4, v_1v_2)$ and $\Sigma_{26}(u_0v_0, u_1u_2, u_3u_4, v_3v_2)$. Any other matching of size four having only one spoke is automorphic to either Σ_{25} or Σ_{26}, and that Σ_{25} is not automorphic to Σ_{26}.

(iii) Let M_4 have two spokes. If spokes are at distance one then let they be u_0v_0 and u_1v_1. Further, out of remaining two edges, only one edge may lie on outer cycle and other may lie on inner inner. Let u_2u_3 lies on outer cycle. Then, the possible automorphic type matchings are $\Sigma_{27}(u_0v_0, u_1v_1, u_2u_3, v_2v_3)$ and $\Sigma_{28}(u_0v_0, u_1v_1, u_2u_3, v_3v_4)$. If the two spokes are at distance two then let they be u_0v_0 and u_2v_2. The only possibility for such matching is $\Sigma_{29}(u_0v_0, u_2v_2, u_3u_4, v_3v_4)$. Any other matching of size four with only two spokes is automorphic to one of Σ_{27}, Σ_{28} and Σ_{29}. Also these matchings are pairwise non-automorphic.

(iv) Let M_4 have three spokes. If one of the spokes is at distance two from the other two spokes, then no edge from the outer or inner cycle can be contained in M_4. Therefore, the only possibility is $\Sigma_{30}(u_0v_0, u_1v_1, u_2v_2, u_3u_4)$.

(v) Let M_4 have four spokes. One of such matchings is $\Sigma_{31}(u_0v_0, u_1v_1, u_2v_2, u_3v_3)$. Any other matching for this case is automorphic to Σ_{31}.

This completes the proof of the lemma. □

Lemma 6.5. *The number of distinct automorphic type matchings of $P(5,1)$ of size five is three.*

Proof. It is clear that a matching M_5 of size five must have at least one spoke. Further, if M_5 has exactly two spokes, then only three vertices are unsaturated in the outer cycle as well as in the inner cycle. However, we must have at least two edges in M_5 either from the outer cycle or from the inner cycle. Therefore, M_5 cannot have exactly two spokes. Similarly, M_5 cannot have four spokes. Thus following are the only possible cases.

(i) Let M_5 have one spoke. There is only one such automorphism type M_5. We denote it by $\Sigma_{32}(u_0v_0, u_1u_2, v_1v_2, u_3u_4, v_3v_4)$.

(ii) Let M_5 have three spokes. There is only one automorphism type of M_5. We denote it by $\Sigma_{33}(u_0v_0, u_1v_1, u_2v_2, u_3u_4, v_3v_4)$.

(iii) Let M_5 have five spokes. There is also only one automorphism type of M_5. We denote it by $\Sigma_{34}(u_0v_0, u_1v_1, u_2v_2, u_3v_3, u_4v_4)$.

This completes the proof of lemma. □

The matchings obtained in the preceding lemmas along with Σ_0 give us 35 different automorphic type matchings of $P(5,1)$ *viz.,* $\Sigma_0, \Sigma_1, \ldots, \Sigma_{34}$. However, some of these 35 matchings may be switching isomorphic to each other. We have the following observations.

- In Σ_7, by resigning at u_0, u_1; we get a matching automorphic to Σ_5. Thus $\Sigma_7 \sim \Sigma_5$.
- In Σ_{11}, by resigning at u_1, u_2, v_1, v_2; we get a matching automorphic to Σ_1. Thus $\Sigma_{11} \sim \Sigma_1$.
- In Σ_{12}, by resigning at u_1, u_2, v_1; we get a matching automorphic to Σ_6. Thus $\Sigma_{12} \sim \Sigma_6$.
- In Σ_{13}, by resigning at u_1, v_1, u_2, v_2, v_3; we get a matching automorphic to Σ_9. Thus $\Sigma_{13} \sim \Sigma_9$.
- In Σ_{14}, by resigning at u_0, u_1, u_4; we get a matching automorphic to Σ_{10}. Thus $\Sigma_{14} \sim \Sigma_{10}$.
- In Σ_{15}, by resigning at u_0, u_1, v_1; we get a matching automorphic to Σ_4. Thus $\Sigma_{15} \sim \Sigma_4$.
- In Σ_{17}, by resigning at u_0, u_1, u_2, v_1, v_2; we get a matching automorphic to Σ_4. Thus $\Sigma_{17} \sim \Sigma_4$.
- In Σ_{18}, by resigning at u_1, u_2, u_0; we get a matching automorphic to Σ_9. Thus $\Sigma_{18} \sim \Sigma_9$.
- In Σ_{20}, by resigning at u_0, u_1, u_2; we get a matching automorphic to Σ_5. Thus $\Sigma_{20} \sim \Sigma_5$.
- In Σ_{21}, by resigning at u_0, u_1, u_2, u_3, u_4; we get a matching automorphic to Σ_{10}. Thus $\Sigma_{21} \sim \Sigma_{10}$.
- In Σ_{22}, by resigning at u_1, v_1, u_2, v_2; we get a matching automorphic to Σ_0. Thus $\Sigma_{22} \sim \Sigma_0$.
- In Σ_{23}, by resigning at u_1, u_2, v_1, v_2, v_3; we get a matching automorphic to Σ_2. Thus $\Sigma_{23} \sim \Sigma_2$.
- In Σ_{24}, by resigning at u_1, u_2, v_2, v_3; we get a matching automorphic to Σ_6. Thus $\Sigma_{24} \sim \Sigma_6$.
- In Σ_{25}, by resigning at $u_0, u_1, u_4, v_0, v_1, v_4$; we get a matching automorphic to Σ_6. Thus $\Sigma_{25} \sim \Sigma_6$.
- In Σ_{26}, by resigning at u_0, u_1, u_4; we get a matching automorphic to Σ_{19}. Thus $\Sigma_{26} \sim \Sigma_{19}$.
- In Σ_{27}, by resigning at u_0, u_1, u_2, v_2; we get a matching automorphic to Σ_8. Thus $\Sigma_{27} \sim \Sigma_8$.
- In Σ_{28}, by resigning at u_0, u_1, u_2; we get a matching automorphic to Σ_{16}. Thus $\Sigma_{28} \sim \Sigma_{16}$.
- In Σ_{29}, by resigning at u_3, v_2, v_3; we get a matching automorphic to Σ_{16}. Thus $\Sigma_{29} \sim \Sigma_{16}$.

- In Σ_{30}, by resigning at u_0, u_1, u_2, u_4; we get a matching automorphic to Σ_6. Thus $\Sigma_{30} \sim \Sigma_6$.
- In Σ_{31}, by resigning at u_0, u_1, u_2, u_3, u_4; we get a matching automorphic to Σ_2. Thus $\Sigma_{31} \sim \Sigma_2$.
- In Σ_{32}, by resigning at v_2, u_2, u_3, v_3; we get a matching automorphic to Σ_2. Thus $\Sigma_{32} \sim \Sigma_2$.
- In Σ_{33}, by resigning at u_0, u_1, u_2, u_3, v_3; we get a matching automorphic to Σ_8. Thus $\Sigma_{33} \sim \Sigma_8$.
- In Σ_{34}, by resigning at u_0, u_1, u_2, u_3, u_4; we get a matching automorphic to Σ_0. Thus $\Sigma_{34} \sim \Sigma_0$.

Thus we are left with 12 different matchings *viz.,* $\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_8, \Sigma_9, \Sigma_{10}, \Sigma_{16}$, and Σ_{19}. The corresponding signified graphs of these 12 matchings are shown in Figure 4, where the label of the vertices correspond to that of Figure 3.

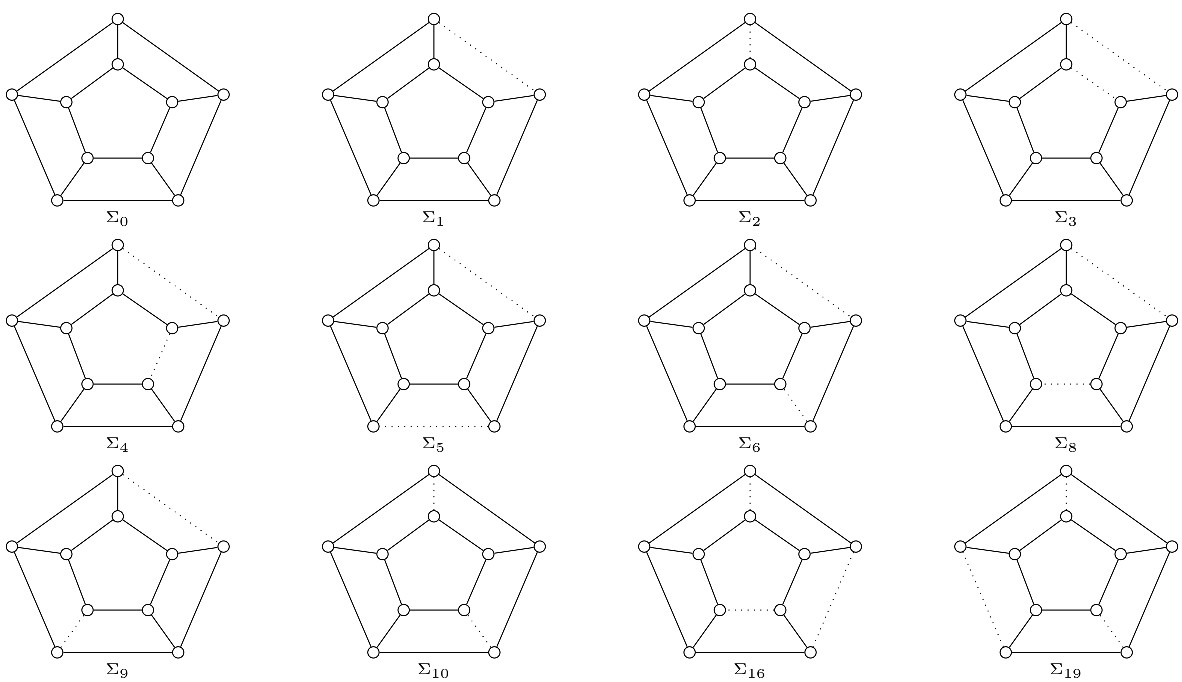


Figure 4: Twelve signed $P(5,1)$.

Theorem 6.1. *There are exactly twelve signed $P(5,1)$ up to switching isomorphism.*

Proof. The number of negative 4-cycles, negative 5-cycles and negative 6-cycles for the 12 signed $P(5,1)$ in Figure 4 are given in Table 2.

From Theorem 2.1 and Table 2, it is easy to see that the twelve signed $P(5,1)$, shown in Figure 4, are non-switching isomorphic. This concludes the proof of the theorem. □

Table 2: Number of negative 4-cycles, 5-cycles, 6-cycles of some signed $P(5,1)$.

	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_8	Σ_9	Σ_{10}	Σ_{16}	Σ_{19}
number of negative C_4	0	1	2	0	2	2	3	2	3	4	4	5
number of negative C_5	0	1	0	2	2	0	1	2	1	0	2	1
number of negative C_6	0	2	2	0	2	4	2	4	4	2	2	0

7. Signings on $P(7,1)$

The graph $P(7,1)$ is shown in Figure 5. From Theorem 4.1, we see that the number of 4-cycles, 6-cycles, 7-cycles and 8-cycles in $P(7,1)$ are $7,7,2$ and 7, respectively. We now find the non-isomorphic matchings of sizes $0,1,2,3,4,5,6$ and 7, up to switching isomorphism. We have the following lemmas to settle the possible cases of matchings of different sizes.

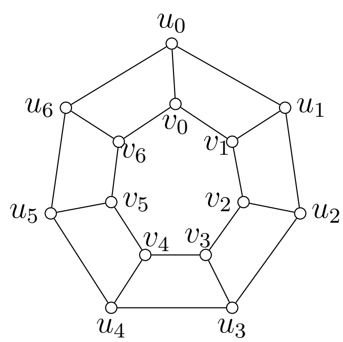


Figure 5: The graph $P(7,1)$.

Lemma 7.1. *Consider the subsets* $\sigma_1 = \{u_0u_1, v_0v_1, v_2u_2\}, \sigma_2 = \{u_0u_1, v_1v_2, u_2u_3\}$, $\sigma_3 = \{u_0u_1, v_1v_2, v_4v_5\}$, $\sigma_4 = \{u_0u_1, v_0v_1, u_3u_4\}$, $\sigma_5 = \{u_0u_1, v_0v_6, u_3u_4\}$, $\sigma_6 = \{u_0v_0, u_1v_1, v_2v_3\}$ *and* $\sigma_7 = \{u_0v_0, u_1v_1, u_2v_2\}$ *of edges of* $P(7,1)$*. If any one of these seven signatures appears in a matching* M_l *of* $P(7,1)$*, where* $l \geq 3$*, then* M_l *is switching equivalent to* $M_{l'}$*, where* $l' \leq l-1$*.*

Proof. Let M_l^j be a matching of size j which contains the set σ_j, where $j = 1,\ldots,7$ and $l \geq 3$. Consider the sets $S_1 = \{u_1, v_1, v_2\}, S_2 = \{u_1, v_2, u_2\}, S_3 = \{u_1, v_2, u_2, u_3, u_4, v_3, v_4\}, S_4 = \{u_1, v_1, u_2, v_2, v_3, u_3\}$, $S_5 = \{v_0, u_1, v_1, u_2, v_2, u_3, v_3\}$, $S_6 = \{v_1, v_2, v_0\}$ and $S_7 = \{u_0, u_1, u_2\}$. If we resign at the vertices belonging to S_j, then we get a signature of size up to $l-1$, *i.e.,* $M_l \sim M_{l'}$, where $l' \leq l-1$. This proves the lemma. □

In Lemma 7.1, the inequality $l' \leq (l-1)$ may be strict. For example, if $M_3 = \sigma_4$, then by resigning at the vertices u_1, v_1, u_2, v_2, v_3 and u_3, we get $M_3 \sim M_1$. The matchings σ_i, where $1 \leq i \leq 7$, are said to be *forbidden* matchings of $P(7,1)$. Let M_0 denotes the matching $\Sigma_1 = \emptyset$ of size zero. Further, there are only two automorphic type matchings of size one, we denote them by $\Sigma_2(u_0u_1)$ and $\Sigma_3(u_0v_0)$. Any other matching of $P(7,1)$ of size one is automorphic to either Σ_2 or Σ_3, and that Σ_2 is not automorphic to Σ_3.

Lemma 7.2. *The number of distinct automorphic type matchings of $P(7,1)$ of size two is* 12*.*

Proof. We classify these matchings by looking at the distance of their edges. Recall that the distance between any two edges of $P(7,1)$ is at most three.

(i) Let the edges of M_2 be at distance one. Five such possible matchings of size two are $\Sigma_4(u_0u_1, v_0v_1)$, $\Sigma_5(u_0u_1, v_1v_2), \Sigma_6(u_0u_1, v_2u_2), \Sigma_7(u_0u_1, u_2u_3)$ and $\Sigma_8(u_0v_0, u_1v_1)$. Note that any matching of $P(7,1)$ of size two contains either two consecutive spokes, or one spoke and one edge from outer (inner) cycle or two edges from the outer (inner) cycle, or one edge from the inner cycle and one from the outer cycle. Each such possible M_2, whose edges are at distance one, is automorphic to one of $\Sigma_8, \Sigma_6, \Sigma_7, \Sigma_4$ and Σ_5. These five matchings are also pairwise non-automorphic.

(ii) Let the edges of M_2 be at distance two. There are only four automorphic type matchings of size two having edges at distance two. We denote them by $\Sigma_9(u_0u_1, v_2v_3), \Sigma_{10}(u_0u_1, v_3u_3)$, $\Sigma_{11}(u_0v_0, v_2u_2)$ and $\Sigma_{12}(u_0u_1, u_3u_4)$. It is easy to see that any other matching of size two whose edges are at distance two is automorphic to one of $\Sigma_9, \Sigma_{10}, \Sigma_{11}$ and Σ_{12}. Further, these matchings are pairwise non-automorphic.

(iii) Let the edges of M_2 be at distance three. There are only three automorphic type matchings of size two whose edges are at distance three. We denote them by $\Sigma_{13}(u_0u_1, v_3v_4)$, $\Sigma_{14}(u_0u_1, v_4u_4)$ and $\Sigma_{15}(u_0v_0, v_3u_3)$. Any other M_2 whose edges are at distance three is automorphic to one of Σ_{13}, Σ_{14} and Σ_{15}. Further, no two of these matchings are automorphic to each other.

This completes the proof. □

Lemma 7.3. *The number of distinct automorphic type matchings of $P(7,1)$ of size three is* 23*.*

Proof. We classify matchings of size three on the basis of the number of spokes contained in it. Since each forbidden matching is a matching of size three and that they are switching equivalent to a matching of size at most two, we consider matchings other than the forbidden matchings.

(i) Let M_3 have no spoke. The possible distinct automorphic type matchings of size three without spokes are denoted by $\Sigma_{16}(u_0u_1, u_2u_3, u_4u_5), \Sigma_{17}(u_0u_1, u_2u_3, v_4v_5)$ and $\Sigma_{18}(u_0u_1, u_4u_3, v_5v_6)$. Any other matching of size three with no spokes is either a forbidden matching or automorphic to one of Σ_{16}, Σ_{17} and Σ_{18}. Further, it is easy to see that they are pairwise non-automorphic.

(ii) Let M_3 have only one spoke, say u_0v_0. Possible automorphic type matchings are $\Sigma_{19}(u_0v_0, u_1u_2, u_3u_4)$, $\Sigma_{20}(u_0v_0, u_1u_2, u_4u_5), \Sigma_{21}(u_0v_0, u_1u_2, u_5u_6), \Sigma_{22}(u_0v_0, u_2u_3, u_4u_5)$, $\Sigma_{23}(u_0v_0, u_1u_2, v_2v_3), \Sigma_{24}(u_0v_0, u_1u_2, v_3v_4), \Sigma_{25}(u_0v_0, u_1u_2, v_4v_5), \Sigma_{26}(u_0v_0, u_1u_2, v_5v_6)$, $\Sigma_{27}(u_0v_0, u_2u_3, v_3v_4), \Sigma_{28}(u_0v_0, u_2u_3, v_4v_5), \Sigma_{29}(u_0v_0, u_2u_3, v_5v_6)$ and $\Sigma_{30}(u_0v_0, u_3u_4, v_5v_6)$. Any other matching of size three containing only one spoke is automorphic to one of these twelve matchings. Further, any two of these matchings are pairwise non-automorphic.

(iii) Let M_3 have only two spokes. If the spokes are consecutive then let they be v_0u_0 and v_1u_1. Thus the only possible automorphic type matching is $\Sigma_{31}(v_0u_0, v_1u_1, u_3u_4)$. If spokes are at distance two then let the spokes be v_0u_0 and v_2u_2. The possible automorphic type matchings are $\Sigma_{32}(v_0u_0, v_2u_2, u_3u_4)$ and $\Sigma_{33}(v_0u_0, v_2u_2, u_4u_5)$. If the spokes are at distance three then let they be v_0u_0 and v_3u_3. The possible automorphic type matchings are $\Sigma_{34}(v_0u_0, v_3u_3, u_1u_2)$ and $\Sigma_{35}(v_0u_0, v_3u_3, u_4u_5)$. Because of the forbidden matchings and the automorphism group of $P(7, 1)$, it is easy to see that any other matching of size three containing only two spokes is automorphic to one of $\Sigma_{31}, \Sigma_{32}, \Sigma_{33}, \Sigma_{34}$ and Σ_{35}. Further, these matchings are pairwise non-automorphic.

(iv) Let M_3 have three spokes. The possible automorphic type matchings of size three having three spokes are $\Sigma_{36}(v_0u_0, v_1u_1, u_3v_3)$, $\Sigma_{37}(v_0u_0, v_1u_1, u_4v_4)$ and $\Sigma_{38}(v_0u_0, v_2u_2, u_4v_4)$. Any matching of size three with three spokes is automorphic to Σ_{36}, Σ_{37} or Σ_{38}. Also, these matchings are pairwise non-automorphic.

This completes the proof of the lemma. □

Lemma 7.4. *The number of distinct automorphic type matchings of $P(7, 1)$ of size four is* 10.

Proof. We classify the matchings of size four on the basis of number of spokes contained in it. If any M_4 contains a forbidden matching then that matching is not considered as a possible candidate for distinct automorphic type matching of size four.

(i) Let M_4 have no spoke. It is clear that any such M_4 has its three edges on outer cycle and the remaining edge on the inner cycle, or two edges on the inner cycle and other two edges on the outer cycle. Therefore, any such M_4 must contain one of σ_2, σ_3 and σ_4. Hence by Lemma 7.1, every matching of size four containing no spoke is equivalent to a matching $M_{l'}$, where $l' \leq 3$.

(ii) Let M_4 have one spoke and let it be u_0v_0. It is clear that the remaining three edges of M_4 either lie on the outer (inner) cycle, or two edges lie on the outer cycle and one edge lies on the inner cycle. If three edges lie on the outer cycle, then one such matching is $\Sigma_{39}(v_0u_0, u_1u_2, u_3u_4, u_5u_6)$. If two edges lie on the outer cycle and one edge lies on the inner cycle, then the possible automorphic type matchings are $\Sigma_{40}(v_0u_0, u_1u_2, u_3u_4, v_5v_6)$ and $\Sigma_{41}(v_0u_0, u_1u_2, v_3v_4, u_5u_6)$. Any other matching of size four containing only one spoke either contains one of the forbidden matchings or automorphic to one of Σ_{39}, Σ_{40} and Σ_{41}. Further, these three matchings are pairwise non-automorphic.

(iii) Let M_4 have two spokes. If the spokes are consecutive then by resigning at the end vertices of the spokes lying on the outer cycle, we find that the matching is equivalent to a matching (signature) of size four. Note that this resultant signature is either equivalent to a matching of size four of case (i) or it is not a matching. Therefore in both cases, it is switching equivalent to a matching $M_{l'}$, where $l' \leq 3$. If the spokes are at distance two or three, then the possible automorphic type matchings of size four are $\Sigma_{42}(v_0u_0, v_2u_2, u_3u_4, u_5u_6)$, $\Sigma_{43}(v_0u_0, v_2u_2, u_3u_4, v_5v_6)$, $\Sigma_{44}(v_0u_0, v_2u_2, u_4u_5, v_5v_6)$, $\Sigma_{45}(v_0u_0, v_3u_3, u_1u_2, u_4u_5)$, $\Sigma_{46}(v_0u_0, v_3u_3, u_1u_2, v_4v_5)$ and $\Sigma_{47}(v_0u_0, v_3u_3, u_4u_5, v_5v_6)$. Any other matching of size four containing only two spokes is automorphic to one of these six matchings. Further, any two of these matchings are pairwise non-automorphic.

(iv) Let M_4 have three spokes. It is clear that a matching of size four cannot contain two or three consecutive spokes up to switchings. Thus the only possible automorphic type matching is $\Sigma_{48}(v_0u_0, v_2u_2, v_4u_4, u_5u_6)$.

(v) Let M_4 have four spokes. By resigning at the vertices from the set $\{u_0, u_1, u_2, u_3, u_4, u_4, u_6\}$, we see that such a matching is equivalent to a matching of size three containing three spokes.

This proves the lemma. □

Theorem 7.1. *Every matching of $P(7,1)$ of size five is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.*

Proof. We prove the theorem by classifying the matchings of size five on the basis of number of spokes contained in it.

(i) Let M_5 have no spoke. Note that three edges of M_5 may lie on the outer (inner) cycle and remaining two edges lie on the inner (outer) cycle. It is easy to see that each such combination of five edges of $P(7,1)$ must contain either σ_2 or σ_4. Therefore by Lemma 7.1, each such M_5 is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.

(ii) Let M_5 have only one spoke and let it be u_0v_0. There are two possibilities for the remaining four edges of M_5.
(a) Three edges of M_5 lie on the outer (inner) cycle and one edge lies on the inner (outer) cycle.
(b) Two edges of M_5 lie on the outer cycle and remaining two edges lie on the inner cycle.
Note that in (a), the three edges lying on the outer cycle must be u_1u_2, u_3u_4 and u_5u_6. Therefore for the fifth edge of M_5, whichever edge we choose from the inner cycle, M_5 will contain either σ_2 or σ_4. Hence by Lemma 7.1, each such M_5 is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.
In (b), let two edges of M_5 lying on the outer cycle be u_1u_2 and u_3u_4. The possibilities for the remaining two edges from the inner cycle are $\{v_1v_2, v_3v_4\}$, $\{v_1v_2, v_4v_5\}$, $\{v_1v_2, v_5v_6\}$, $\{v_2v_3, v_4v_5\}$, $\{v_2v_3, v_5v_6\}$ and $\{v_3v_4, v_5v_6\}$. In all of these possible cases, it is easy to see that M_5 contains a forbidden matching. Hence by Lemma 7.1, each such M_5 is switching equivalent to some matching $M_{l'}$, where $l' \leq 4$. In the similar way, it can be proved that any other matching of size five, containing one spoke and two edges from both outer as well as inner cycles, is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.

(iii) Let M_5 have two spokes. If M_5 has two consecutive spokes, i.e., spokes at distance two, then by switching at their end vertices lying on outer cycle, we find that M_5 is switching equivalent to either a signature of $P(7,1)$ of size five with no spokes or a matching of size five with no spoke. In both cases, M_5 is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.
If two spokes of M_5 are at distance two then let they be u_0v_0 and u_2v_2. The remaining three edges have to be chosen from $\{u_3u_4, u_4u_5, u_5u_6, v_3v_4, v_4v_5, v_5v_6\}$. We can take two edges from outer cycle and one edge from inner cycle or vice-versa. If the two edges from the outer cycle are u_3u_4 and u_5u_6, then for all possible choices of the edge from the inner cycle, M_5 must contain either σ_1 or σ_2. Similarly, if two edges are taken from the inner cycle, M_5 will contain some forbidden matching. Hence each such M_5 is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$. Similarly, it can be shown that if the two spokes of M_5 are at distance three, then also M_5 is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.

(iv) Let M_5 have three spokes. It is clear from part (iii) that, out of the three spokes, no two spokes can be at distance one. Hence the only possibility for such a matching of size five is $\{u_0v_0, u_2v_2, u_4v_4, u_5u_6, v_5v_6\}$. But this matching contains σ_1, hence by Lemma 7.1, we get that this M_5 is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.

(v) Let M_5 have four spokes. Obviously, resigning at $u_0, u_1, u_2, u_3, u_4, u_5$ and u_6, each such matching of size five becomes switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.

(vi) Let M_5 have five spokes. Resigning at $u_0, u_1, u_2, u_3, u_4, u_5$ and u_6, we see that each such matching of size five is switching equivalent to a matching of size two.

This completes the proof of theorem. □

Note that any matching of $P(7, 1)$ of size six or seven contains some M_5. So by Theorem 7.1, we get the following corollary.

Corollary 7.1. *Every matching of $P(7, 1)$ of size six or seven is switching equivalent to a matching $M_{l'}$, where $l' \leq 4$.*

In the preceding lemmas, along with the signatures Σ_1, Σ_2 and Σ_3, we get 48 distinct automorphic type matchings of $P(7, 1)$ of different sizes. However, among these 48 automorphic type matchings, some of them may be switching isomorphic to each other. Now we attempt to eliminate such switching isomorphic matchings. We have the following observations.

- In Σ_8, by resigning at u_0, u_1; we get a matching of size two automorphic to Σ_7. Therefore $\Sigma_8 \sim \Sigma_7$.
- In Σ_{17}, by resigning at u_1, v_1, u_2, v_2 and using the automorphism γ; we get a matching automorphic to Σ_{16}. Thus $\Sigma_{17} \sim \Sigma_{16}$.
- In Σ_{18}, by resigning at u_0, v_0, u_6, v_6 and using the automorphisms δ_3 followed by δ_5; we get a matching automorphic to Σ_{17}. Thus $\Sigma_{18} \sim \Sigma_{17}$.
- In Σ_{20}, by resigning at u_1, u_0; we get a matching automorphic to Σ_{19}. Thus $\Sigma_{20} \sim \Sigma_{19}$.
- In Σ_{21}, by resigning at u_1, u_0, u_6; we get a matching automorphic to Σ_{11}. Thus $\Sigma_{21} \sim \Sigma_{11}$.
- In Σ_{25}, by resigning at u_0, u_1, and using the automorphism δ_1; we get a matching automorphic to Σ_{24}. Thus $\Sigma_{25} \sim \Sigma_{24}$.
- In Σ_{26}, by resigning at v_1, v_0, v_6, u_1; we get a matching automorphic to Σ_{23}. Thus $\Sigma_{26} \sim \Sigma_{23}$.
- In Σ_{29}, by resigning at v_6, v_0 and using the automorphism γ; we get a matching automorphic to Σ_{24}. Thus $\Sigma_{29} \sim \Sigma_{24}$.
- In Σ_{30}, by resigning at v_6, v_0 and using the automorphism γ; we get a matching automorphic to Σ_{25}. Thus we get $\Sigma_{30} \sim \Sigma_{25}$.
- In Σ_{31}, by resigning at u_0, u_1; we get a matching automorphic to Σ_{16}. Thus $\Sigma_{31} \sim \Sigma_{16}$.

- In Σ_{34}, by resigning at u_2u_3; we get a matching automorphic to Σ_{32}. Thus $\Sigma_{34} \sim \Sigma_{32}$.
- In Σ_{36}, by resigning at u_0, u_1 and using the automorphism δ_5; we get a matching automorphic to Σ_{19}. Thus $\Sigma_{36} \sim \Sigma_{19}$.
- In Σ_{37}, by resigning at u_0, u_1; we get a matching automorphic to Σ_{22}. Thus $\Sigma_{37} \sim \Sigma_{22}$.
- In Σ_{39}, by resigning at u_6, u_0, u_1; we get a matching automorphic to Σ_{33}. Thus $\Sigma_{39} \sim \Sigma_{33}$.
- In Σ_{40}, by resigning at $u_0, u_1, u_4, u_5, u_6, v_4, v_5$ and using γ; we get a matching automorphic to Σ_{33}. Thus $\Sigma_{40} \sim \Sigma_{33}$.
- In Σ_{41}, by resigning at u_0, u_1, u_6 and using γ; we get a matching automorphic to Σ_{33}. Thus $\Sigma_{41} \sim \Sigma_{33}$.
- In Σ_{42}, by resigning at $u_0, u_1, u_2, u_3, u_6, v_4$ and v_5; we get matching automorphic to Σ_{38}. Thus $\Sigma_{42} \sim \Sigma_{38}$.
- In Σ_{44}, by resigning at u_0, u_5, u_6, v_5 and using δ_4; we get a matching automorphic to Σ_{46}. Thus $\Sigma_{44} \sim \Sigma_{46}$.
- In Σ_{45}, by resigning at u_0, u_1; we get a matching automorphic to Σ_{42}. Thus $\Sigma_{45} \sim \Sigma_{42}$.
- In Σ_{46}, by resigning at u_2, u_3 and using δ_5; we get a matching automorphic to Σ_{43}. Thus $\Sigma_{46} \sim \Sigma_{43}$.
- In Σ_{47}, by resigning at u_4, v_5, v_4, v_3 and using γ; we get a matching automorphic to Σ_{44}. Thus $\Sigma_{47} \sim \Sigma_{44}$.

Hence we are left with the following matchings: $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7, \Sigma_8, \Sigma_9, \Sigma_{10}, \Sigma_{11}, \Sigma_{12}, \Sigma_{13}, \Sigma_{14}$, $\Sigma_{15}, \Sigma_{18}, \Sigma_{19}, \Sigma_{22}, \Sigma_{23}, \Sigma_{25}, \Sigma_{27}, \Sigma_{32}, \Sigma_{33}, \Sigma_{36}, \Sigma_{37}, \Sigma_{41}$ and Σ_{47}. The corresponding signed graphs of these matchings are shown in Figure 6, where the label of the vertices correspond to that of Figure 5.

Theorem 7.2. *There are exactly* 27 *different signed* $P(7,1)$ *up to switching isomorphism.*

Proof. Let $|C_4^-|, |C_6^-|, |C_7^-|$ and $|C_8^-|$ denote the number of negative 4-cycles, negative 6-cycles, negative 7-cycles and negative 8-cycles of a signed graph. These numbers for the signed graphs shown in Figure 6 are given in Table 3 and Table 4. From Table 3, Table 4 and Theorem 2.1, it is clear that all these 27 signed $P(7,1)$ are switching non-isomorphic. This concludes the proof of theorem. □

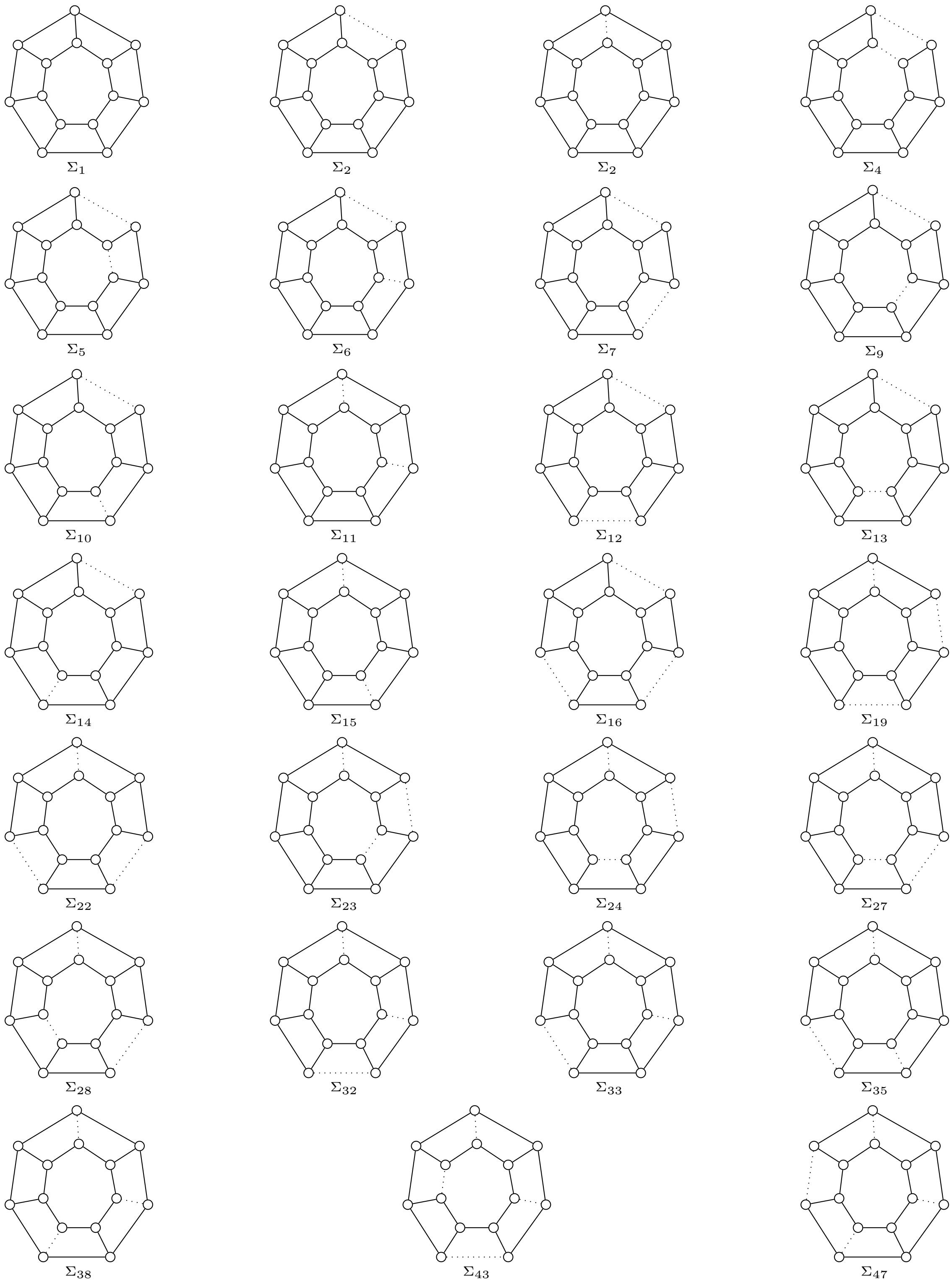


Figure 6: Twenty seven signed $P(7, 1)$.

Table 3: The number of negative 4, 6, 7, and 8-cycles of some signed graphs of Figure 6.

	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}
$\lvert C_4^-\rvert$	0	1	2	0	2	3	2	2	3	4	2	2	3	4
$\lvert C_6^-\rvert$	0	2	2	0	2	2	4	4	4	2	4	4	4	4
$\lvert C_7^-\rvert$	0	1	0	2	2	1	0	2	1	0	0	2	1	0
$\lvert C_8^-\rvert$	0	3	2	0	2	3	4	4	3	4	6	6	5	2

Table 4: The number of negative 4, 6, 7 and 8-cycles of some signed graphs of Figure 6.

	Σ_{16}	Σ_{19}	Σ_{22}	Σ_{23}	Σ_{24}	Σ_{27}	Σ_{28}	Σ_{32}	Σ_{33}	Σ_{35}	Σ_{38}	Σ_{43}	Σ_{47}
$\lvert C_4^-\rvert$	3	4	4	4	4	4	4	5	5	5	6	6	7
$\lvert C_6^-\rvert$	6	4	6	2	4	4	6	2	4	4	2	2	0
$\lvert C_7^-\rvert$	1	0	0	2	2	2	2	1	1	1	0	2	1
$\lvert C_8^-\rvert$	5	4	2	4	4	2	2	5	3	1	4	4	6

8. Conclusions and Remarks

In the Sections 5, 6 and 7, we have found the exact number of non-isomorphic signatures of $P(3,1), P(5,1)$ and $P(7,1)$ up to switching. However this number for the general case is still unknown. So, it is natural to pose the following problem.

Problem 1. *What is the exact number of non-switching isomorphic signatures on $P(2n+1,1)$ for all $n \geq 4$ up to switching ?*

We have a partial answer towards the solution of Problem 1, that is, we give the answer for non-isomorphic matchings (or minimum signatures) of size two of $P(2n+1,1)$ for all $n \geq 4$. We have the following theorem.

Theorem 8.1. *Up to switching isomorphism, the number of matchings of $P(2n+1,1)$ of size two is $4n-1$, where $n \geq 4$.*

Proof. From Theorem 4.2, we know that edges of any M_2 can be at maximum distance n. Four matchings of size two, whose edges are at distance one are $M_2^{11} = \{u_0u_1, v_0v_1\}$, $M_2^{12} = \{u_0u_1, u_2v_2\}$, $M_2^{13} = \{u_0u_1, v_1v_2\}$ and $M_2^{14} = \{u_0u_1, u_2u_3\}$. For any matching of size two whose edges lie either on the inner cycle or on the outer cycle of $P(2n+1,1)$, there exists some automorphism ρ_k of $P(2n+1,1)$ which maps the matching onto M_2^{14}. If a matching of size two contains any two spokes which are at distance one, then by resigning at their end points lying on the outer cycle, we see that resultant matching is automorphic to M_2^{14}. Similarly, it can be shown that all other matchings of size two whose edges are at distance one are automorphic to one of M_2^{11}, M_2^{12} and M_2^{13}.

Further, the number of negative 4-cycles in $M_2^{11}, M_2^{12}, M_2^{13}$ and M_2^{14} are $0, 3, 2$ and 2, respectively. The number of negative $2n+1$-cycles in $M_2^{11}, M_2^{12}, M_2^{13}$ and M_2^{14} are $2, 1, 2$ and 0, respectively. These numbers of negative cycles show that $M_2^{11}, M_2^{12}, M_2^{13}$ and M_2^{14} are pairwise non-isomorphic. Hence up to switching isomorphism, $M_2^{11}, M_2^{12}, M_2^{13}$ and M_2^{14} are the only matchings of size two whose edges are at distance one. Similarly, For each i, one can show up to switching isomorphism that there are exactly four matchings of size two whose edges are at distance i, where $2 \leq i < n$.

Three matchings of size two whose edges are at distance n are $M_2^{n1} = \{u_0u_1, u_{n+1}v_{n+1}\}$, $M_2^{n2} = \{u_0u_1, v_nv_{n+1}\}$ and $M_2^{n3} = \{u_0v_0, u_nv_n\}$. It is clear that any other matching of size two whose edges are at distance n is automorphic to one of M_2^{n1}, M_2^{n2} and M_2^{n3}. Further, the number of negative 4-cycles in M_2^{n1}, M_2^{n2} and M_2^{n3} are $3, 2$ and 4, respectively. Thus these three matchings are pairwise non-isomorphic.

It is easy to see that any two matchings of size two whose edges are at different distances are different up to switching isomorphisms. This concludes the proof of theorem. □

Having proved this theorem, we propose the following problem.

Problem 2. *Can we find the number of non-isomorphic matchings of sizes* $3, \ldots, 2n+1$ *in* $P(2n+1, 1)$ *for all* $n \geq 4$*?*

In Section 5, we noticed that $P(3, 1)$ has no matching (or minimum signature) of size 3, up to switching isomorphism. In Section 6, we noticed that $P(5, 1)$ has no matching of sizes 4 and 5, up to switching isomorphism. In Section 7, we noticed that $P(7, 1)$ has no matching of sizes $5, 6$ and 7, up to switching isomorphism. On the basis of these observations, we propose the following conjecture.

Conjecture 1. *Any matching in* $P(2n+1, 1)$ *can be of size at most* $n+1$*, where* $n \geq 4$*.*

Now Problem 2 can be reformulated as follows.

Problem 3. *Can we find the number of non-isomorphic matchings of size* $3, 4, 5, \ldots, n+1$ *in* $P(2n+1, 1)$ *for all* $n \geq 4$ *?*

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