Electronic Journal of Graph Theory and Applications 9 (2) (2021), 419–431

Electronic Journal of Graph Theory and Applications

Tarantula graphs are determined by their Laplacian spectrum

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Abstract

A graph G is said to be determined by its Laplacian spectrum (DLS) if every graph with the same Laplacian spectrum is isomorphic to G. A graph which is a collection of hexagons (lengths of these cycles can be different) all sharing precisely one vertex is called a spinner graph. A tree with exactly one vertex of degree greater than 2 is called a starlike tree. If a spinner graph and a starlike tree are joined by merging their vertices of degree greater than 2, then the resulting graph is called a tarantula graph. It is known that spinner graphs and starlike trees are DLS. In this paper, we prove that tarantula graphs are determined by their Laplacian spectrum.

Keywords: tarantula graph, Laplacian matrix, Laplacian spectrum, L-cospectral
Mathematics Subject Classification : 05C50
DOI: 10.5614/ejgta.2021.9.2.14

1. Introduction

As usual $G = (V(G), E(G))$ is a simple graph having $n = n(G)$ vertices and $m = m(G)$ edges, with $V(G) = \{v_1, v_2, \dots, v_n\}$. The complement of G is denoted by $\overline{G}$. The degree sequence of G, denoted by $\deg(G)$, is the sequence of vertex degrees of G; in fact $\deg(G) = (d_1, d_2, \dots, d_n)$

Received: 08 November 2019, Revised: 23 March 2021, Accepted: 19 April 2021.

in which $d_i = d_i(G) = d_G(v_i)$, for $i = 1, \dots, n$, is the degree of the vertex v_i so that $d_1 \geq d_2 \geq \cdots \geq d_n$. For $i = 0, 1, 2, \dots, n-1$, let $n_i = n_i(G)$ denote the number of vertices of degree i in G.

Let $A(G)$ and $D(G) = \mathrm{Diag}(d_1, d_2, \dots, d_n)$ denote the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The *Laplacian matrix* of G is defined as $L(G) = A(G) - D(G)$. The polynomial $\varphi_{L(G)}(\mu) = \det(\mu \mathbb{I}_n - L(G))$, where $\mathbb{I}_n$ is the identity matrix of order n, is called the Laplacian characteristic polynomial of G. Any root of $\varphi_{L(G)}(\mu)$ is called a *Laplacian eigenvalue* of G. The multi-set of Laplacian eigenvalue of G is called the *Laplacian spectrum* or L*-spectrum* of G. Note that the eigenvalues of $L(G)$ are all real non-negative, since it is a symmetric, positive semidefinite matrix. We denote its eigenvalues in the non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The Laplacian matrix of G is a major tool for enumerating spanning trees of G and has numerous applications [18, 37]. Two graphs G and H is said L*-cospectral* if they have the same L-spectrum, and a graph G is *determined by its Laplacian spectrum*, abbreviated by DLS, if no other graphs are L-cospectral with G. Only a few graphs with very special structures have been found to be determined by their spectra (DS, for short) (see [1]–[16], [19], [28]–[32], [34], [35] and the references cited therein).

The coalescence of two graphs G_1 and G_2, with respect to $u_1 \in V(G_1)$ and $u_2 \in V(G_1)$, is the graph obtained by identifying u_1 and u_2 in the disjoint union of G_1 and G_2. We denote it by $(G_1 \circ G_2)(u_1, u_2)$. In the case when it dose not make deference which vertex in G_1 and G_2 is identified to obtain a coalescence, we denote this graph by $G_1 \circ G_2$. This operation is extended, inductively, to any arbitrary number of graphs. For instance, the coalescence of cycles $C_{n_1}, \dots, C_{n_p}$ (or $C_{n_1} \circ \cdots \circ C_{n_a}$) is called an a-rose graph. Specially, $\underbrace{C_6 \circ \cdots \circ C_6}_{a \text{ times}}$ is a rose graph which is called an spinner graph and is denoted by H_a.

An starlike tree is defined as a tree with a unique vertex v of degree greater than 2. We denote by $S(t_1, \dots, t_b)$ the starlike tree with maximum degree b such that

$$S(t_1, \dots, t_b) - v = P_{t_1} \cup P_{t_2} \cup \cdots \cup P_{t_b},$$

where v is the vertex of degree b, and $t_1, t_2, \dots, t_b$ are any positive integers. We may describe an starlike tree as a coalescence of $P_{t_1+1}, P_{t_2+1}, \dots, P_{t_b+1}$. In fact, if v_i is a specific pendant vertex of P_{t_i+1}, for $i = 1, \dots, b$, then we have

$$S(t_1, \dots, t_b) = P_{t_1+1} \circ P_{t_2+1} \circ \cdots \circ P_{t_b+1}(v_1, v_2, \dots, v_b).$$

A tarantula graph, $\mathfrak{T}(a, t_1, \dots, t_b)$, $a, b \geq 1$ is a graph of order $n = 5a + t_1 + \cdots + t_b + 1$, which consists of a hexagons and b paths of lengths $t_1, t_2, \dots, t_b$ sharing a common vertex. Note that $\mathfrak{T}(a, t_1, \dots, t_b) = H_a \circ T(u, v)$, where u and v are, respectively, the unique vertices of H_a and $T = T_{t_1, t_2, \dots, t_b}$ with the maximum degrees $2a$ and b, respectively, see Figure 1.

van Dam and Haemers [38] conjectured that almost all graphs are DLS. However, very few graphs are known to have that property, and so discovering new classes of such graphs is an interesting problem.

We are interested in DLS graphs being a coalescence of DLS graphs. Note that the coalescence of two DLS graphs is not, necessarily DLS, see Figure 2. All paths, cycles, starlike trees, triangle free 2-graph are DLS [26, 36, 41, 38]. In [24] it was shown that apart from two exceptional cases

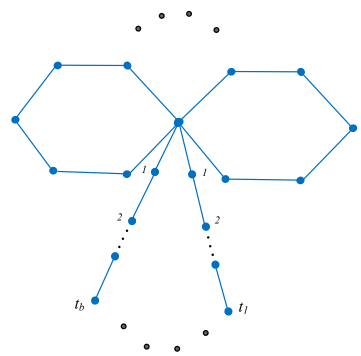

Figure 1. A tarantula graph $\mathfrak{T}(a, t_1, \dots, t_b)$

of order 6 and 7, all roses are DLS. In [15] it was shown that $\mathfrak{T}(2, 1, \dots, 1)$ is DLS. In [40] it was shown that $\mathfrak{T}(3, t_1)$ is DLS. In [35] it was shown that $\mathfrak{T}(3, t_1, \dots, t_b)$ is DLS.

Figure 2. Tow non-isomorphic graphs with the same Laplacian spectrum $\{0, 3 - \sqrt{5}, 2, 3, 3, 3 + \sqrt{5}\}$, [39, 24]

The following theorem is our main result.

Theorem 1.1. *All tarantula graphs and their complements are* DLS.

The rest of this article is organized as follows: Section 2 contains preliminary results on the Laplacian spectrum of a graph, while Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section, we recall some previously established results being used to prove Theorem 1.1. Note that the trace of a matrix M is denoted by $\mathrm{tr}(M)$.

Lemma 2.1. *Let M and N be two matrices of size n. Then the following are equivalent:*

(i) *M and N are cospectral;*
(ii) *M and N have the same characteristic polynomial;*
(iii) $\mathrm{tr}\,(M^i) = \mathrm{tr}\,(N^i)$ *for $i = 1, 2, \dots, n$.*

Lemma 2.2 ([27, 34, 38, 42])**.** *Let G be graph of size m with the Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Then the following hold:*

1. *the number of vertices of* G *is equal to* n*;*
2. $2m = \sum_{i=1}^{n} \mu_i$*;*
3. *the number of spanning trees of* G *is equal to* $\prod_{i=1}^{n-1} \mu_i$*;*
4. *the number of components of* G *is equal to the multiplicity of* $\mu_n = 0$*;*
5. $\operatorname{tr}(L(G)^2) = \sum\limits_{i=1}^{n} {\mu_i}^2 = 2m + \sum_{v \in V(G)} d_G^2(v)$.

The next lemma relates the Laplacian spectrum of a graph and its complements.

Lemma 2.3 ([20])**.** *Let* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ *and* $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ *be the Laplacian spectrum of* G *and* $\overline{G}$*, respectively. Then* $\overline{\mu}_i = n - \mu_{n-i}$ *for* $i = 1, 2, \ldots, n-1$.

As a consequence of Lemma 2.3, we have the following fact:

Corollary 2.1. *The complement of a* DLS *graph is also* DLS.

Proof. Let G be a DLS graph and let H be a graph L-cospectral with $\overline{G}$. Then it follows from Lemma 2.7 that $\overline{H}$ is L-cospectral with G. Therefore, $\overline{H} \cong G$, since G is DLS. Consequently, $H \cong \overline{G}$, as desired. □

We denote by $\mathcal{N}_G(C_3)$ and $\mathcal{W}_G(i)$, the number triangles of G, and the number of closed walks of length i in G, respectively.

The next lemma enables us to compute the number of closed walks of lengths 2, 3, and 4 in G:

Lemma 2.4 ([14, 42])**.** *Let* G *be a graph with* m *edges. Then the following hold:*

1. $\mathcal{W}_G(2) = 2m$*;*
2. $\mathcal{W}_G(3) = \operatorname{tr}(A^3(G)) = 6\mathcal{N}_G(C_3)$*;*
3. $\mathcal{W}_G(4) = 2m + 4\mathcal{N}_G(P_3) + 8\mathcal{N}_G(C_4)$.

Lemma 2.5 ([33])**.** *Let* G *be a graph with* n *vertices and* m *edges, and let* $\varphi(G) = \sum\limits_{i=1}^{n} l_i \mu^i$ *be the Laplacian characteristic polynomial of* G*. Then*

(i) $l_0 = 1, \qquad l_1 = -2m, \qquad l_2 = 2m^2 - m - \frac{1}{2}\sum\limits_{i=1}^{n} d_i^2(G)$*;*

(ii) $$l_3 = \frac{1}{3}\left(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2(G) - \sum_{i=1}^{n} d_i^3(G) - 3\sum_{i=1}^{n} d_i^2(G) + 6\mathcal{N}_G(C_3)\right).$$

It follows from Lemma 2.1 that L-cospectral graphs share the same characteristic polynomial. Hence the corresponding coefficients are also equal. Thus, the following lemma follows from the Lemma 2.5 and Lemma 2.4:

Lemma 2.6 ([26])**.** *If* G *and* H *are two* L*-cospectral graphs, then*

$$\operatorname{tr}(A^3(G)) - \sum_{i=1}^{n}(d_i^3(G) - 2)^3 = \operatorname{tr}(A^3(H)) - \sum_{i=1}^{n}(d_i^3(H) - 2)^3.$$

The following fact is a direct consequence of Lemma 2.4 and Lemma 2.6.

Corollary 2.2. *Let G and H be L-cospectral graphs such that $\deg(G) = \deg(H)$. Then $\mathcal{N}_G(C_3) = \mathcal{N}_H(C_3)$.*

Lemma 2.7 ([23])**.** *Let G be a non-empty graph with n vertices. Then*

$$\mu_1(G) \geq d_1(G) + 1. \tag{1}$$

Furthermore, if G is connected, then the equality in (1) holds if and only if $d_1(G) = n - 1$.

A graph G is called *regular* if $d_1(G) = \cdots = d_n(G)$. A bipartite graph is called *semi-regular* if the degrees of vertices in each part, are constant.

For an arbitrary vertex u of G, we set $\theta_G(u) = \sum_{v \in N_G(u)} \frac{d_G(v)}{d_G(u)}$, where $N_G(u)$ denotes the set of neighbors of u in G.

Lemma 2.8 ([25, 27])**.** *Let G be a connected graph. Then*

$$\mu_1(G) \leq \max_{u \in V(G)} d_G(u) + \theta_G(u),$$

with equality if and only if G is a regular or a semi-regular bipartite graph.

Lemma 2.9 ([14, 25])**.** *If G is a non-empty graph, then $\mu_1(G) \leq d_1(G) + d_2(G)$; and if G is connected, then $\mu_2(G) \geq d_2(G)$.*

Lemma 2.10 ([20])**.** *Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of a symmetric $n \times n$ matrix M, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$ be the eigenvalues of a principal sub-matrix of M of size $n - 1$. Then $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_{n-1} \geq \mu_n$.*

Lemma 2.11 ([21])**.** *Let G be a graph and $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\} \subseteq V(G)$ such that*

$$N_G(u_1) = N_G(u_2) = \cdots = N_G(u_k) = \{w_1, w_2, \ldots, w_p\}.$$

Let G^* be the graph obtained from G by adding q ($1 \leq q \leq \binom{k}{2}$) edges among $\{u_1, u_2, \ldots, u_k\}$. Then the eigenvalues of $L(G^*)$ are as follows: those eigenvalues of $L(G)$ being equal to p are incremented by $\lambda_i(G^*[X])$, $i = 1, 2, \ldots, k-1$, and the remaining eigenvalues are the same, where $X = \{u_1, u_2, \ldots, u_k\}$.

Lemma 2.12 ([22])**.** *No two non-isomorphic starlike trees are L-cospectral.*

Note that in the proof of Lemma 2.12, the following observation was proved:

Lemma 2.13. *If $S_1 = S(l_1, \ldots, l_t)$ and $S_2 = S(j_1, \ldots, j_t)$ are two non-isomorphic starlike trees, then $\mu_1(S_1) \neq \mu_1(S_2)$, where $l_1 \geq l_2 \geq \cdots \geq l_t \geq 1$ and $j_1 \geq j_2 \geq \cdots \geq j_t \geq 1$.*

3. Proof of Theorem 1.1

In this section, it is proved that all tarantula graphs are DLS.
By an straightforward calculation we obtain the following fact:

Lemma 3.1. *Let* $G = \mathfrak{T}(a, t_1, \ldots, t_b)$ *be a tarantula graph. Then*

(i) $n(G) = 5a + 1 + \sum_{i=1}^{b} t_i, \qquad m(G) = n(G) + a - 1$;

(ii) $\deg(G) = (2a + b, \underbrace{2, \ldots, 2}_{n(G)-(b+1) \text{ times}}, \underbrace{1, \ldots, 1}_{b \text{ times}})$.

Lemma 3.2. *If* H *is a graph* L*-cospectral with* $G = \mathfrak{T}(a, t_1, \ldots, t_b)$*, then*

(i) $2a + b + 1 \le \mu_1(H) \le 2a + b + 2$,

(ii) $\mu_2(H) < 4$.

Proof. (i) By Lemma 2.7, $\mu_1(G) \ge 2a + b + 1$, and by Lemma 2.8, $\mu_1(G) \le 2a + b + 2$. This implies that $2a + b + 1 \le \mu_1(H) = \mu_1(G) \le 2a + b + 2$.

(ii) Let v be the vertex with maximum degree of G, and let M_v be the $(n-1) \times (n-1)$ principal sub-matrix of $L(G)$ formed by deleting the row and column corresponding to v. Since M_v contains negative entries, we consider $|M_v|$ which is obtained by taking the absolute value of the entries of M_v. Now M_v is reducible, but it has $a+b$ irreducible sub-matrices that correspond to the components of $G - v$. On the other hand, each of these components has spectral radius strictly less than 4, so one can conclude that the largest eigenvalue of $|M_v|$ is less than 4, and so is that of M_v. By Lemma 2.10, $\mu_2(G) < 4$ and so $\mu_2(H) < 4$, as desired. □

Theorem 3.1. *If* H *is a graph being* L*-cospectral with* $G = \mathfrak{T}(a, t_1, \ldots, t_b)$*, then they have the same degree sequence, i. e.,* $\deg(H) = \deg(G)$*, and* $\mathcal{N}_H(C_3) = 0$.

Proof. By Lemma 3.2(ii), $\mu_2(H) < 4$, and thus it follows from Lemma 2.9 that $d_2(H) \le 3$. Since H and G are L-cospectral, by Lemma 2.6, H is also connected, and has the same order, size, and sum of the squares of its degrees as G. Therefore, by Lemma 2.6, we have

$$\sum_{i=1}^{d_1(H)} n_i = n(G), \tag{2}$$

$$\sum_{i=1}^{d_1(H)} i n_i = 2m(G), \tag{3}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n_1(G) + 4n_2(G) + d_1^2(G), \tag{4}$$

Recall that by Lemma 3.1,

$$n(G) = 5a + 1 + \sum_{i=1}^{b} t_i, \quad m(G) = n(G) + a - 1,$$
$$n_1(G) = b, \quad n_2(G) = n(G) - (b + 1),$$

and $d_1(G) = 2a + b$. By adding (2), (3), and (4) with coefficients $2, -3, 1$, respectively, we get:

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i(G) = 4a^2 + 4ab - 6a + (b - 1)(b - 2). \tag{5}$$

By Lemma 3.1, $\sum_{i=1}^{n} (d_i(G) - 2)^3 = ((2a + b - 2)^3 - b)$. Hence, it follows from Lemma 2.6 that

$$\mathcal{N}_H(C_3) = \frac{\sum_{i=1}^{n} (d_i(H) - 2)^3 - ((2a + b - 2)^3 - b)}{6}. \tag{6}$$

By Lemma 3.2, $2a+b+1 \le \mu_1(H) \le 2a+b+2$. It follows from Lemma 2.7 that $d_1(H)+1 \le \mu_1(H) = \mu_1(G) \le 2a + b + 2$, and so $d_1(H) \le 2a + b + 1$. Moreover, it follows from Lemma 3.2 and Lemma 2.9 that

$$2a + b + 1 \le \mu_1(G) = \mu_1(H) \le d_1(H) + d_2(H) \le d_1(H) + 3,$$

which implies that $d_1(H) \ge 2a + b - 2$. As a result, $2a + b - 2 \le d_1(H) \le 2a + b + 1$. We claim that $d_1(H) = 2a + b$. To prove our claim, we consider the following cases:

Case 1. $d_1(H) = 2a + b + 1$
Clearly, $n_{2a+b+1} = 1$, since $d_1(H) = 2a + b + 1 > 3 \ge d_2(H)$. From (5), we deduce that

$$((2a + b + 1)^2 - 3(2a + b + 1) + 2) + 2n_3 = 4a^2 + 4ab - 6a + (b - 1)(b - 2). \tag{7}$$

As a result, $n_3 = -2a - b + 1 < 0$, which is impossible.

Case 2. $d_1(H) = 2a + b$
If $b = 0$, then for $a = 1$ we have $H = C_6$ and in this case, the problem is clear, since cycles are DLS.
For $a \ge 2$ and $b = 0$, we have

$$4a^2 - 6a + 2 + 2n_3 = 4a^2 - 6a + 2, \tag{8}$$

which implies that $n_3 = 0$. It follows from (2) and (3) that $n_1 = 0$ and $n_2 = n-1$. Therefore, the degrees of H are the same as those of G and also $d_1(H) = 2a > 3 \ge d_2(H)$.
If $a = b = 1$, then $d_1(H) = 3$. By (5), $n_3 = 1$. If $a \ne 1$ or $b \ne 1$, then $d_1(H) = 2a + b > 3 \ge d_2(H)$. As a result, for any natural numbers a, b we have $n_{2a+b} = 1$. Consequently, by (5) we have

$$((2a + b)^2 - 3(2a + b) + 2) + 2n_3 = 4a^2 + 4ab - 6a + b^2 - 3b + 2, \tag{9}$$

from which we have $n_3 = 0$. Combining (3) and (4), we obtain that $n_1 = b$ and $n_2 = n-(b+1)$. Therefore, $\deg(H) = \deg(G)$. Hence, by Corollary 2.2, $\mathcal{N}_H(C_3) = \mathcal{N}_G(C_3) = 0$.

Case 3. $d_1(H) = 2a + b - 1$

3.1. $n_{2a+b-1} = 1$.

In this case we have

$$((2a + b - 1)^2 - 3(2a + b - 1) + 2) + 2n_3 = 4a^2 + 4ab - 6a + (b - 1)(b - 2), \quad (10)$$

Hence, $n_3 = 2a + b - 2$. It follows from (3) and (4) that $n_1 = 2a + 2b - 3$ and $n_2 = n - 4a - 3b + 4$. Now, from (6) it is easy to see that

$$\mathcal{N}_H(C_3) = \frac{(2 - b)(b - 3) + (-4a^2 - 4ab + 10a)}{2}.$$

Let us prove that $\mathcal{N}_H(C_3) < 0$. To do so, assume that $g(a, b) = -2a(2a + 2b - 5)$ and $f(b) = (2 - b)(b - 3)$. It is easy to see that $l(b) \geq 0$ if is either 2 or 3 and $l(b) < 0$ otherwise. Therefore, for $b \geq 3$, $\mathcal{N}_H(C_3) = \dfrac{f(b) + g(a, b)}{2} < 0$, which is impossible. Consequently, there are two subcases to be considered:

3.1.1. If $b = 2$, $\mathcal{N}_H(C_3) = -2a^2 + a < 0$, which is impossible.

3.1.2. If $b = 1$, $\mathcal{N}_H(C_3) = -2a^2 + 3a - 1$. For $a \geq 2$, $\mathcal{N}_H(C_3) < 0$, which is impossible. For $a = 1$, $n_{2a+b-1} = n_2 = 1$ and so by (5) we obtain that $2 = 0$, which is impossible.

3.2 $n_{2a+b-1} \geq 2$.

In this case we have

$$2a + b - 1 = d_1(H) = d_2(H) \leq 3,$$

and so

$$(a, b) \in \Big\{(1, 0), (1, 1), (1, 2)\Big\}.$$

Consider the following subcases:

3.2.1. $(a, b) = (1, 2)$. In this case, $d_1(H) = d_2(H) = 3$. Hence, by (5) $n_3 = 3$. Combining (3) and (4), we find that the roots are $n_1 = 3$ and $n_2 = n - 6$. It follows from (6) that $\mathcal{N}_H(C_3) = -1$, a contradiction.

3.2.2. $(a, b) = (1, 1)$. In this case, $d_1(H) = d_2(H) = 2$. As before, we will have a contradiction.

3.2.3. $(a, b) = (1, 0)$. In this case, $1 = d_1(H) = d_2(H)$, and hence $H = K_2$. But, $(a, b) = (1, 0)$ means that $n(H) = n \geq 4$, a contradiction.

Case 4. $d_1(H) = 2a + b - 2$

4.1. $n_{2a+b-2} = 1$.

In this case,

$$2a + b - 2 = d_1(H) > 3 \geq d_2(H).$$

From (5), and by an straightforward calculation, we obtain that:

$$((2a + b - 2)^2 - 3(2a + b - 2) + 2) + 2n_3 = 4a^2 + 4ab - 6a + (b - 1)(b - 2). \quad (11)$$

As a result, $n_3 = 4a + 2b - 5$. It follows from (3) and (4) that $n_2 = n - 8a - 5b + 11$ and $n_1 = 4a + 3b - 7$. Furthermore, by (6) $\mathcal{N}_H(C_3) = -4a(a + b - 3) - (b - 3)^2$. We put

$$g(a, b) = -6a(a + b - 3), \quad \text{and} \quad f(q) = -(b - 3)^2.$$

If $b \geq 2$, then $f(b) + g(a, b) < 0$ and so $\mathcal{N}_H(C_3) < 0$, a contradiction. So we need to consider the following cases:

4.1.1. $b = 1$. For $a \geq 2$, $\mathcal{N}_H(C_3) < 0$, which is impossible. If $a = 1$, then $d_1(H) = 1 > 3$, which is impossible.

4.1.2. $b = 0$. Then $f(0) + g(a, 0) < 0$ and so $\mathcal{N}_H(C_3) < 0$, a contradiction.

4.2. $n_{2a+b-2} \geq 2$.
In this case,

$$d_1(H) = d_2(H) = 2a + b - 2 \leq 3.$$

Hence

$$(a, b) \in \Big\{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (3, 1)\Big\}.$$

We have the following subcases:

4.2.1. $(b, a) = (0, 1)$. As a result $d_1(H) = d_2(H) = 0$, a contradiction, since H is connected.

4.2.2. $(b, a) = (0, 2)$. So $d_1(H) = d_2(H) = 2$ and so the degree sequence of H consists of either 1 or 2, which means that H is either a path or a cycle. Consequently, by (5) we get $0 = 6$, which is impossible.

4.2.3. $(b, a) = (1, 1)$. Obviously, $n = n(H) \geq 6$. Moreover, $1 = d_1(H) = d_2(H)$, which implies that $H = K_2$, since H is a connected graph. This is clearly a contradiction.

4.2.4. $(b, a) = (1, 2)$. Therefore, $3 = d_1(H) = d_2(H)$. It follows from (2), (3) and (5) that $n_1 = 4$, $n_3 = 6$ and $n_2 = n - 10$. Finally, it follows from (6) that $\mathcal{N}_H(C_3) = -4$, which is impossible.

4.2.5. $(b, a) = (2, 1)$. Therefore, $2 = d_1(H) = d_2(H)$, a contradiction.

4.2.6. $(b, a) = (3, 1)$. As a result, $3 = d_1(H) = d_2(H)$. By (2), (3) and (4), we obtain that $n_1 = n_3 = 6$ and $n_2 = n - 12$, which is impossible by (6).

□

Let x and y be, respectively, the unique vertices of $G_1 = C_{k_1} \circ \cdots \circ C_{k_a}$ and $G_2 = S(l_1, \ldots, l_b)$ with maximum degree; Consider $G = G_1 \circ G_2(x, y)$ with maximum degree v. For each $1 \leq i \leq a$, let e_i be an arbitrary edge of C_{k_i} being adjacent to v. Then $G - \{e_1, \ldots, e_a\}$ is an starlike tree which is denoted by $\mathcal{S}_G$.

We are now ready to finalize the proof of Theorem 1.1. Let $G = \mathfrak{T}(a, t_1, t_2, \ldots, t_b)$ be a tarantula graph having a hexagons and b paths of lengths t_i, $i = 1, 2, \ldots, b$. Assume that H is a graph L-cospectral with G. It follows from Theorem 3.1 that $\deg(H) = \deg(G)$ and $\mathcal{N}_H(C_3) = 0$.

Moreover, sine G is connected, it follows from Lemma 2.2 that H is also connected. Let v be the unique vertex of degree $2a+b$ in H. Then, $H-v$ has maximum degree at most 2. We claim that $H-v$ does not contain any cycles; otherwise, the connectedness of H implies that it must have another vertex of degree greater than 2, which is impossible. Consequently, $H-v$ must be a forest each component of which is a path. Now due to the fact

$$d_H(v)=2a+b, \quad \mathcal{N}_H(C_3)=0, \quad n_1=b, \quad \deg(H)=(2a+b, \underbrace{2,\dots,2}_{n(G)-(b+1)\text{ times}}, \underbrace{1,\dots,1}_{b\text{ times}}),$$

we conclude that there exist natural numbers $k_1,\dots,k_a \geq 2$ such that

$$H \cong (C_{k_1} \circ C_{k_2} \circ \cdots \circ C_{k_a}) \circ S(t_1,\dots,t_b)(x,y),$$

where x and y are, respectively, the unique vertices of $C_{k_1} \circ \cdots \circ C_{k_a}$ and $S(t_1,\dots,t_b)$ with maximum degree; Consider starlike trees $\mathcal{S}_G$ and $\mathcal{S}_H$. It follows from Lemma 2.11 that $\mu_1(\mathcal{S}_G)=\mu_1(G)$ and $\mu_1(\mathcal{S}_H)=\mu_1(H)$, from which we have $\mu_1(\mathcal{S}_G)=\mu_1(\mathcal{S}_H)$ since $\mu_1(H)=\mu_1(G)$. Thus by Lemma 2.13, we have $G\cong H$, which as direct consequence of Corollary 2.1, we have $\overline{G}\cong\overline{H}$, as desired.

4. Conclusion

Let G be a tarantula graph. In this article, we show that tarantula graphs are determined by their Laplacian Spectrum (DLS) trough the fact that a graph which is L-cospectral with a tarantula graph is a triangle-free graph with the same vertex degree sequence as G.

Let us put forward the following questions for further research in the future:

Question 1. *Are tarantula graphs determined by their adjacency spectrum (*DAS*)?*

Question 2. *Are tarantula graphs determined by their signless Laplacian spectrum (*DQS*)?*

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