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16-vertex graphs with automorphism groups A_4 and A_5 from the icosahedron

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Abstract

The article deals with the problem of finding vertex-minimal graphs with a given automorphism group. We exhibit two undirected 16-vertex graphs having automorphism groups A_4 and A_5. It improves Babai's bound for A_4 and the graphical regular representation bound for A_5. The graphs are constructed using projectivisation of the vertex-face graph of the icosahedron.

Keywords: graph, icosahedron, hemi-icosahedron, automorphism group, alternating group
Mathematics Subject Classification : 05C25, 05E18, 05C35.

This article addresses a problem in graph representation theory of finite groups - finding undirected graphs with a given automorphism group and minimal number of vertices. Denote by $\mu(G)$ the minimal number of vertices of undirected graphs having automorphism group isomorphic to G, $\mu(G) = \min\limits_{\Gamma:Aut(\Gamma)\simeq G} |V(\Gamma)|$. It is known [1] that $\mu(G) \leq 2|G|$, for any finite group G which is not cyclic of order $3, 4$ or 5. See Babai [2] for an exposition of this area. There are groups which admit a graphical regular representation, for such groups $\mu(G) \leq |G|$. For some recent work see [4].

For alternating groups A_n $\mu(A_n)$ is known for $n \geq 13$, see Liebeck [6]. If $n \equiv 0$ *or* $1 (mod\ 4)$, then $\mu(A_n) = 2^n - n - 2$. Additionally, for $n \geq 5$ A_n admits a graphical regular representation, see [8]. Thus for A_5 the best published estimate until now seemed to be $\mu(A_5) \leq 60$.

In this paper we exhibit graphs $\Gamma_i = (V, E_i)$, $i \in \{4, 5\}$, such that $|V| = 16$ and $Aut(\Gamma_i) \simeq A_i$.

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Γ_4 (also denoted Ξ_I) improves Babai's bound for A_4. Γ_5 (also denoted Π_I) has fewer vertices than the graphical regular representation of A_5. Γ_5 is listed in [3] together with the order of its automorphism group. The new graphs are based on projectivisation of the vertex-face incidence relation of the regular icosahedron.

We use standard notation for undirected graphs, see Diestel [5]. A bipartite graph Γ with vertex partition sets V_1 and V_2 is denoted as $\Gamma = (V_1, V_2, E)$. Given a polyhedron P, we denote its vertex, edge and face sets as $V = V(P)$, $E = E(P)$ and $F = F(P)$, respectively. We can think of P as the triple (V, E, F). If S is a subset of $\mathbb{R}^3$ not containing the origin, then its image under the projectivisation map to $P(\mathbb{R}^3)$ is denoted by $\pi(S)$ or $[S]$, $[S] = \bigcup_{x \in S}[x]$.

1. Main results

In this section we define objects used for our construction - projective vertex-face graphs. We prove that the automorphism group of the projective vertex-face graph of the regular icosahedron is A_5. We further show that after adding three extra edges we get a graph with the automorphism group A_4.

1.1. Vertex-face graphs of polyhedra

Definition 1.1. Let $P = (V, E, F)$ be a polyhedron. An undirected bipartite graph $\Gamma_P = (V, F, I)$ is the **vertex-face graph of** P if $v \sim f$ iff $v \in V$, $f \in F$ and $v \in f$. In other words, Γ_P corresponds to the vertex-face incidence relation in $V \times F$.

Definition 1.2. Let $S = (V, E, F)$ be a centrally symmetric polyhedron. Let S be positioned in $\mathbb{R}^3$ so that its center is at $(0, 0, 0)$. We call the undirected bipartite graph $\Pi_S = ([V], [F], I_p)$ **projective vertex-face graph** if for any $v_p \in [V]$, $f_p \in [F]$ we have $v_p \sim f_p$ iff $v \in f$ for some $v \in \pi^{-1}(v_p)$ and $f \in \pi^{-1}(f_p)$.

1.2. Projective vertex-face graph of the icosahedron and A_5

Let $I = (V, E, F)$ be the regular icosahedron. Define $\Gamma_5 = \Pi_I$, it is shown in Fig.1, an adjacency matrix of Π_I is given in Appendix A. Π_I can be interpreted in terms of the hemi-icosahedron, see [7].

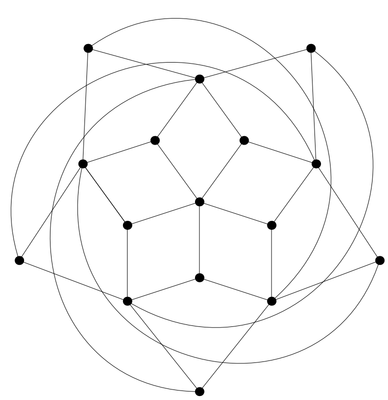

Fig.1. - Π_I.

Proposition 1.1. *Let* I *be the regular icosahedron. Then* $Aut(\Pi_I) \simeq A_5$.

Proof. We prove that $Rot(I) \simeq Aut(\Pi_I)$ in two steps. First we show that there is a subgroup in $Aut(\Pi_I)$ isomorphic to $Rot(I)$ - the group of rotational symmetries of I, rotations of $\mathbb{R}^3$ preserving V and E. It is known that $Rot(I) \simeq A_5$. There is an injective group morphism $f : Rot(I) \xrightarrow{f_1} Aut(\Gamma_I) \xrightarrow{f_2} Aut(\Pi_I)$. $f_1 : Rot(I) \to Aut(\Gamma_I)$ maps every $\rho \in Rot(I)$ to $f_1(\rho) \in Aut(\Gamma_I)$ which is the permutation of $V \cup F$ induced by ρ: $f_1(\rho)(x) = \rho(x)$ for any $x \in V \cup F$. Rotations of I preserve the vertex-face incidence relation and f_1 is a group morphism. $f_2 : Aut(\Gamma_I) \to Aut(\Pi_I)$ maps every $\varphi \in Aut(\Gamma_I)$ to $\varphi_P \in Aut(\Pi_I)$ defined by the rule $\varphi_P([x]) = [\varphi(x)]$ for any $x \in V(\Gamma_I)$. Projectivization and composition commute therefore f_2 is a group morphism. f is injective since there is no nontrivial rotation of I sending each vertex to another vertex in the same projective class.

In the second step we prove that $|Aut(\Pi_I)| \leq 60$ by a counting argument. Every vertex $v \in [V]$ is contained in a subgraph $\sigma(v)$ shown in Fig.2.

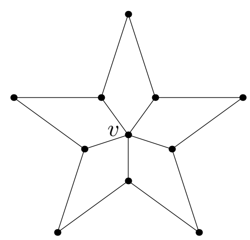

Fig.2. - $\sigma(v)$.

All Π_I-vertices in $[V]$ have degree 5, all Π_I-vertices in $[F]$ have degree 3. It follows that $[V]$ and $[F]$ both are unions of $Aut(\Pi_I)$-orbits. v can be mapped by a Π_I-automorphism in at most 6 possible ways. After fixing the image of v it follows by $Aut(\Pi_I)$-invariance of $[V]$ that the subgraph $\sigma(v)$ can be mapped in at most 10 ways. Any permutation of $[V]$ by an automorphism determines a unique permutation of $[F]$. Thus $|Aut(\Pi_I)| \leq 60$. We have proved that $Aut(\Pi_I) = f(Rot(I)) \simeq A_5$. ☐

Remark 1.1. A graph isomorphic to Π_I is listed without discussion of its construction and automorphism group in [3] as ET16.5.

1.3. A modification of the projective vertex-face graph of the icosahedron and A_4

Since A_5 has subgroups isomorphic to A_4, we can try to modify Π_I so that the automorphism group of the modified graph is isomorphic to A_4. We find generators for a subgroup $H \leq Rot(I)$, such that $H \simeq A_4$, and add three extra edges to Π_I which are permuted only by elements of H.

Denote by I_1 the polyhedral (1-skeleton) graph of I, $Aut(I_1) \simeq Sym(I) \simeq A_5 \times \mathbb{Z}_2$.

Proposition 1.2. *Choose a* 6*-subset of vertices* $W = \{O, A, B, C, D, E\} \subseteq V(I)$ *such that* $I_1[W]$ *is isomorphic to the* 5*-wheel, see Fig.3.*

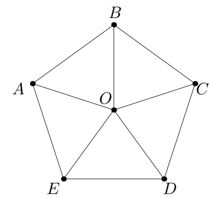

Fig.3. - $I_1[W]$.

Define an undirected graph $\Gamma_4 = \Xi_I = ([V] \cup [F], I_p \cup J)$ *by adding three edges to* Π_I*:* $J = \{[A] \sim [C], [B] \sim [O], [D] \sim [E]\}$*, see Fig.4, Fig.5 and Appendix B. Then* $Aut(\Xi_I) \simeq A_4$*.*

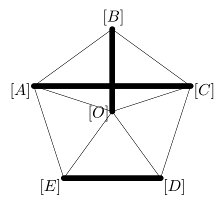

Fig.4. - the extra edges.

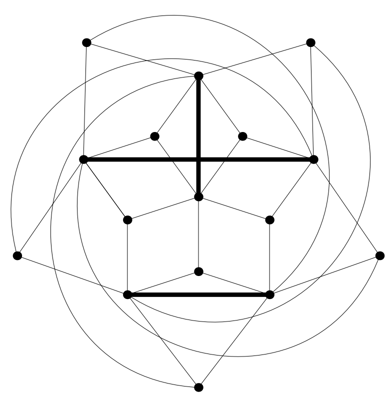

Fig.5. - Ξ_I*.*

Proof. Consider the subgroup $H = \langle a, b \rangle \leq Rot(I)$ generated by two rotations: a - a rotation of order 2 around the line passing through the center of the edge OB and the center of I, b - a rotation of order 3 around the line passing through the center of the face OCD and the center of I. We prove that $H \simeq A_4$ and $f(H) = Aut(\Xi_I)$ where f is as in Proposition 1.1.

To prove that $H \simeq A_4$ we investigate subgroups of A_5 generated by two elements of order 2 and 3. If $H' = \langle a', b' \rangle \leq A_5$, $ord(a') = 2$, $ord(b') = 3$, then there are 3 possibilities for the isomorphism type of the functional graph ("cycle type") of the pair (a', b'): $(a_1, b_1) = ((12)(34), (345))$, $(a_2, b_2) = ((12)(34), (134))$ or $(a_3, b_3) = ((12)(34), (135))$. It can be checked that $\langle a_1, b_1 \rangle \simeq \Sigma_3$, $\langle a_2, b_2 \rangle \simeq A_4$, $\langle a_3, b_3 \rangle \simeq A_5$. Additionally, $ord(a_1 b_1) = 2$, $ord(a_2 b_2) = 3$, $ord(a_3 b_3) = 5$. Now, in our case $ord(ab) = 3$, thus $H = \langle a, b \rangle \simeq \langle a_2, b_2 \rangle \simeq A_4$.

Next we prove that $Aut(\Xi_I) = f(H)$. Note that O, A, B, C, D, E in Fig.3 and Fig.4 represent $[V]$.

First we prove that $f(H) \leq Aut(\Xi_I)$. Ξ_I differs from Π_I by three extra edges. Elements of $f(H)$ permute Π_I-edges so we only need to check that they permute the new edges. The restrictions of $f(a)$ and $f(b)$ to $[V]$ are, respectively, $([O][B])$ and $([O][C][D])([A][E][B])$ (in cycle notation). It follows that $f(b)$ cyclically permutes the three extra edges and $f(a)$ fixes them.

To prove that $Aut(\Xi_I) \leq f(H)$ we observe that only $[F]$-type vertices have degree 3 in both Π_I and Ξ_I, only V-type vertices have degree 5 in Π_I. Thus any $Aut(\Xi_I)$-element as a permutation of $[V] \cup [F]$ belongs to $Aut(\Pi_I)$ and thus is the f-image of a $Rot(I)$-element. We show that for any rotation $r' \in Rot(I) \backslash H$, $f(r')$ does not permute the three extra edges and thus $f(r') \notin Aut(\Xi_I)$. We have that $Rot(I) = \langle a, b, c\rangle$ where c is any rotation of order 5. Since $|Rot(I) : H| = 5$ it follows that any element of $Rot(I)$ is in form $c^n h$ where $h \in \langle a, b\rangle = H$. Let c be the rotation around the line passing through the center of I and O corresponding to the vertex permutation $(ABCDE)$. The edge $[O] \sim [B]$ is the only extra edge having $[O]$ as a vertex, all edges from $[O]$ are rotationally permuted by $f(c^n)$, see Fig.4. It follows that nontrivial elements $f(c^n)$ do not permute the three extra edges in Ξ_I. □

Remark 1.2. If D is the dodecahedron then $\Pi_D \simeq \Pi_I \simeq A_5$.

2. Appendices

A - An adjacency matrix of Π_I

Remark 2.1. In the standard ordering vertices $\{1, ..., 10\}$ correspond to $[F]$ and vertices $\{11, .., 16\}$ correspond to $[V]$.

										1	1	1			
										1	1			1	
										1		1	1		
										1			1		1
										1				1	1
											1	1			1
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1	1	1	1	1											
1	1				1	1	1								
1		1			1			1	1						
		1	1			1	1	1							
	1			1		1		1	1						
			1	1	1		1		1						

B - An adjacency matrix of Ξ_I

										1	1	1			
										1	1			1	
										1		1	1		
										1			1		1
										1				1	1
											1	1			1
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1	1	1	1	1									**1**		
1	1				1	1	1							**1**	
1		1			1			1	1						**1**
		1	1			1	1	1		**1**					
	1			1		1		1	1		**1**				
			1	1	1		1		1			**1**			

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