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On strict-double-bound numbers of graphs and cut sets

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Abstract

For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* of P is the graph $\mathrm{sDB}(P)$ on $V(\mathrm{sDB}(P)) = X$ for which vertices u and v of $\mathrm{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{\ n\ ;\ \ \mathrm{sDB}(P) \cong G \cup \overline{K}_n \text{ for some poset } P\}$. We obtain an upper bound of strict-double-bound numbers of graphs with a cut-set generating a complete subgraph. We also estimate upper bounds of strict-double-bound numbers of chordal graphs.

Keywords: strict-double-bound graph, strict-double-bound number, cut-set, chordal graph
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1. Introduction

In this paper we consider finite graphs with no loops and no multiple edges, and finite posets. For a graph G and $S \subseteq V(G)$, $\langle S \rangle_V$ is the induced subgraph on S and $G - S = \langle V(G) - S \rangle_V$. The graph $\overline{K}_n$ is a graph with n vertices and no edges.

A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. A family $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_m\}$ is an *edge clique cover* of G if each Q_i is a clique of G and for each $uv \in E(G)$, there exists $Q_i \in \mathcal{Q}$ such that $u, v \in Q_i$.

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A partially ordered set (*poset*) $P = (X, \leq_P)$ consists of a non-empty set X and a binary relation $\leq_P$ on X which satisfy reflexive law, anti-symmetric law and transitive law:

1. For all $u \in X$, $u \leq_P u$: reflexive law.
2. If $u \leq_P v$ and $v \leq_P u$, then $u = v$: anti-symmetric law.
3. If $u \leq_P v$ and $v \leq_P w$, then $u \leq_P w$: transitive law.

For $u, v \in P$, u and v are *comparable* if $u \leq_P v$ or $v \leq_P u$, and otherwise u and v are *incomparable*.

For a poset P, let $\mathrm{Max}(P)$ be the set of all maximal elements of P and $\mathrm{Min}(P)$ be the set of all minimal elements of P. For a poset P and an element $v \in V(P)$, $U_P(v) = \{u \in V(P); v \leq_P u\}$ and $L_P(v) = \{u \in V(P); u \leq_P v\}$. For a poset P and elements u and v of P, $u \parallel v$ denotes that u is incomparable with v in P.

McMorris and Zaslavsky [6] introduced concepts of some kinds of graphs on posets, that is, upper bound graphs, strict upper bound graphs, double bound graphs and strict-double-bound graphs. Langley et. al [4] and Scott [10] dealt with interval strict upper bound graphs and chordal strict upper bound graphs. Cheston and Jap [1] studied upper bound graphs from the viewpoint of algorithms.

We consider strict-double-bound graphs and strict-double-bound numbers. For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* (sDB-*graph*) of P is the graph $\mathrm{sDB}(P)$ on $V(\mathrm{sDB}(P)) = X$ for which vertices u and v of $\mathrm{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. We say that a graph G is a *strict-double-bound graph* if there exists a poset whose strict-double-bound graph is isomorphic to G. Note that maximal elements and minimal elements of a poset P are isolated vertices of $\mathrm{sDB}(P)$. So, a connected graph with at least two vertices is not a strict-double-bound graph. Scott [11] showed the following result.

Theorem 1.1 (Scott [11]). *Any graph that is the disjoint union of a non-trivial component and enough number of isolated vertices is a strict-double-bound graph.*

Therefore, we introduced the strict-double-bound number of a graph in [8]. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{\, n\,;\ \mathrm{sDB}(P) \cong G \cup \overline{K}_n \text{ for some poset } P\}$.

Scott [11] obtained the following result, using a concept of transitive double competition numbers.

Theorem 1.2 (Scott [11]). *For a non-trivial connected graph G and a minimal edge clique cover $\mathcal{Q}$ of G,* $\left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil \leq \zeta(G) \leq |\mathcal{Q}| + 1$.

In [7] we obtain the following result.

Proposition 1.1 (Ogawa et. al [7]). *Let G be a connected graph with at least two vertices and P a poset with* $\mathrm{sDB}(P) \cong G \cup \overline{K}_{\zeta(G)}$. *Then* $|\mathrm{Max}(P) \cup \mathrm{Min}(P)| = \zeta(G)$.

By Theorem 1.2, we obtained that $\zeta(K_n) = 2$ for $n \geq 2$. We already obtained strict-double-bound numbers of $K_{1,n}$, P_n, C_n and W_n in [3] and [8]. We also gave an upper bound of strict-double-bound numbers of non-trivial trees in [8].

The *sum* $G + H$ of two graphs G and H is the graph with the vertex set $V(G + H) = V(G) \cup V(H)$ and the edge set $E(G + H) = E(G) \cup E(H) \cup \{uv\,; u \in V(G), v \in V(H)\}$. In [3] we also obtained the following result on the sum operation.

Theorem 1.3 (Konishi et. al [3])**.** *For a graph G with at least two vertices and no isolated vertices, $\zeta(K_n + G) = \zeta(G)$ for $n \geq 1$.*

We consider another operation on graphs. The *union* $G \cup H$ of two graphs G and H is the graph with the vertex set $V(G \cup H) = V(G) \cup V(H)$ and the edge set $E(G \cup H) = E(G) \cup E(H)$. In this paper, we consider graphs with a cut-set generating a complete graph. Using concepts of cut-sets and union of graphs, we estimate a strict-double-bound number of a graph.

2. Cut-vertices and strict-double-bound numbers

In this section we consider connected graphs and cut-vertices. We obtain the following result. For a graph G, $k(G)$ is the number of connected components. For a graph G, a vertex v of G is called a *cut-vertex* if $k(G - v) > k(G)$.

Theorem 2.1. *Let G be a connected graph with a cut-vertex s and $G - s$ has two components G_1 and G_2. For $i = 1, 2$, let $H_i = \langle V(G_i) \cup \{s\}\rangle_V$ and $P(H_i)$ be a poset such that $\mathrm{sDB}(P(H_i)) \cong H_i \cup \overline{K}_{\zeta(H_i)}$. Then $\zeta(G) \leq \zeta(H_1) + \zeta(H_2) - 1$.*

Proof. For $i = 1, 2$, let α_i be a minimal element of $P(H_i)$ such that $\alpha_i \leq_{P(H_i)} s$. We construct a poset Q as follows:

1. $V(Q) = V(P(H_1)) \cup V(P(H_2)) - \{\alpha_2\}$,
2. $x \leq_Q x$ for all $x \in V(Q)$,
3. for $x \in \mathrm{Min}(P(H_1)) \cup (\mathrm{Min}(P(H_2)) - \{\alpha_2\})$ and $y \in V(Q)$, $x \leq_Q y$ if $x \leq_{P(H_i)} y$,
4. for $x \in V(Q)$ and $y \in \mathrm{Max}(P(H_1)) \cup \mathrm{Max}(P(H_2))$, $x \leq_Q y$ if $x \leq_{P(H_i)} y$,
5. for $x \in V(Q) - (\mathrm{Max}(P(H_1)) \cup \mathrm{Min}(P(H_1)) \cup \mathrm{Max}(P(H_2)) \cup \mathrm{Min}(P(H_2)))$ and $\gamma \in \mathrm{Max}(P(H_2))$, $\alpha_1 \leq_Q x \leq_Q \gamma$ and $\alpha_1 \leq_Q \gamma$ if $\alpha_2 \leq_{P(H_2)} x \leq_{P(H_2)} \gamma$.

Note that $H_1 \cup H_2 = G$. We show that $\mathrm{sDB}(Q) \cong G \cup \overline{K}_m$, where $m = \zeta(H_1) + \zeta(H_2) - 1$. We consider the following cases.

Case 1. $u, v \in V(P(H_i)) - (\mathrm{Max}(P(H_i)) \cup \mathrm{Min}(P(H_i)))$ and $uv \in E(H_i)$ $(i = 1, 2)$
Then there exist $a \in \mathrm{Min}(P(H_i))$ and $b \in \mathrm{Max}(P(H_i)) \subseteq \mathrm{Max}(Q)$ such that $a \leq_{P(H_i)} u \leq_{P(H_i)} b$ and $a \leq_{P(H_i)} v \leq_{P(H_i)} b$.

Subcase 1.1. $a \neq \alpha_2$
Then $a \in \mathrm{Min}(Q)$. So $a \leq_Q u \leq_Q b$ and $a \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$.

Subcase 1.2. $a = \alpha_2$
Then $\alpha_1 \leq_Q u \leq_Q b$ and $\alpha_1 \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$. **Case 2.** $u, v \in V(P(H_i)) - (\mathrm{Max}(P(H_i)) \cup \mathrm{Min}(P(H_i)))$ and $uv \notin E(H_i)$ $(i = 1, 2)$

If $L_{P(H_2)}(u) \cap L_{P(H_2)}(v) = \emptyset$ for $u, v \in V(P(H_2))$, then $\alpha_2 \notin L_{P(H_2)}(u) \cap L_{P(H_2)}(v)$ and $L_Q(u) \cap L_Q(v) = \emptyset$. Thus for $uv \notin E(H_i)$, $L_Q(u) \cap L_Q(v) = \emptyset$ or $U_Q(u) \cap U_Q(v) = \emptyset$. So $uv \notin E(\mathrm{sDB}(Q))$.

Case 3. $u \in V(P(H_1)) - (\mathrm{Max}(P(H_1)) \cup \mathrm{Min}(P(H_1)))$ and $v \in V(P(H_2)) - (\mathrm{Max}(P(H_2)) \cup \mathrm{Min}(P(H_2)))$

Then $U_Q(u) \subseteq U_{P(H_1)}(u)$, $U_Q(v) \subseteq U_{P(H_2)}(v)$ and $U_{P(H_1)}(u) \cap U_{P(H_2)}(v) = \emptyset$. Thus $U_Q(u) \cap U_Q(v) = \emptyset$ and $uv \notin E(\mathrm{sDB}(Q))$.

Therefore $\mathrm{sDB}(Q) \cong G \cup \overline{K}_m$, where $m = \zeta(H_1) + \zeta(H_2) - 1$. □

3. Cut-sets and strict-double-bound numbers

In this section we consider connected graphs and cut-sets inducing complete subgraphs. For a graph G, a vertex subset S of $V(G)$ is called a *cut-set* if $k(G - S) > k(G)$. For a poset P and $S \subseteq V(P)$, $\mathrm{Max}(P; S) = (\bigcup_{v \in S} U_P(v)) \cap \mathrm{Max}(P)$, $\mathrm{Min}(P; S) = (\bigcup_{v \in S} L_P(v)) \cap \mathrm{Min}(P)$ and $\mathrm{NoMin}(P; S) = \{c \in \mathrm{Min}(P) \ ; \ c \parallel v \text{ for all } v \in S\}$. Then $\mathrm{NoMin}(P; S) = \mathrm{Min}(P) - \mathrm{Min}(P; S)$.

We obtain the following result.

Theorem 3.1. *Let G be a connected graph with a cut-set S, where the induced subgraph $\langle S \rangle_V$ is a complete subgraph and $G - S$ has components $G_1, G_2, \ldots, G_k$. For $i = 1, 2, \ldots, k$, let $H_i = \langle V(G_i) \cup S \rangle_V$ and $P(H_i)$ be a poset such that $\mathrm{sDB}(P(H_i)) \cong H_i \cup \overline{K}_{\zeta(H_i)}$. Then $\zeta(G) \leq \sum_{i=1}^{k} |\mathrm{Max}(P(H_i))| + \max\{|\mathrm{Min}(P(H_i); S)| \ ; \ i = 1, 2, \ldots, k\} + \max\{|\mathrm{NoMin}(P(H_i); S)| \ ; \ i = 1, 2, \ldots, k\}$.*

Proof. For $i = 1, 2, \ldots, k$, let $\mathrm{Min}(P(H_i); S) = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,p_i}\}$ and $\mathrm{NoMin}(P(H_i); S) = \{\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,q_i}\}$. We assume that for $i = 1, 2, \ldots, k$, $|\mathrm{Min}(P(H_1); S)| \geq |\mathrm{Min}(P(H_i); S)|$ and $|\mathrm{NoMin}(P(H_t); S)| \geq |\mathrm{NoMin}(P(H_i); S)|$. We construct a poset Q as follows:

1. $V(Q) = \bigcup_{i=1}^{k} V(P(H_i)) - \bigcup_{i \neq 1} \mathrm{Min}(P(H_i); S) - \bigcup_{i \neq t} \mathrm{NoMin}(P(H_i); S)$
   $= (\bigcup_{i=1}^{k} (\mathrm{Max}(P(H_i)) \cup V(H_i))) \cup \mathrm{Min}(P(H_1); S) \cup \mathrm{NoMin}(P(H_t); S)$,
2. $x \leq_Q x$ for all $x \in V(Q)$,
3. for $x \in \mathrm{Min}(P(H_1); S) \cup \mathrm{NoMin}(P(H_t); S)$ and $y \in V(Q)$, $x \leq_Q y$ if $x \leq_{P(H_i)} y$,
4. for $x \in V(Q)$ and $y \in \bigcup_{i=1}^{k} \mathrm{Max}(P(H_i))$, $x \leq_Q y$ if $x \leq_{P(H_i)} y$,
5. for $i = 1, 2, \ldots, k$, if $w \in V(P(H_i)) - (\mathrm{Max}(P(H_i)) \cup \mathrm{Min}(P(H_i)))$, $\alpha_{i,j} \in \mathrm{Min}(P(H_i); S)$, $\gamma \in \mathrm{Max}(P(H_i))$ and $\alpha_{i,j} \leq_{P(H_i)} w \leq_{P(H_i)} \gamma$, then $\alpha_{1,j} \leq_Q w \leq_Q \gamma$ and $\alpha_{1,j} \leq_Q \gamma$,
6. for $i = 1, 2, \ldots, k$, if $w \in V(P(H_i)) - (\mathrm{Max}(P(H_i)) \cup \mathrm{Min}(P(H_i)))$, $\beta_{i,j} \in \mathrm{NoMin}(P(H_i); S)$, $\gamma \in \mathrm{Max}(P(H_i))$ and $\beta_{i,j} \leq_{P(H_i)} w \leq_{P(H_i)} \gamma$, then $\beta_{t,j} \leq_Q w \leq_Q \gamma$ and $\beta_{t,j} \leq_Q \gamma$.

Note that $H_1 \cup H_2 \cup \ldots \cup H_k = G$. We show that $\mathrm{sDB}(Q) \cong G \cup \overline{K}_m$, where $m = (\sum_{i=1}^{k} |\mathrm{Max}(P(H_i))|) + |\mathrm{Min}(P(H_1); S)| + |\mathrm{NoMin}(P(H_t); S)|$. We consider the following cases.

Case 1. $u, v \in V(P(H_1)) - (\mathrm{Max}(P(H_1)) \cup \mathrm{Min}(P(H_1)))$ and $uv \in E(H_1)$

Then there exist $a \in \mathrm{Min}(P(H_1))$ and $b \in \mathrm{Max}(P(H_1)) \subseteq \mathrm{Max}(Q)$ such that $a \leq_{P(H_1)} u \leq_{P(H_1)} b$ and $a \leq_{P(H_1)} v \leq_{P(H_1)} b$.

Subcase 1.1. $a \in \mathrm{Min}(P(H_1); S)$
Then $a \in \mathrm{Min}(Q)$. So $a \leq_Q u \leq_Q b$ and $a \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$.

Subcase 1.2. $a \in \mathrm{NoMin}(P(H_1); S)$ Then $a = \beta_{1,j}$. So $\beta_{t,j} \leq_Q u \leq_Q b$ and $\beta_{t,j} \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$.

Case 2. $u, v \in V(P(H_t)) - (\mathrm{Max}(P(H_t)) \cup \mathrm{Min}(P(H_t)))$ and $uv \in E(H_t)$
Then there exist $a \in \mathrm{Min}(P(H_t))$ and $b \in \mathrm{Max}(P(H_t)) \subseteq \mathrm{Max}(Q)$ such that $a \leq_{P(H_t)} u \leq_{P(H_t)} b$ and $a \leq_{P(H_t)} v \leq_{P(H_t)} b$.

Subcase 2.1. $a \in \mathrm{Min}(P(H_t); S)$
Then $a = \alpha_{t,j}$. So $\alpha_{1,j} \leq_Q u \leq_Q b$ and $\alpha_{1,j} \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$.

Subcase 2.2. $a \in \mathrm{NoMin}(P(H_t); S)$
Then $a \in \mathrm{Min}(Q)$. So $a \leq_Q u \leq_Q b$ and $a \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$.

Case 3. $u, v \in V(P(H_i)) - (\mathrm{Max}(P(H_i)) \cup \mathrm{Min}(P(H_i)))$, $uv \in E(H_i)$ and $i \neq 1, t$
Then there exist $a \in \mathrm{Min}(P(H_i))$ and $b \in \mathrm{Max}(P(H_i)) \subseteq \mathrm{Max}(Q)$ such that $a \leq_{P(H_i)} u \leq_{P(H_i)} b$ and $a \leq_{P(H_i)} v \leq_{P(H_i)} b$.

Subcase 3.1. $a \in \mathrm{Min}(P(H_i); S)$
Then $a = \alpha_{i,j}$. So $\alpha_{1,j} \leq_Q u \leq_Q b$ and $\alpha_{1,j} \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$.

Subcase 3.2. $a \in \mathrm{NoMin}(P(H_i); S)$
Then $a = \beta_{i,j}$. So $\beta_{t,j} \leq_Q u \leq_Q b$ and $\beta_{t,j} \leq_Q v \leq_Q b$. And $uv \in E(\mathrm{sDB}(Q))$.

Case 4. $u, v \in V(P(H_i)) - (\mathrm{Max}(P(H_i)) \cup \mathrm{Min}(P(H_i)))$ and $uv \notin E(H_i)$
Then $L_{P(H_i)}(u) \cap \mathrm{Min}(P(H_i)) = \{\alpha_{i,l_1}, \alpha_{i,l_2}, \ldots, \alpha_{i,l_s}\} \cup \{\beta_{i,f_1}, \beta_{i,f_2}, \ldots, \beta_{i,f_d}\}$ and $L_{P(H_i)}(v) \cap \mathrm{Min}(P(H_i)) = \{\alpha_{i,g_1}, \alpha_{i,g_2}, \ldots, \alpha_{i,g_o}\} \cup \{\beta_{i,h_1}, \beta_{i,h_2}, \ldots, \beta_{i,h_r}\}$. Thus $L_Q(u) = \{\alpha_{1,l_1}, \alpha_{1,l_2}, \ldots, \alpha_{1,l_s}\}$ $\cup$ $\{\beta_{t,f_1}, \beta_{t,f_2}, \ldots, \beta_{t,f_d}\}$ and $L_Q(v) = \{\alpha_{1,g_1}, \alpha_{1,g_2}, \ldots, \alpha_{1,g_o}\} \cup \{\beta_{t,h_1}, \beta_{t,h_2}, \ldots, \beta_{t,h_r}\}$. Since $uv \notin E(H_i)$, $L_{P(H_i)}(u) \cap L_{P(H_i)}(v) = \emptyset$ or $U_{P(H_i)}(u) \cap U_{P(H_i)}(v) = \emptyset$. So $L_Q(u) \cap L_Q(v) = \emptyset$ or $U_Q(u) \cap U_Q(v) = \emptyset$. Thus $uv \notin E(\mathrm{sDB}(Q))$.

Case 5. $u \in V(P(H_i)) - (\mathrm{Max}(P(H_i)) \cup \mathrm{Min}(P(H_i)) \cup S)$, $v \in V(P(H_j)) - (\mathrm{Max}(P(H_j)) \cup \mathrm{Min}(P(H_j)) \cup S)$ and $i \neq j$
Then $U_Q(u) \subseteq U_{P(H_i)}(u)$, $U_Q(v) \subseteq U_{P(H_j)}(v)$ and $U_{P(H_i)}(u) \cap U_{P(H_j)}(v) = \emptyset$. Thus $U_Q(u) \cap U_Q(v) = \emptyset$ and $uv \notin E(\mathrm{sDB}(Q))$.

Case 6. $u, v \in S$
Since $S \subseteq V(P(H_1)) - (\mathrm{Max}(P(H_1)) \cup \mathrm{Min}(P(H_1)))$ and $uv \in E(H_1)$, $uv \in E(\mathrm{sDB}(Q))$ by Case 1.

Thus $\mathrm{sDB}(Q) \cong G \cup \overline{K}_m$, where $m = \sum_{i=1}^{k} |\mathrm{Max}(P(H_i))| + |\mathrm{Min}(P(H_1); S)| + |\mathrm{NoMin}(P(H_t); S)|$. Therefore $\zeta(G) \leq \sum_{i=1}^{k} |\mathrm{Max}(P(H_i))| + |\mathrm{Min}(P(H_1); S)| + |\mathrm{NoMin}(P(H_t); S)|$ $= \sum_{i=1}^{k} |\mathrm{Max}(P(H_i))| + \max_{i=1,2,\ldots,k} |\mathrm{Min}(P(H_i); S)| + \max_{i=1,2,\ldots,k} |\mathrm{NoMin}(P(H_i); S)|$. □

4. Chordal graphs

A graph is called a *chordal graph* if every cycle of length greater than 3 has a chord. We already know the following result in [2].

Theorem 4.1. *For a graph G, G is a chordal graph if and only if every minimal cut-set induces a complete subgraph of G.*

Since a minimal cut-set of a chordal graph generates a complete subgraph, we have the following result on chordal graphs by Theorem 3.1.

Proposition 4.1. *Let G be a connected chordal graph.*
(1) If G is a complete graph, then $\zeta(G) = 2$.
(2) If G is a non-complete graph, then $\zeta(G) \leq \sum_{i=1}^{k} |\operatorname{Max}(P(H_i))| + \max\{\, |\operatorname{Min}(P(H_i); S)|\,;\, i = 1, 2, \ldots, k\} + \max\{\, |\operatorname{NoMin}(P(H_i); S)|\,;\, i = 1, 2, \ldots, k\}$, where $S \subseteq V(G)$ is a minimal cut-set, G_1, G_2. …, G_k are components of $G - S$, $H_i = \langle V(G_i) \cup S\rangle_V$ for $i = 1, 2, \ldots, k$ and $P(H_i)$ is a poset such that $\operatorname{sDB}(P(H_i)) \cong H_i \cup \overline{K}_{\zeta(H_i)}$ for $i = 1, 2, \ldots, k$.

5. *k*-trees

In this section we consider k-trees. A *k-tree* is a chordal graph that can be constructed from a complete graph K_k by a sequence of vertex additions in which the neighborhood of each new vertex is a complete subgraph with k vertices of the current graph. Further k-trees other than complete graphs are called *non-clique k-trees*. And k-trees are connected graphs. In [5] and [9] Lin et. al reported some properties of k-trees.

Let $G(K_k; v_1, v_2, \ldots, v_m)$ be a k-tree with the vertex additions sequence $v_1, v_2, \ldots, v_m$. Let P_Z be a poset with $V(P_Z) = \{z_1, u_0, u_1, \ldots, u_k, z_2\}$, $z_1 \leq_{P_Z} u_j \leq_{P_Z} z_2$, $z_1 \leq_{P_Z} z_2$, $z_1 \leq_{P_Z} z_1$, $u_j \leq_{P_Z} u_j$ and $z_2 \leq_{P_Z} z_2$ for $j = 0, 1, \ldots, k$. Then $\operatorname{sDB}(P_Z) \cong K_{k+1} \cup \overline{K}_2$. We obtain the following result by Theorem 3.1.

Proposition 5.1. *Let $G(K_k; v_1, v_2, \cdots, v_m)$ be a k-tree with the vertex addition sequence $v_1, v_2, \cdots, v_m$. Then $\zeta(G(K_k; v_1, v_2, \ldots, v_m)) \leq m + 1$.*

Proof. The proof is by induction on the length of a vertex addition sequence. Since $G(K_k; v_1) \cong K_{k+1}$, $\zeta(G(K_k; v_1)) = \zeta(K_{k+1}) = 2 \leq 1+1$. By induction hypothesis, $\zeta(G(K_k; v_1, v_2, \ldots, v_{m-1})) \leq m$. Let P be a poset such that $\operatorname{sDB}(P) \cong G(K_k; v_1, v_2, \ldots, v_{m-1}) \cup \overline{K}_n$, where $n = \zeta(G(K_k; v_1, v_2, \ldots, v_{m-1})) \leq m$. Let S be the neighborhood of v_m of $G(K_k; v_1, v_2, \ldots, v_m)$. Then S is a cut-set of $G(K_k; v_1, v_2, \ldots, v_m)$ and generates a complete subgraph with k vertices. Using the proof methods of Theorem 3.1, we can construct a poset Q from P and P_Z such that $\operatorname{sDB}(Q) \cong G(K_k; v_1, v_2, \ldots, v_m) \cup \overline{K}_l$. Since $\operatorname{sDB}(P_Z) \cong K_{k+1} \cup \overline{K}_2$, $\operatorname{NoMin}(P_Z; S) = \emptyset$, $\operatorname{Min}(P_Z; S) = \{z_1\}$ and $\operatorname{Max}(P_Z) = \{z_2\}$, $l = \zeta(G(K_k; v_1, v_2, \ldots, v_{m-1})) + 1 \leq m + 1$. □

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