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On central-peripheral appendage numbers of uniform central graphs

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Abstract

In a uniform central graph (UCG) the set of eccentric vertices of a central vertex is the same for all central vertices. This collection of eccentric vertices is the centered periphery. For a pair of graphs (C, P) the central-peripheral appendage number, $A_{\mathrm{ucg}}(C, P)$, is the minimum number vertices needed to be adjoined to the graphs C and P in order to construct a uniform central graph H with center $V(C)$ and centered-periphery $V(P)$. We compute $A_{\mathrm{ucg}}(C, P)$ in terms of the radius and diameter of P, and whether or not C is a complete graph. In the process we show $A_{\mathrm{ucg}}(C, P) \leq 6$ if $\mathrm{diam}(P) > 2$. We also provide structure theorems for UCGs in terms of the centered periphery.

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1. Introduction

An appendage number of a graph G is the minimum number of vertices to be added to G to obtain a supergraph H of G so that H satisfies some prescribed properties. There is no restriction on the number of edges. Appendage numbers are a measurement of how efficiently one can construct a graph with desired properties. In this paper we study appendage numbers where the supergraph is what is known as a uniform central graph.

Let $G = (V(G), E(G))$ be a simple graph. The eccentricity of a vertex v, denoted by $e_G(v)$, is defined by $\max\{d(u, v) : u \in V(G)\}$ where $d(u, v)$ is the distance between two vertices u and

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v. When the graph is clear from context we will just write $e_G(v) = e(v)$. A vertex x is called an eccentric vertex of v if $d(v, x) = e(v)$; and the set of all eccentric vertices of v is denoted by $EC(v)$. The radius of G, $r(G)$ or r, is the minimum eccentricity of the vertices in G and the diameter of G, $\mathrm{diam}(G)$, is the maximum eccentricity of the vertices in G.

A (u, v)-path is called a geodesic, when the length of the path is $d(u, v)$. For a central vertex u, a (u, v)-geodesic is called a radial path if the length of the path is $r(G)$. For any nonempty subset S of vertices in G, $\langle S \rangle$ represents the subgraph of G induced by S. For terminology not defined in this paper, the reader is referred to [6].

Let $\mathcal{Z}(G) = \{z \in V(G) | e(z) = r(G)\}$ be the center of the graph G. We now introduce a new term called the centered periphery.

Definition The *centered periphery* of G is

$$\mathcal{CP}(G) = \bigcup_{z \in \mathcal{Z}(G)} EC(z).$$

This is the set of vertices that are far from at least one vertex in the center. The periphery, which is the set of maximally eccentric vertices, coincides with centered periphery for some graphs. However, in many cases they differ. For instance, the graph in Figure 1 has periphery $\{p_0, p_1, p_2, p_5, p_6, p_7\}$ while the centered periphery is $\{p_1, \ldots, p_6\}$.

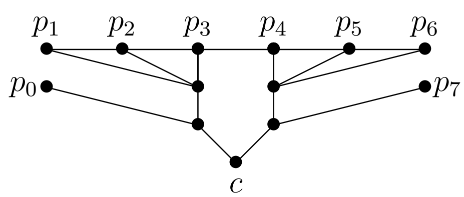

Figure 1: Periphery $\neq$ Centered Periphery

A graph G is called a *uniform central graph*, or UCG, if $EC(c)$ is same for all central vertices c of G, i.e. $EC(c) = \mathcal{CP}(G)$ for every vertex c in the center. The term was introduced in [3] where it is shown that a radial path in a uniform central graph contains only one central vertex. However, the structure of UCGs have not been extensively studied and there is no known classification of UCGs.

So far all papers involving UCGs have only used centers to study UCGs and have not looked at the centered periphery [2, 3, 4, 5]. However, it appears the structure of a UCG relies on both the center and the centered periphery. In this paper we study an appendage number that relies on both the center, and the centered periphery. In doing so we show how the structure of UCGs relies on both substructures. With this in mind we consider a UCG G to consist of three types of vertices, the center $\mathcal{Z}(G)$, the centered periphery $\mathcal{CP}(G)$, and the "intermediate" vertices defined by $\mathcal{I}(G) = V(G) - (\mathcal{Z}(G) \cup \mathcal{CP}(G))$.

We now turn to appendage numbers. Buckley, Miller and Slater [1] studied the appendage number of a graph G, where G is the center of its supergraph. These results have been extended to UCGs. In [4] Gu determined the appendage number for G where G is the center and supergraph is a UCG. More recently S. Klavžar, K. Narayankar, and S. Lokesh [5] looked at appendage numbers of G where the supergraph is a UCG and G is any subgraph.

In this paper we study the appendage number for a pair of given graphs (C, P), where the supergraph H is a uniform central graph satisfying $\langle \mathcal{Z}(H) \rangle = C$ and $\langle \mathcal{CP}(H) \rangle = P$. We denote the *central-peripheral appendage number*, $A_{\text{ucg}}(C, P)$, as the minimum number of "intermediate" vertices needed to construct such a uniform central graph H. Note, that by construction, C and P will be disjoint in H. By convention $A_{\text{ucg}}(C, P) = \infty$ if there is no graph satisfying the above conditions. We summarize our main results as follows.

Theorem 6.1. Let $C =< v >$ and P be any graph with $r(P) \geq 2$. Then $A_{\text{ucg}}(C, P) = 0$.

Theorem 6.2. When $n \geq 2$ and a graph P

$$A_{\text{ucg}}(K_n, P) = \begin{cases} 2 & \text{if } \text{diam}(P) \geq 4 \text{ and } r(P) \geq 3 \\ 3 & \text{if } \text{diam}(P) \geq 3 \text{ and } r(P) = 2 \end{cases}$$

Furthermore, for all $t \in \mathbb{N}$ there is a graph P with $\text{diam}(P) = r(P) = 2$ and $A_{\text{ucg}}(K_n, P) \geq t$.

Theorem 6.3. For a graph P, if C is a graph with $\text{diam}(C) > 1$ then

$$A_{\text{ucg}}(C, P) = \begin{cases} 4 & \text{if } \text{diam}(P) = \infty \\ 5 & \text{if } 5 \leq \text{diam}(P) < \infty \\ 5 \text{ or } 6 & \text{if } \text{diam}(P) = 4 \text{ and } r(P) = 4 \\ 6 & \text{if } \text{diam}(P) = 4 \text{ and } r(P) = 2 \\ 6 & \text{if } \text{diam}(P) = 3 \end{cases}$$

Furthermore, for all $t \in \mathbb{Z}$ there is a graph P with $\text{diam}(P) = r(P) = 2$ and $A_{\text{ucg}}(C, P) \geq t$.

These results show that it takes more vertices to construct UCGs with centered periphery that have small radius and diameter. In fact, when $\text{diam}(P) = r(P) = 2$, depending on the structure of P, one needs arbitrarily large number of intermediate vertices. This shows the importance of the centered periphery in understanding UCGs.

We also draw attention to the fact that our results depend on whether or not C is a complete graph, which matches the results of Gu [4]. In light of this we organize our paper in the following way. Section 2 focuses on results that do not depend on the center. These results are then applied to graphs with specific centers; complete graphs in Section 3 and all other graphs in Section 4 to classify central-peripheral appendage numbers. The results in Sections 3 and 4 are given in terms of sizes of various coverings of P. In Section 5 we compute the sizes of these coverings. Section 6 summarizes the previous work and proves Theorems 6.1, 6.2, and 6.3. Finally, in Section 7, using different techniques than Gu [4], we obtain her results as a corollary of our results.

2. General Centers

This section develops results about the structure of a uniform central graph H in terms of $P = \langle \mathcal{CP}(H) \rangle$. We do this by studying coverings of the centered periphery.

A *covering* of a graph $G = (V, E)$ is a set $\overline{V} = \{V_1, \dots, V_k\}$ where $V_i \subset V$ with $\cup V_i = V$. We say k is the size of the covering. A *subcovering* of a covering $\overline{V}$ is a subset of $\overline{V}$ that is also a covering.

Throughout this paper we are interested in coverings satisfying various properties. The simplest and most important of these conditions is condition A. Let P be a graph with covering $\overline{P} = \{P_1, \dots, P_k\}$. We say the pair $(P, \overline{P})$ satisfies

Condition A if for each $1 \leq i \leq k$, there is a vertex $p \notin P_i$ satisfying $d(P_i, p) \geq 2$.

When there is no confusion to the graph P, we simply say $\overline{P}$ satisfies condition A.

Elements of a cover can overlap. It is useful to minimize this overlap in some sense.

Lemma 2.1. *Let $\{P_1, \dots, P_k\}$ be a covering of P satisfying condition A. Then there is a subcovering $\{P'_1, \dots, P'_\eta\}$ satisfying condition A and for each i ($1 \leq i \leq \eta$)*

$$P'_i \not\subset \bigcup_{j \neq i} P'_j.$$

That is, for each i there exists $\tilde{p}_i \in P'_i$ such that $\tilde{p}_i \notin P'_j$ for all $j \neq i$.

Proof. Suppose there exists an i satisfying

$$P_i \subset \bigcup_{j \neq i} P_j.$$

Then $\{P_j : j \neq i\}$ is a subcovering satisfying condition A. □

The main idea of this paper is to take the question of determining $A_{\text{ucg}}(C, P)$ and to transfer it to questions about the size of coverings of P satisfying various conditions. This is possible because for any UCG there exists a natural covering on P. It turns out this covering satisfies condition A. These coverings are used to obtain lower and upper bounds on $A_{\text{ucg}}(C, P)$. To do this we first introduce some notation.

For a uniform central graph H we define a stratification of $V(H)$ in terms of the distance from the center. Let

$$D_m = \{u \in V(G) : d(u, \mathcal{Z}(H)) = m\}$$

for $0 \leq m \leq r = r(H)$. Then $D_0 = \mathcal{Z}(H)$ and $D_r = \mathcal{CP}(H)$. For an enumeration $D_1 = \{x_1, \dots, x_k\}$, let $\mathcal{L}_i$ be the set of all radial paths from C to P containing x_i

Lemma 2.2. *Let H be a uniform central graph with $\langle \mathcal{CP}(H) \rangle = P$. Then the graph H induces a covering on P satisfying condition A.*

Proof. Let H be a uniform central graph with $\langle \mathcal{CP}(H) \rangle = P$. Furthermore, let $C = \langle \mathcal{Z}(H) \rangle$ and $D_1 = \{x_1, \dots, x_k\}$ for some k. Let $P_i = V(P) \cap V(\mathcal{L}_i)$. Then $\{P_1, \dots, P_k\}$ is a covering of P since each radial path from C to P contains a vertex in D_1.

Assume $\{P_1, \dots, P_k\}$ does not satisfy the condition A. Then there is an i satisfying that for every vertex $p \notin P_i$ there is a $p' \in P_i$ with $d(p, p') \leq 1$. This shows $x_i \in V(C)$, a contradiction.

Since $x_i \in D_1$, there is a central vertex c with $d(x_i, c) = 1$. For each vertex v of H not in P, $d(c, v) \leq r(H) - 1$ and hence $d(x_i, v) \leq r(H)$. If p is a vertex in P_i, $d(x_i, p) = r(G) - 1$ from the construction of P_i.

If p is a vertex in P but not in P_i, then there is a vertex $p' \in P_i$ with $d(p, p') = 1$ and so

$$d(x_i, p) \leq d(x_i, p') + d(p', p) = (r(H) - 1) + 1 = r(H)$$

and so $x_i \in V(C)$, a contradiction. □

Corollary 2.1. *If H is a UCG with $P = \langle \mathcal{CP}(H) \rangle$, then $r(P) > 1$.*

Proof. If $r(P) = 1$, then there is $q \in V(P)$ with $e_P(q) = 1$. Let $\overline{P} = \{P_1, \ldots, P_k\}$ be the induced cover of P. Without loss of generality we may assume $q \in P_1$. Then for all $p \in V(P)$, $d(p, P_1) \leq 1$ and condition A is not met. Hence by Lemma 2.2 P is not a centered periphery for any UCG. □

Corollary 2.2. *If P is a graph with $r(P) \leq 1$, then $A_{\mathrm{ucg}}(C, P) = \infty$ for all graphs C.*

For H, a UCG with $C = \langle \mathcal{Z}(H) \rangle$ and $P = \langle \mathcal{CP}(H) \rangle$, we use the induced covering from Lemma 2.2 to gain a better understanding of the structure of $\mathcal{I}(H)$.

Let $D_1 = \{x_1, \ldots, x_k\}$ and $\{P_1, \ldots, P_k\}$ be the induced covering of P defined as in Lemma 2.2. Without loss of generality assume $\{P_1, \ldots, P_{k'}\}$ with $k' \leq k$, is subcovering with non-empty elements. Also let $\overline{P} = \{P_1, \ldots, P_{k''}\}$ and $\{\tilde{p}_1, \ldots, \tilde{p}_{k''}\}$, with $k'' \leq k'$, be as in Lemma 2.1. For $1 \leq m \leq r(H) - 1$ we define the following subsets of D_m.

$$D'_m = D_m \cap \bigcup_{i=1}^{k'} V(\mathcal{L}_i) \text{ and } D''_m(\overline{P}) = D_m \cap \bigcup_{i=1}^{k''} V(\mathcal{L}_i).$$

D_m and D'_m are well-defined with respect to H, however $D''_m(\overline{P})$ depends on a choice of subcover. When this choice is clear from context, we simply use D''_m.

The following proposition describes the structure of $\mathcal{I}(H)$ that is the key to the rest of this paper.

Proposition 2.1. *Let H be a UCG with $C = \langle \mathcal{Z}(H) \rangle$, $P = \langle \mathcal{CP}(H) \rangle$, $\overline{P} = \{P_1, \ldots, P_{k''}\}$ and $\{\tilde{p}_1, \ldots, \tilde{p}_{k''}\}$ as in Lemma 2.1 with respect to the induced covering from Lemma 2.2. Then any $(x_i, \tilde{p}_i)$-geodesic and $(x_j, \tilde{p}_j)$-geodesic are disjoint if $i \neq j$.*

Proof. For L_i a $(x_i, \tilde{p}_i)$-geodesic and L_j a $(x_j, \tilde{p}_j)$-geodesic with $i \neq j$, assume there exists a $y \in V(L_i) \cap V(L_j)$. Let L'_i be the subpath of L_i from x_i to y and L'_j the subpath of L_j from y to $\tilde{p}_j$. The concatenation of L'_i and L'_j yields a $(x_i, \tilde{p}_j)$-geodesic and so $\tilde{p}_j \in P_i$, a contradiction. □

Corollary 2.3. *With the assumptions of Proposition 2.1, $|D''_1| \leq |D''_m|$ for $1 \leq m \leq r(H) - 1$.*

Proof. For each $1 \leq i \leq k''$ there exists L_i, a $(x_i, \tilde{p}_i)$-geodesic. Let $\mathcal{L} = \{L_1, \ldots, L_{k''}\}$ and $F_m = V(\mathcal{L}) \cap D''_m$. By Proposition 2.1 $|F_1| = |F_m|$. Then by the construction of F_m

$$|D''_1| = |F_1| = |F_m| \leq |D''_m|.$$

□

We often work with UCGs with few intermediate vertices. For such UCGs we can say more about their structure.

Lemma 2.3. *Let H be a UCG with $P = \langle \mathcal{CP}(H) \rangle$ and $r = r(H)$. Furthermore, let $\overline{P} = \{P_1, \ldots, P_{k''}\}$ be a subcovering of the induced covering and $\{\tilde{p}_1, \ldots, \tilde{p}_{k''}\}$ an associated set of vertices as in Lemma 2.1. If $|D''_1| = |D''_{r-1}|$ then for each $P_i \in \overline{P}$ there is a unique $y_i \in D''_{r-1}$ such that y_i is adjacent to every vertex in P_i. Furthermore, if $i \neq j$, then $y_i \neq y_j$.*

Proof. Let $D''_1 = \{x_1, \ldots, x_{k''}\}$. For each i, let L_i be an $(x_i, \tilde{p}_i)$-geodesic and let y_i be the vertex on L_i adjacent to $\tilde{p}_i$. Then $y_i \in D''_{r-1}$. By Proposition 2.1 $y_i \neq y_j$ when $i \neq j$. Furthermore $D''_{r-1} = \{y_1, \ldots, y_{k''}\}$. We claim y_i is adjacent to every vertex in P_i.

For a vertex $p \in P_i$ there is a $c \in \mathcal{Z}(H)$ and a (c, p)-radial path L that contains x_i. Let y be the vertex on L adjacent to p. Since $y \in D''_{r-1}$, $y = y_j$ for some $1 \leq j \leq k''$. Since y_j is adjacent to $\tilde{p}_j$, there is a $(x_i, \tilde{p}_j)$-geodesic of length $r - 1$. Therefore, $\tilde{p}_j \in P_i$ which means $j = i$, $y = y_i$ and y_i is adjacent to p. □

Lemma 2.4. *Let H be a UCG with $C = \langle \mathcal{Z}(H) \rangle$ and $P = \langle \mathcal{CP}(H) \rangle$ and assume the induced covering $\overline{P} = \{P_1, \ldots, P_k\}$ satisfies the conditions of the subcovering in Lemma 2.1. That is $D_1 = D''_1$. Then each vertex in C is adjacent to each vertex in D_1.*

Proof. Let $D_1 = \{x_1, \ldots, x_k\}$ and $\{\tilde{p}_1, \ldots, \tilde{p}_k\}$ be an associated set of vertices to $\overline{P}$ as in Lemma 2.1. For each vertex c in the center and each $\tilde{p}_i$, there is a $(c, \tilde{p}_i)$-radial path. By construction of $\overline{P}$ and definition of $\tilde{p}_i$, each $(c, \tilde{p}_i)$-radial path must contain x_i. Therefore, c and x_i are adjacent. □

For a graph P, let $\mathrm{cov}_A(P)$ be the smallest size of a covering of P satisfying condition A. From the definition of condition A, it follows that $\mathrm{cov}_A(P) \geq 2$ for any graph P with $r(P) > 1$. The following result is critical in obtaining lower bounds on $A_{\mathrm{ucg}}(C, P)$.

Proposition 2.2. *If H is a uniform central graph with $C = \langle \mathcal{Z}(H) \rangle$, $\langle \mathcal{CP}(H) \rangle = P$, $r = r(H)$ and $\kappa = \mathrm{cov}_A(P)$, then $|\mathcal{I}(H)| \geq \kappa(r - 1)$.*

Proof. Let $\{P_1, \ldots, P_{k''}\}$ be a subcover of the induced cover on P from H from Lemmas 2.2 and 2.1. By definition of κ and corollary 2.3 it follows that

$$\kappa \leq k'' = |D''_1| \leq |D''_m| \leq |D_m|.$$

Since

$$\mathcal{I}(H) = \bigcup_{m=1}^{r-1} D_m$$

the result follows. □

Proposition 2.2 is used to obtain bounds on the radius of a UCG.

Corollary 2.4. *If H is a uniform central graph with $\langle \mathcal{CP}(H) \rangle = P$, $\kappa = \mathrm{cov}_A(P)$ and $|\mathcal{I}(H)| \leq s\kappa + t$ where $s, t \in \mathbb{N}$ and $t < \kappa$, then $r(H) \leq s + 1$.*

Proof. Observing $(s+1)\kappa > s\kappa + t$ the result follows from the contrapositive of Proposition 2.2. □

We now construct a graph with center C and centered periphery P. Let C and P be graphs, and $\rho \in \mathbb{N}$. Suppose $\overline{P} = \{P_1, \ldots, P_k\}$ is a covering of P. Define a graph $G = \mathcal{G}(C, P, \overline{P}, \rho)$ as follows:

$$V(G) = V(C) \cup V(P) \cup \{x_{i,j} : 0 \leq i \leq k, 1 \leq j \leq \rho\}$$

and ab is an edge in G if and only if one of the following occurs

1. ab is an edge of C.
2. ab is an edge of P.
3. a is a vertex of C and $b = x_{i,1}$ for some i with $0 \leq i \leq k$.
4. $a \in P_i$ and $b = x_{i,\rho}$ for some i with $1 \leq i \leq k$.
5. $a = x_{i,j}$ and $b = x_{i,j+1}$ for i and j with $0 \leq i \leq k, 1 \leq j \leq \rho - 1$

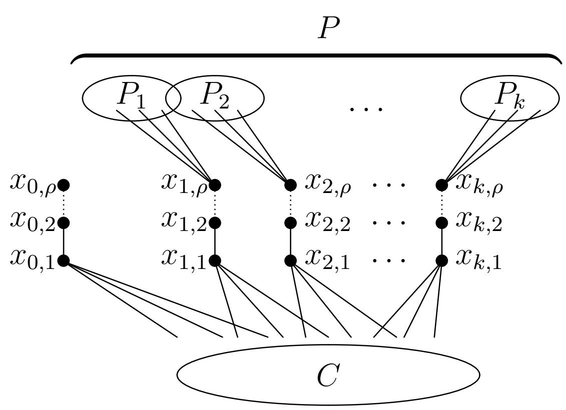

Figure 2: $G = \mathcal{G}(C, P, \overline{P}, \rho)$

Proposition 2.3. *For two given graphs C and P, let $\overline{P} = \{P_1, \ldots, P_k\}$ be a covering of P satisfying condition A, and $\rho \geq \min\{\text{diam}(C), 2\}$. Then $G = \mathcal{G}(C, P, \overline{P}, \rho)$ is a UCG with radius $\rho + 1$, $\langle \mathcal{Z}(G) \rangle = C$ and $\langle \mathcal{CP}(G) \rangle = P$.*

Proof. From the construction of G, we can show the following:

i) For all $j = 1, \ldots, \rho$ and any vertex p in P, $d(x_{0,j}, p) \geq \rho + 2$, and so $e(x_{0,j}) \geq \rho + 2$ and $e(p) \geq \rho + 2$.

ii) For all i and j with $1 \leq i \leq k$ and $2 \leq j \leq \rho$, $d(x_{i,j}, x_{0,\rho}) \geq \rho + 2$, and so $e(x_{i,j}) \geq \rho + 2$.

iii) Since the covering $\overline{P}$ satisfies condition A, for each i with $1 \leq i \leq k$ there exists a vertex p in P_j for some $j \neq i$ satisfying $d_P(P_i, p) \geq 2$. Thus $d(x_{i,1}, p) \geq \rho + 2$ and $e(x_{i,1}) \geq \rho + 2$.

iv) For a vertex c in C, $d(c, x_{i,j}) \leq \rho$, $d(c, p) = \rho + 1$ for all p in P, and $d(c, c') \leq 2$ for any vertex c' in C. This implies $e(c) = \rho + 1$ and $EC(c) = V(P)$.

Therefore, $G = \mathcal{G}(C, P, \overline{P}, \rho)$ is a UCG with radius $\rho + 1$, $\langle \mathcal{Z}(G) \rangle = C$ and $\langle \mathcal{CP}(G) \rangle = P$. □

We end this section with a result about conditions when a spanning subgraph of a UCG is still a UCG.

Lemma 2.5. *Let H be a UCG with $V(C) = \mathcal{Z}(H)$ and let G be a spanning subgraph of H. Then G is a UCG with $V(C) = \mathcal{Z}(G)$ and $\mathcal{CP}(G) = \mathcal{CP}(H)$ if for all $c \in V(C)$ and $x \in V(G)$, $d_G(c, x) = d_H(c, x)$.*

Proof. Since $E(G) \subset E(H)$, the eccentricity of a vertex v in G is greater than or equal to the eccentricity of v in H. The assumption that $d_G(c, x) = d_H(c, x)$ for all $c \in V(C)$ and $x \in V(G)$ implies $\mathcal{Z}(G) = V(C)$ and $EC(c)$ in G is $\mathcal{CP}(H)$. Hence, G is a UCG with $V(C) = \mathcal{Z}(G)$ and $\mathcal{CP}(G) = \mathcal{CP}(H)$. □

3. When $C = K_n$ with $n \geq 2$

In this section we compute appendage numbers when the center C is a complete graph K_n with $n \geq 2$ in terms of the size of a smallest covering for the centered periphery.

We introduce a new condition on coverings. Let P be a graph with covering $\overline{P} = \{P_1, \ldots, P_k\}$. We say $(P, \overline{P})$ satisfies

Condition B if for every $1 \leq i \leq k$ and for each $p \in P_i$ either

1. there is a vertex $p' \notin P_i$ with $d(p, p') \geq 3$, or
2. there is a $j \neq i$ satisfying $d(p, P_j) \geq 2$.

Once again, we say the covering $\overline{P}$ satisfies condition B when there is no confusion about P. Also define $\text{cov}_{AB}(P)$ to be the smallest size of a covering of P satisfying both condition A and condition B.

Before determining appendage numbers it is necessary to prove the existence of coverings satisfying conditions A and B.

Lemma 3.1. *Let P be a graph with $r(P) \geq 2$. Then there exists a covering of P satisfying conditions A and B.*

Proof. Let $V(P) = \{p_1, \ldots, p_k\}$. Then $\overline{P} = \{\{p_1\}, \ldots, \{p_k\}\}$ is a covering of P and $\overline{P}$ satisfies conditions A and B because $r(P) \geq 2$. □

We now find the appendage numbers in terms of $\kappa = \text{cov}_A(P)$.

Proposition 3.1. $\kappa \leq A_{\text{ucg}}(K_n, P) \leq \kappa + 1$ *where* $\kappa = \text{cov}_A(P)$.

Proof. By Lemma 3.1 there is a covering of P satisfying condition A, so let $\overline{P} = \{P_1, \ldots, P_\kappa\}$ be a smallest covering of P satisfying condition A. By Proposition 2.3, the graph $G = \mathcal{G}(K_n, P, \overline{P}, 1)$ is a UCG with radius 2 and $|\mathcal{I}(G)| = \kappa + 1$, and thus $A_{\text{ucg}}(K_n, P) \leq \kappa + 1$. The lower bound follows from Proposition 2.2.

□

Theorem 3.1. *Let* $\kappa = \text{cov}_A(P)$. *Then* $A_{\text{ucg}}(K_n, P) = \kappa$ *if and only if* $\text{cov}_{AB}(P) = \kappa$.

Proof. Suppose $\text{cov}_{AB}(P) = \kappa$ and let $\overline{P} = \{P_1, \ldots, P_\kappa\}$ be a smallest covering of P with respect to conditions A and B. Also, let $G = \mathcal{G}(K_n, P, \overline{P}, 1) - \{x_{0,1}\}$. Since $\overline{P}$ satisfies condition B, one can verify that for G, $e(p) \geq 3$ for all $p \in V(P)$. Using a similar argument to Proposition 2.3 we can show G is a UCG with $r(G) = 2$, $\langle \mathcal{CP}(G) \rangle = P$, $\langle \mathcal{Z}(G) \rangle = K_n$ and $|\mathcal{I}(G)| = \kappa$.

Now assume $A_{\text{ucg}}(K_n, P) = \kappa$. Let H be a UCG with $P = \langle \mathcal{CP}(H) \rangle$, $\langle \mathcal{Z}(H) \rangle = K_n$ and $|\mathcal{I}(H)| = \kappa$. From corollary 2.4, $r(H) = 2$. Let $\{P_1, \ldots, P_k\}$ be the induced covering of P as in Lemma 2.2. We know

$$\kappa \leq k = |D_1| = |\mathcal{I}(H)| = \kappa$$

and so $k = \kappa$. Let $D_1 = \{x_1, \ldots, x_k\}$, where x_i is associated to P_i.

Suppose p is a vertex in P_i. Since $e(p) \geq 3$ there is an $x \in V(H)$ with $d_H(p, x) \geq 3$. Note, $x \notin P_i$ because all vertices in P_i are adjacent to x_i. If $x \in P_j$ for some $j \neq i$, then condition B(1) is satisfied by definition. Finally, if $x = x_j \in D_1$ for some $j \neq i$, then $d_P(p, P_j) \geq 2$ and so $\overline{P}$ satisfies condition B(2). □

In Section 5 we determine when $\text{cov}_A(P) = \text{cov}_{AB}(P)$.

4. When $C \neq K_n$

For this section assume that C is not a complete graph and so $\text{diam}(C) \geq 2$. The results in this section mirror those of when $C = K_n$, however conditions on the coverings of P, as well as the proofs, are more technical.

Proposition 4.1. *For a given pair of graphs* (C, P) , $2\kappa \leq A_{\text{ucg}}(C, P) \leq 2\kappa + 2$ *where* $\kappa = \text{cov}_A(P)$.

Proof. For a given pair of graphs C and P, suppose H is a UCG with $\langle \mathcal{Z}(H) \rangle = C$ and $\langle \mathcal{CP}(H) \rangle = P$. Then $r(H) \geq \text{diam}(C) + 1 \geq 3$. From corollary 2.2 we obtain a lower bound $A_{\text{ucg}}(C, P) \geq 2\kappa$.

For the upper bound consider $G = \mathcal{G}(C, P, \overline{P}, 2)$ where $\overline{P}$ is a smallest covering of P satisfying condition A. By Proposition 2.3, G is a UCG with $|\mathcal{I}(G)| = 2\kappa + 2$ and so $A_{\text{ucg}}(C, P) \leq 2\kappa + 2$. □

In the proof of Proposition 4.1 we use the fact that $\mathcal{G}(C, P, \overline{P}, 2)$ is a UCG so long as $\overline{P}$ satisfies condition A. We now consider a modification of this graph, and determine conditions that this new graph is a UCG.

Let $\overline{P} = \{P_1, \ldots, P_k\}$ be a covering of P and let

$$G = \mathcal{G}(C, P, \overline{P}, 2) - \{x_{0,1}, x_{0,2}\}.$$

By construction, $d(c,p) = 3$ for all $c \in V(C)$, $p \in V(P)$, and $d(c,x) \leq 2$ for $x \in V(G) - V(P)$. So G is a UCG with center C and centered periphery P if and only if for all $u \in V(G) - V(C)$, $e(u) \geq 4$.

Consider the case $u = x_{i,1}$. If G is a UCG, $\overline{P}$ must satisfy condition A. This implies $e(x_{i,1}) \geq 4$ for all $1 \leq i \leq k$.

Next, consider the case $u = x_{i,2}$ for some $1 \leq i \leq k$. Since $e(u) \geq 4$ there exists a $v \in V(G)$ such that $d(u,v) \geq 4$. Because $v \notin V(C)$ and $d(u, x_{j,1}) \leq 3$, either $v \in V(P)$ or $v = x_{j,2}$ for some $j \neq i$. If $v \in V(P)$ then $v \notin P_i$ and $d(v, P_i) \geq 3$. If $v = x_{j,2}$, then $d(P_i, P_j) \geq 2$.

Finally, consider the case $u \in P_i$ for some i. If G is a UCG, then $e(u) \geq 4$, and so there is a $v \in V(G)$ such that $d(u,v) \geq 4$. Since $v \notin V(C)$ and $v \notin P_i$ because $d(p,p') \leq 2$ for all $p, p' \in P_i$, it follows that either $v = x_{j,1}$, $v = x_{j,2}$ or $v \in P_j$ for some $j \neq i$. If $v = x_{j,1}$ then $d(u, P_j) \geq 2$ for $j \neq i$. If $v = x_{j,2}$ then $d(u, P_j) \geq 3$. Finally, if $v \in P_j$ for some $j \neq i$, then for all $p \in P_j$

$$4 \leq d(u,v) \leq d(u,p) + d(p,v) \leq d(u,p) + 2.$$

We conclude $d(u, P_j) \geq 2$. This analysis gives rise to two new conditions on $\overline{P}$.

Let $\overline{P} = \{P_1, \ldots, P_k\}$ be a covering of a graph P. If for all $1 \leq i \leq k$ we say $(P, \overline{P})$ satisfies

Condition A′ if either

1. there is a $p \notin P_i$ satisfying $d(P_i, p) \geq 3$, or
2. there is a $j \neq i$ satisfying $d(P_i, P_j) \geq 2$.

Condition B′ if for each $p \in P_i$ there is a $j \neq i$ satisfying $d(p, P_j) \geq 2$.

Condition A′ arises from $e(x_{i,2}) \geq 4$ and condition B′ from $e(p) > 4$, for $p \in V(P)$. Note condition A′ implies condition A, and condition B′ implies condition B. We often abuse notation and say the covering $\overline{P}$ satisfies a specified condition.

Let $\mathrm{cov}_{A'}(P)$ be the smallest size of the covering of P satisfying condition A′ and $\mathrm{cov}_{A'B'}(P)$ the smallest size of the covering of P satisfying conditions A′ and B′.

The arguments used to determine conditions A′ and B′ are bi-directional. This implies the following proposition.

Proposition 4.2. *Let C and P be graphs, $\overline{P}$ a covering of P, and $G = \mathcal{G}(C, P, \overline{P}, 2) - \{x_{0,1}, x_{0,2}\}$. Then G is a UCG with center C and centered periphery P if and only if $\overline{P}$ satisfies conditions A′ and B′.*

We are now ready to relate appendage numbers to coverings.

Proposition 4.3. *Let $\kappa = \mathrm{cov}_A(P)$. For a graph P with $r(P) > 1$, $\mathrm{cov}_{A'B'}(P) = \kappa$ if and only if $A_{\mathrm{ucg}}(C, P) = 2\kappa$.*

Proof. First, assume $\mathrm{cov}_{A'B'}(P) = \kappa$. Let $\overline{P} = \{P_1, \ldots, P_\kappa\}$ be a covering of P satisfying conditions A′ and B′. Then by Proposition 4.2 the graph

$$G = \mathcal{G}(C, P, \overline{P}, 2) - \{x_{0,1}, x_{0,2}\}$$

is a UCG and $|\mathcal{I}(G)| = 2\kappa$ and so $A_{\text{ucg}}(C, P) \leq 2\kappa$. However, by Proposition 4.1, $A_{\text{ucg}}(C, P) \geq 2\kappa$ and the result follows.

Next, assume $A_{\text{ucg}}(C, P) = 2\kappa$. Let H be a UCG with $P = \langle \mathcal{CP}(H) \rangle$, $C = \langle \mathcal{Z}(H) \rangle$ and $|\mathcal{I}(H)| = 2\kappa$. Note that $r(H) = 3$ by corollary 2.4.

Let $\{P_1, \ldots, P_k\}$ be the induced covering of P as in Lemma 2.2. There exists a subcover $\overline{P} = \{P_1, \ldots, P_{k''}\}$ and an associated set of vertices $\{\tilde{p}_1, \ldots, \tilde{p}_{k''}\}$ as in Lemma 2.1. Assume $D_1'' = \{x_1, \ldots, x_{k''}\}$, and by corollary 2.3 $|D_2''| \geq k''$. Because $\kappa = \text{cov}_A(P)$ and $\overline{P}$ satisfies condition A, we know

$$2\kappa \leq 2k'' \leq |D_1''| + |D_2''| \leq |\mathcal{I}(H)| = 2\kappa,$$

and so $k'' = |D_1''| = |D_2''| = \kappa$. This implies $D_1 = D_1'', D_2 = D_2''$ and $\kappa = k'' = k$.

We now show H contains a spanning subgraph isomorphic to

$$G = \mathcal{G}(C, P, \overline{P}, 2) - \{x_{0,1}, x_{0,2}\}.$$

First, for each $\tilde{p}_i$ defined above and each central vertex c there is a $(c, \tilde{p}_i)$-radial path. By construction of $\tilde{p}_i$ this path must contain x_i, and thus x_i is adjacent to every c in the center by Lemma 2.4.

By Lemma 2.1 there is an enumeration of $D_2 = \{y_1, \ldots, y_k\}$ so that y_i is adjacent to both x_i and each vertex in P_i. Therefore, $G = \mathcal{G}(C, P, \overline{P}, 2) - \{x_{0,1}, x_{0,2}\}$ is isomorphic to a spanning subgraph of H. By Lemma 2.5, G is a UCG and by Proposition 4.2, $\overline{P}$ satisfies conditions A$'$ and B$'$.

□

We now move on to understand when $A_{\text{ucg}}(C, P) = 2\kappa + 1$. Let H be a UCG such that $\langle \mathcal{Z}(H) \rangle = C$, $\langle \mathcal{CP}(H) \rangle = P$ and $|\mathcal{I}(H)| = 2\kappa + 1$. By Proposition 4.3, $\kappa \neq \text{cov}_{A'B'}(P)$. Also, by corollary 2.4 $r(H) = 3$, and hence $|D_1| + |D_2| = 2\kappa + 1$. Let $\{P_1, \ldots, P_k\}$ be the induced covering of P through $D_1 = \{x_1, \ldots, x_k\}$ as in Lemma 2.2. Without loss of generality let $\overline{P} = \{P_1, \ldots, P_{k''}\}$, with $k'' \leq k$, be a subcover with an associated set of vertices $\{\tilde{p}_1, \ldots, \tilde{p}_{k''}\}$ as in Lemma 2.1. Then

$$2\kappa \leq 2k'' \leq |D_1''| + |D_2''| \leq |\mathcal{I}(H)| = 2\kappa + 1$$

and hence $\kappa = k''$.

Since

$$\kappa = |D_1''| \leq |D_1| \text{ and } \kappa = |D_1''| \leq |D_2''| \leq |D_2|,$$

either $|D_1| = \kappa + 1$ or $|D_2| = \kappa + 1$.

We first address when $|D_1| = \kappa + 1$ and $|D_2| = \kappa$.

Proposition 4.4. *Assume $A_{\text{ucg}}(C, P) = 2\kappa + 1$, and let H be a UCG such that $\langle \mathcal{Z}(H) \rangle = C$, $\langle \mathcal{CP}(H) \rangle = P$, and $|\mathcal{I}(H)| = 2\kappa + 1$ where $\kappa = \text{cov}_A(P)$. Furthermore, assume $|D_1| = \kappa + 1$. Then $\text{cov}_{A'}(P) = \kappa$.*

Proof. By corollary 2.3 it follows that

$$\kappa = |D_1''| \leq |D_2''| \leq |D_2| = \kappa.$$

Therefore $D_2 = D_2''$ and $|D_1''| = \kappa$. Since $k = |D_1| = \kappa + 1$, $x_k \in D_1$ but $x_k \notin D_1''$.

Next, we prove $x_k \notin D_1'$ by showing if $x_k \in D_1'$, then $\overline{P}$ satisfies both conditions A$'$ and B$'$. Hence $A_{\text{ucg}}(C, P) = 2\kappa$ by Proposition 4.3.

Assume $x_k \in D_1'$, that is $D_1 = D_1'$. Because $|D_2| = |D_2''| = \kappa = |D_1''|$ we may assume $D_2 = \{y_1, \ldots, y_\kappa\}$ such that each y_i is adjacent to x_i and all vertices of P_i by Lemma 2.3. To understand the structure of P_k, define the indexing set

$$I = \{i : x_k \text{ is adjacent to } y_i \text{ for some } y_i \in D_2\}.$$

From the definition of D_1' and I,

$$P_k = \bigcup_{i \in I} P_i.$$

For each $j \notin I$ every radial path to $\tilde{p}_j$ must contain x_j. Hence, each central vertex $c \in V(C)$ is adjacent to $x_j \in D_1$ for $j \notin I$. A similar argument shows that for each $c \in V(C)$ and each $i \in I$, c is adjacent to either x_k or x_i.

We now show $\overline{P}$ satisfies conditions A$'$ and B$'$.

First, assume condition B$'$ fails. Then there exists an ι and a vertex $p \in P_\iota$ such that $d_P(p, P_j) \leq 1$ for all $j \neq \iota$. Therefore, there is a $p_j \in P_j$ such that $d(p, p_j) \leq 1$, and so the following hold:

i) $d(p, x_j) \leq d(p, p_j) + d(p_j, x_j) \leq 3$.

ii) $d(p, y_j) \leq d(p, p_j) + d(p_j, y_j) \leq 2$.

iii) $d(p, p') \leq d(p, p_j) + d(p_j, p') \leq 3$ for all $p' \in P_j$.

iv) $d(p, x_\iota) = 2$.

v) $d(p, y_\iota) = 1$.

vi) $d(c, p) = 3$ for $c \in V(C)$.

If $\iota \in I$, then $d(p, x_k) = 2$. If $\iota \notin I$, then for a $j \in I$

$$d(p, x_k) \leq d(p, p_j) + d(p_j, x_k) \leq 3$$

since $d(p_j, x_k) = 2$. Therefore $e(p) = 3$, a contradiction, and so $\overline{P}$ satisifies condition B$'$.

Next, assume condition A$'$ fails. Then there exists an ι such that $d_P(P_\iota, p') \leq 2$ for all $p' \in V(P) - P_\iota$ and $d_P(P_\iota, P_j) \leq 1$ for all $j \neq \iota$. Then $d(y_\iota, p) \leq 3$ for all $p \in V(P)$ and $d(y_\iota, y_j) \leq 3$ for all $j \neq \iota$. We obtain a contradiction by showing y_ι is in the center.

If $\iota \notin I$, then for each $x_j \in D_1$ there is a $c \in V(C)$ that is adjacent to x_j. Then $y_\iota - x_\iota - c - x_j$ is a path and $d(y_\iota, x_j) \leq 3$. Similarly when $\iota \in I$ and $j \notin I$, $d(y_\iota, x_j) \leq 3$. If $\iota, j \in I$ then

$y_\iota - x_k - y_j - x_j$ is a path and so $d(y_\iota, x_j) \leq 3$. Finally, $d(y_\iota, c) = 2$ for all $c \in V(C)$, and so $e(y_\iota) = 3$, a contradiction and $\overline{P}$ satisfies condition A′.

Since $\overline{P}$ satisfies conditions A′ and B′, $A_{\text{ucg}}(C, P) = 2\kappa$ by Proposition 4.3. This contradicts the assumptions of the proposition and hence $x_k \notin D_1'$ and so $D_1' = D_1''$.

Then x_k is not on a radial path because $x_k \notin D_1'$. Therefore, vertices adjacent to x_k are in C or D_1''. Furthermore each $y_i \in D_2$ satisfies $e(y_i) \geq 4$ from H being a UCG with $r(H) = 3$. Therefore, there exists a $u \in V(G)$ with $d(y_i, u) \geq 4$. Because $d(c, y_i) = 2$ for all $c \in V(C)$, we know $u \notin V(C)$.

Because $D_1' = D_1''$, each $(c, \tilde{p}_i)$-radial path contains x_i, and hence each c is adjacent to x_i. It follows that for a $c \in V(C)$ adjacent to $x_j \in D_1$ that

$$d(y_i, x_j) \leq d(y_i, c) + d(c, x_j) = 3,$$

which means $u \notin D_1$.

If $u \in V(P)$ then $d(u, P_i) \geq 3$ and condition A′(1) is satisfied for i. If $u = y_j \in D_2''$ then $d_P(P_i, P_j) \geq 2$ and condition A′(2) is satisfied for i. Since these hold for each i, $\overline{P}$ satisfies condition A′ and $\text{cov}_{A'}(P) = \kappa$. □

A weak converse of Proposition 4.4 also holds.

Proposition 4.5. *If P is a graph with $r(P) > 1$, $\kappa = \text{cov}_A(P) \neq \text{cov}_{A'B'}(P)$ and $\text{cov}_{A'}(P) = \text{cov}_A(P)$, then $A_{\text{ucg}}(C, P) = 2\kappa + 1$.*

Proof. Assume $\text{cov}_{A'B'}(P) \neq \kappa$ but $\text{cov}_{A'}(P) = \kappa$. Let $\overline{P} = \{P_1, \dots, P_\kappa\}$ be a smallest covering with respect to condition A′ and

$$G = \mathcal{G}(C, P, \overline{P}, 2) - \{x_{0,2}\}.$$

We claim G is a UCG with $C = \langle \mathcal{Z}(G) \rangle$, $P = \langle \mathcal{CP}(G) \rangle$ and $|\mathcal{I}(G)| = 2\kappa + 1$.

By Proposition 4.2 the graph $G' = G - \{x_{0,1}\}$ is a UCG if and only if $\overline{P}$ satisfies conditions A′ and B′. In G', if $\overline{P}$ satisfies conditions A′, then $e(x_{i,1}) \geq 4$ and $e(x_{i,2}) \geq 4$. This still holds in G. Furthermore, for each vertex $p \in V(P)$, $d(x_{0,1}, p) = 4$ in G, and so G is a UCG. Therefore, $A_{\text{ucg}}(C, P) \leq 2\kappa + 1$, but by Proposition 4.3 $A_{\text{ucg}}(C, P) > 2\kappa$. □

The case when $|D_2| = \kappa + 1$ is more complicated. To understand this case we consider a new graph G, and determine new conditions for when G is a UCG.

For two graphs C and P, let $\overline{P} = \{P_1, \dots, P_k\}$ be a covering of P, and $\overline{Q} = \{Q_0, Q_1, P_2, \dots, P_k\}$ be a covering such that $Q_0 \cup Q_1 = P_1$.

Define a graph $G = \mathcal{G}'(C, P, \overline{Q})$ as follows (see Figure 3):

$$V(G) = V(C) \cup V(P) \cup \{x_i, y_j :\ 1 \leq i \leq k, 0 \leq j \leq k\}$$

and ab is an edge in G if and only if one of the following occurs

i) ab is an edge of C.

ii) ab is an edge of P.

iii) a is a vertex of C and $b = x_i$ for some i with $1 \leq i \leq k$.

iv) $a \in P_j$ and $b = y_j$ for some j with $2 \leq j \leq k$.

v) $a \in Q_l$ and $b = y_l$ for some $l = 0, 1$.

vi) $a = x_i$ and $b = y_i$ for $1 \leq i \leq k$

vii) $a = x_1$ and $b = y_0$.

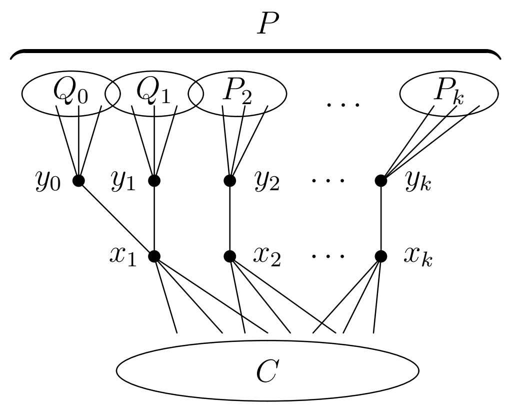

Figure 3: $G = \mathcal{G}'(C, P, \overline{Q})$

We now determine conditions on $\overline{Q}$ for G to be a UCG with $C = \langle \mathcal{Z}(G) \rangle$ and $P = \langle \mathcal{CP}(G) \rangle$. By construction for all $c \in V(C)$ and $p \in V(P)$, $d(c, p) = 3$, and $d(c, x) \leq 2$ for all $x \in V(G) - V(P)$. So G is a UCG with center C and centered periphery P if and only if for all $u \in V(G) - V(C)$, $e(u) \geq 4$. Then there exists a vertex v with $d(u, v) \geq 4$.

First consider the case $u = x_i$. In every UCG the induced covering satisfies condition A. This implies $v \in V(P)$ and $e(x_i) \geq 4$ for all $1 \leq i \leq k$. Note that, in G the induced covering is $\overline{P}$, not $\overline{Q}$.

Next, consider the case $u = y_i$ for some $2 \leq i \leq k$. Since $v \notin V(C)$ and $d(v, x_j) \leq 3$ for $1 \leq j \leq k$, either $v \in V(P)$ or $v = y_j$ for $0 \leq j \leq k$ and $j \neq i$. The one of the following holds.

i) If $v \in V(P)$ then $v \notin P_i$ and $d(P_i, v) \geq 3$.

ii) If $v = y_j$, $2 \leq j \leq k$ and $j \neq i$, then $d(P_i, P_j) \geq 2$.

iii) If $v = y_l$ for $l = 0$ or 1, then $d(P_i, Q_l) \geq 2$.

Now, consider the case $u = y_0$ or y_1. Without loss of generality we assume $u = y_0$. Since $v \notin V(C)$, $v \neq x_j$ for $1 \leq j \leq k$, $v \neq y_1$ and $v \notin Q_0 \cup Q_1 = P_1$, the following must hold.

i) If $v \in V(P)$ then $v \notin P_1$ and $d(Q_0, v) \geq 3$.

ii) If $v = y_j$, $2 \leq j \leq k$ and $j \neq i$, then $d(Q_0, P_j) \geq 2$.

Now, consider the case $u = p \in P_i$ for some $2 \leq i \leq k$. Once again, if G is a UCG, then $e(p) \geq 4$, and so there exists a $v \in V(G)$ such that $d(u,v) \geq 4$. We know $v \notin V(C)$. Also, $v \notin P_i$ since $d(p,p') \leq 2$ for all $p' \in P_i$. Also note, that for $j \geq 2$ and $j \neq i$, if $d(u, y_j) \geq 4$ then $d(u, x_j) \geq 4$. This implies that we do not need to determine the conditions for $d(u, y_j) \geq 4$. Given this, one of the following must hold.

i) If $v = x_j$ for $j \neq i$ and $2 \leq j \leq k$, then $d(u, P_j) \geq 2$.

ii) If $v = x_1$, then $d(u, Q_0) \geq 2$ and $d(u, Q_1) \geq 2$.

iii) If $v = y_l$ for $l = 0$ or 1, then $d(u, Q_l) \geq 3$.

iv) If $v = p' \in P_j - P_i$ for $2 \leq j \leq k$ and $j \neq i$, then $d(u,p') \geq 4$ and $d(u, P_j) \geq 2$.

v) If $v = p' \in Q_l - P_i$ for $l = 0$ or 1, then $d(p, Q_l) \geq 2$ and $d(u,p') \geq 4$.

The last case to consider, without loss of generality, is $u = p \in Q_0$. Since $d(u,v) \leq 3$ if $v \in V(C)$, $v = x_1, y_0$ or y_1 or $v \in Q_0$, the following must hold.

i) If $v = x_j$, then $2 \leq j \leq k$ and $d(u, P_j) \geq 2$.

ii) If $v = y_j$ for $2 \leq j \leq k$, then $d(u, P_j) \geq 3$.

iii) If $v = p' \in P_j - Q_0$ for $2 \leq j \leq k$, then $d(u, P_j) \geq 2$ and $d(u,p') \geq 4$.

iv) If $v = p' \in Q_1 - Q_0$, then $d(u,p') \geq 4$ and $d(u, Q_1) \geq 2$.

The above discussion is in terms of $\overline{Q}$, however the rest of the paper is in terms of $\overline{P}$. For this reason now summarize the discussion in terms of two technical conditions on $\overline{P}$.

Let $\overline{P} = \{P_1, \ldots, P_k\}$ be a covering of a graph P. For a given ι and two sets Q_0 and Q_1 such that $Q_0 \cup Q_1 = P_\iota$, let

$$\overline{Q} = \overline{P} \cup \{Q_0, Q_1\} - \{P_\iota\}.$$

We say $(P, \overline{P}, \overline{Q})$ satisfies

Condition A$''$

1. if for each $i \neq \iota$ one of the following holds
   (a) there is a $p \notin P_i$ satisfying $d(P_i, p) \geq 3$, or
   (b) there is a $j \neq \iota$ satisfying $d(P_i, P_j) \geq 2$, or
   (c) there is an $l = 0$ or 1 such that $d(P_i, Q_l) \geq 2$,
2. and if for all $l = 0, 1$ either
   (a) there is a $p \notin P_\iota$ satisfying $d(Q_l, p) \geq 3$, or
   (b) there is a $j \neq \iota$ satisfying $d(Q_l, P_j) \geq 2$.

Condition B$''$

1. if for each $p \in P_i$, $i \neq \iota$ one of the following holds
   (a) there is a $j \neq \iota$ satisfying $d(p, P_j) \geq 2$, or
   (b) $d(p, Q_0) \geq 2$ and $d(p, Q_1) \geq 2$, or
   (c) there is an $l = 0$ or 1 such that $d(p, Q_l) \geq 3$, or
   (d) there $l = 0$ or 1 such that $d(p, Q_l) \geq 2$ and a $p' \in Q_l$ so that $d(p, p') \geq 4$.
2. and if for each $l = 0, 1$ and each $p \in Q_l$, either
   (a) there is a $j \neq \iota$ satisfying $d(p, P_j) \geq 2$, or
   (b) there exists $p' \in P_\iota - Q_l$ such that $d(p, p') \geq 4$ and $d(p, Q_{l'}) \geq 2$, where $l' = 0$ or 1 but $l' \neq l$.

When there exists an ι, Q_0 and Q_1 such that $(P, \overline{P}, \overline{Q})$ satisfies condition A$''$ (*resp.* condition B$''$), we say $(P, \overline{P})$ or simply $\overline{P}$ satisfies condition A$''$ (condition B$''$). Without loss of generality we may renumber $\overline{P}$ so that $\iota = 1$. Note that A$'$ and B$'$ imply A$''$ and B$''$ by taking $\iota = 1$ and letting $Q_0 = P_1$ and $Q_1 = \emptyset$. However, condition A$''$ does not imply condition A. So let $\text{cov}_{AA''B''}(P)$ be the smallest size of the covering $\overline{P}$ of P satisfying conditions A, A$''$ and B$''$.

Similar to the discussion of conditions A$'$ and B$'$, the arguments used to determine conditions A$''$ and B$''$ from the graph $\mathcal{G}(C, P, \overline{Q})$ are bi-directional. We summarize the discussion in the following proposition.

Proposition 4.6. *For graphs C and P, and a triple $(P, \overline{P}, \overline{Q})$, $G = \mathcal{G}'(C, P, \overline{Q})$ is a UCG if and only if $\overline{P}$ satisfies condition A and $(P, \overline{P}, \overline{Q})$ satisfies conditions A$''$ and B$''$.*

We are now ready to prove analogous results to Propositions 4.4 and 4.5.

Proposition 4.7. *Assume $A_{\text{ucg}}(C, P) = 2\kappa + 1$, and let H be a UCG with $\langle \mathcal{Z}(H) \rangle = C$, $\langle \mathcal{CP}(H) \rangle = P$, and $|\mathcal{I}(H)| = 2\kappa + 1$ where $\kappa = \text{cov}_A(P)$. Furthermore, assume $|D_2| = \kappa + 1$. Then $\text{cov}_{AA''B''}(P) = \kappa$.*

Proof. Let H be a UCG with $C = \langle \mathcal{Z}(H) \rangle$, $P = \langle \mathcal{CP}(H) \rangle$, $|\mathcal{I}(H)| = 2\kappa + 1$ and $|D_2| = \kappa + 1$. Since $|D_1| = \kappa$ and $\kappa \leq |D_1''| \leq |D_1|$, it follows that $|D_1| = |D_1''|$. We prove this proposition by studying the structure of a spanning subgraph.

Let $D_1 = \{x_1, \ldots, x_\kappa\}$ and $\{\tilde{p}_1, \ldots, \tilde{p}_\kappa\}$ be a set of vertices associated to the induced cover $\overline{P}$. By Proposition 2.1 there exists an enumeration $\{y_0, \ldots, y_\kappa\}$ of D_2 such that the vertex y_j is adjacent to x_j and $\tilde{p}_j$ for each j, $1 \leq j \leq \kappa$. We may also assume y_0 is adjacent to x_1.

We now define a different cover of P. For each i, $0 \leq i \leq \kappa$, let

$$Q_i = \{p \in V(P) : p \text{ is adjacent to } y_i\}.$$

Let $P_1' = Q_0 \cup Q_1$ and $P_i' = Q_i$ for $2 \leq i \leq \kappa$, $\overline{P}' = \{P_1', \ldots, P_\kappa'\}$ and $\overline{Q} = \{Q_0, \ldots, Q_\kappa\}$. Since $|D_1| = |D_1''|$ every vertex of C is adjacent to vertex in D_1 by Lemma 2.4. Therefore,

$G = \mathcal{G}'(C, P, \overline{Q})$ is isomorphic to a spanning subgraph of H, and is a UCG by Lemma 2.5. By Proposition 4.6 $\overline{P}'$ satisfies conditions A, A$''$ and B$''$ which means

$$\kappa \leq \text{cov}_{AA''B''}(P) \leq |\overline{P}'| = \kappa.$$

We conclude $\text{cov}_{AA''B''}(P) = \kappa$.

□

Proposition 4.8. *If P is a graph with $r(P) > 1$, $\kappa = \text{cov}_A(P) \neq \text{cov}_{A'B'}(P)$ and $\text{cov}_{AA''B''}(P) = \text{cov}_A(P)$, then $A_{\text{ucg}}(C, P) = 2\kappa + 1$.*

Proof. Assume $\text{cov}_{A'B'}(P) \neq \kappa$ but $\text{cov}_{AA''B''}(P) = \kappa$. By Proposition 4.3 $A_{\text{ucg}}(C, P) > 2\kappa$, so we need to show $A_{\text{ucg}}(C, P) \leq 2\kappa + 1$. Let $\overline{P}$ be a smallest covering of P with respect to condition A such that there is a refined cover $\overline{Q}$ where the pair $(\overline{P}, \overline{Q})$ is smallest with respect to conditions A$''$ and B$''$. Without loss of generality we may assume $\iota = 1$. By Proposition 4.6 $G = \mathcal{G}(C, P, \overline{Q})$ is a UCG with $C = \langle \mathcal{Z}(G) \rangle$, $P = \langle \mathcal{CP}(G) \rangle$ and $|\mathcal{I}(G)| = 2\kappa + 1$, which implies $A_{\text{ucg}}(C, P) \leq 2\kappa + 1$. □

The following theorem summarizes Propositions 4.1, 4.3, 4.4, 4.5, 4.7, and 4.8.

Theorem 4.1. *If P is a graph with $r(P) > 1$ and $\text{cov}_A(P) = \kappa$, then following holds:*

1. $A_{\text{ucg}}(C, P) = 2\kappa$ *if and only if* $\text{cov}_{A'B'}(P) = \kappa$.
2. $A_{\text{ucg}}(C, P) = 2\kappa + 1$ *if and only if* $\kappa \neq \text{cov}_{A'B'}(P)$ *and either*
   (a) $\text{cov}_{A'}(P) = \kappa$ *or*
   (b) $\text{cov}_{AA''B''}(P) = \kappa$.
3. $A_{\text{ucg}}(C, P) = 2\kappa + 2$ *otherwise.*

5. Coverings

In this section we determine when $\kappa = \text{cov}_A(P)$ is the size of smallest coverings with respect to the other conditions described in Sections 3 and 4. To do this we introduce one last set of notation. For a graph G, a vertex V in G and $s \in \mathbb{N}$ let

$$N_s[v] = \{x \in V(G) | d(v, x) \leq s\}$$

be the closed s-neighborhood of v. When $s = 1$ we simply let $N_1[x] = N[x]$.

Proposition 3.1 and Theorem 3.1 determine $A_{\text{ucg}}(K_n, P)$ up to knowing when $\text{cov}_A(P) = \text{cov}_{AB}(P)$. We now determine conditions for a graph P to satisfy $\text{cov}_A(P) = \text{cov}_{AB}(P)$.

Proposition 5.1. *If P is a graph with $\text{diam}(P) \geq 3$, then $\text{cov}_A(P) = 2$.*

Proof. Suppose x is a vertex in P satisfying $e(x) \geq 3$. Let $P_1 = N[x]$ and $P_2 = V(P) - P_1$. Then $d(x, P_2) \geq 2$. Since $e(x) \geq 3$, P_2 contains a vertex y satisfying $d(x, y) \geq 3$, and so $d(y, P_1) \geq 2$. Thus, $\{P_1, P_2\}$ is a covering of P satisfying condition A. □

Proposition 5.2. *If P is a graph satisfying* $\operatorname{diam}(P) \geq 4$ *and* $r(P) \geq 3$*, then* $\operatorname{cov}_{AB}(P) = 2$.

Proof. Let x and y be vertices in P satisfying $d(x, y) = 4$. We construct P_1 and P_2 recursively. Initialize $P_1 := N[x]$ and $P_2 := N[y]$. Note that $d(x, N[y]) = d(y, N[x]) = 3$. For a vertex z in $V(P) - (P_1 \cup P_2)$, update P_1 and P_2 as follows.

i) If there is a vertex $p \in P_2$ satisfying $d(z, p) \geq 3$, then let $P_1 := P_1 \cup \{z\}$.

ii) Else if there is a vertex $p \in P_1$ satisfying $d(z, p) \geq 3$, then let $P_2 := P_2 \cup \{z\}$.

iii) Else there is a vertex $p \in V(P) - (P_1 \cup P_2)$ satisfying $d(z, p) = 3$, since $r(P) = 3$. Let $P_1 := P_1 \cup \{z\}$ and $P_2 := P_2 \cup \{p\}$.

Continue until all vertices of P have been accounted for. By construction $\{P_1, P_2\}$ is a covering of P satisfying condition B. Since $N[x] \subset P_1$, $d(x, P_2) \geq 2$ and similarly $d(y, P_1) \geq 2$. Thus $\{P_1, P_2\}$ satisfies condition A. □

Proposition 5.3. *If P is a graph with $r(P) = 2$, then* $\operatorname{cov}_{AB}(P) \neq 2$.

Proof. Suppose there is a covering $\{P_1, P_2\}$ of P satisfying conditions A and B. Assume P_1 contains a vertex z with $e(z) = 2$. By condition B, $d(z, P_2) = 2$ and so $N(z) \cap P_2 = \emptyset$. If p is a vertex in P_2, then $d(z, p) = 2$ and so $d(P_1, p) = 1$, a contradiction to condition A.

□

Proposition 5.4. *For every $\alpha \in \mathbb{N}$, there exists a graph P with $r(P) = \operatorname{diam}(P) = 2$ and* $\operatorname{cov}_A(P) = 2\alpha$.

Proof. Construct $P^\alpha = (V^\alpha, E^\alpha)$ as follows. Let

$$V^\alpha = \{e_i, f_i : 1 \leq i \leq \alpha\}$$

and

$$E^\alpha = \{e_i e_j, f_i f_j, e_i f_j : 1 \leq i, j \leq \alpha, i \neq j\}.$$

Note, P^α is a complete graph minus a perfect matching. It follows that $r(P^\alpha) = \operatorname{diam}(P^\alpha) = 2$.

We now show $\operatorname{cov}_A(P^\alpha) = 2\alpha$. By Lemma 3.1 there exists a covering

$$\{P_1^\alpha, P_2^\alpha, \ldots, P_k^\alpha\}$$

of P^α satisfying condition A. Suppose $e_1 \in P_1^\alpha$. If P_1^α contains a vertex other than e_1, then $d(P_1^\alpha, x) = 1$ for all $x \in V^\alpha - P_1^\alpha$, a contradiction to condition A. Therefore $P_1^\alpha = \{e_1\}$. Similarly $|P_i^\alpha| = 1$ for all $i = 1, 2, \ldots, k$ and $\operatorname{cov}_A(P^\alpha) = 2\alpha$. □

Propositions 5.1 through 5.3 determine when $\operatorname{cov}_A(P) = \operatorname{cov}_{AB}(P)$ for all graphs P with $r(P) > 1$ except those with $r(P) = \operatorname{diam}(P) = 2$ and those with $r(P) = \operatorname{diam}(P) = 3$. Proposition 5.4 gives insight into richness of the case $r(P) = \operatorname{diam}(P) = 2$. However, we do not have any definitive results for $r(P) = \operatorname{diam}(P) = 3$. This is discussed further in Section 6.

For a non-complete graph C, Theorem 4.1 relates $A_{\text{ucg}}(C, P)$ to $\operatorname{cov}_{A'B'}(P)$, $\operatorname{cov}_{A'}(P)$, and $\operatorname{cov}_{AA''B''}(P)$. Because of Proposition 5.4, we only consider P with $\operatorname{diam}(P) > 2$. By Proposition 5.1 we need to understand when the smallest coverings are of size 2.

Proposition 5.5. *P is a graph with* $\mathrm{cov}_{A'B'}(P) = 2$ *if and only if P is disconnected.*

Proof. Assume P is disconnected. Let P_1 be the vertices of a connected component of P and let $P_2 = V(P) - P_1$. Since, for all $u \in P_1$ and $v \in P_2$, $d(u, v) = \infty$ it follows that $\{P_1, P_2\}$ satisfies conditions A′ and B′.

Next, assume P is connected but $\overline{P} = \{P_1, P_2\}$ is a covering. Since P is connected, $d(P_1, P_2) = 1$ and there is a $p_1 \in P_1$ and $p_2 \in P_2$ with $d(p_1, p_2) = 1$. Therefore $d(p_1, P_2) = 1$ and $d(p_2, P_1) = 1$ and condition B′ fails, and so $\mathrm{cov}_{A'B'}(P) > 2$. □

Proposition 5.6. *For a graph P,* $\mathrm{cov}_{A'}(P) = 2$ *if and only if* $\mathrm{diam}(P) \geq 5$.

Proof. Assume $\mathrm{diam}(P) \geq 5$. Then there exist $u, v \in V(P)$ with $d(u, v) \geq 5$. Let $P_1 = \{p \in V(P) : d(u, p) \leq 2\}$ and $P_2 = V(P) - P_1$. Then $d(u, P_2) \geq 3$. Also $d(v, P_1) \geq 3$ because if there is a $p \in P_1$ with $d(v, p) \leq 2$, then $d(u, v) \leq d(u, p) + d(p, v) \leq 4$, a contradiction. So $\{P_1, P_2\}$ satisfies condition A′.

Next, assume there is a covering $\{P_1, P_2\}$ of P satisfying condition A′. Then there is a $u \in P_1$ such that $d(u, P_2) \geq 3$. Then $N_2[u] \subset P_1$ and $N_2[u] \cap P_2 = \emptyset$. Similarly, P_2 contains a vertex v such that $d(v, P_1) \geq 3$, and so $N_2[v] \subset P_1$ and $N_2[v] \cap P_1 = \emptyset$. Therefore $d(u, v) \geq 5$. □

Proposition 5.7. *If P is a graph with* $\mathrm{diam}(P) \leq 3$*, then* $\mathrm{cov}_{AA''B''}(P) \neq 2$.

Proof. Let P be a graph with $\mathrm{diam}(P) \leq 3$. Suppose there is a covering $\overline{P} = \{P_1, P_2\}$ of P satisfying condition A such that for $\iota = 1$ the triple $(P, \overline{P}, \overline{Q})$ satisfies conditions A″ and B″, where $\overline{Q} = \{Q_0, Q_1, P_2\}$ with $P_1 = Q_0 \cup Q_1$. For each $p \in P_1$, $d(p, P_2) \geq 2$ by condition B″(2) because $\mathrm{diam}(P) \leq 3$. This implies $d(P_1, P_2) \geq 2$. However, $d(P_1, P_2) \leq 1$ since P is connected, a contradiction. □

Proposition 5.8. *If P is a graph with $r(P) = 2$, then* $\mathrm{cov}_{AA''B''}(P) \neq 2$.

Proof. Let P be a graph with $r(P) = 2$. Suppose there is a covering $\overline{P} = \{P_1, P_2\}$ of P satisfying condition A such that for $\iota = 1$ the triple $(P, \overline{P}, \overline{Q})$ satisfies conditions A″ and B″, where $\overline{Q} = \{Q_0, Q_1, P_2\}$ with $P_1 = Q_0 \cup Q_1$.

For a central vertex $c \in V(P)$ either $c \in P_1$ or $c \in P_2$.

Suppose $c \in P_1$. Without loss of generality assume $c \in Q_0$. By condition B″(2), $d(c, P_2) \geq 2$. In fact, $d(c, P_2) = 2$ because c is central, and so there exists a $p \in P_2$ with $d(c, p) = 2$. Let $c - x - p$ be a geodesic. Then $x \in P_1$, $e(x) \leq 3$ and $d(x, P_2) = 1$. Therefore, $\{P_1, P_2\}$ does not satisfy condition B''-2 for x, a contradiction.

Next, suppose $c \in P_2$. By the hypotheses c does not satisfy B″(1a), B″(1c), and B″(1d). Therefore, $d(c, Q_0) \geq 2$ and $d(c, Q_1) \geq 2$. Since c is a central vertex, $d(c, Q_0) = 2$ and $d(c, Q_1) = 2$. Then there exist $q_0 \in Q_0$ and $q_1 \in Q_1$, satisfying $d(c, q_0) = 2$ and $d(c, q_1) = 2$. Let $c - x - q_0$ and $c - y - q_1$ be paths. Then both x and y are in P_2 and $d(P_2, Q_i) \leq 1$ for $i = 0, 1$. Therefore $\{P_1, P_2\}$ does not satisfy condition A″(1).

□

To fully understand $A_{\text{ucg}}(C, P)$ there are still two cases left to consider, when $\text{diam}(P) = 4$ and $r(P) = 3$, and $\text{diam}(P) = 4$ and $r(P) = 4$. In these cases $\text{cov}_{A'B'}(P) \neq 2$ and $\text{cov}_{A'}(P) \neq 2$. When $\text{diam}(P) = r(P) = 4$, $\text{cov}_{AA''B''}(P) = 2$ depends on P. To show that there exist graphs P with $\text{cov}_{AA''B''}(P) \neq 2$, we introduce the following lemma.

Lemma 5.1. *If P is a graph with $\text{diam}(P) = 4$ and $\text{cov}_{AA''B''}(P) = 2$, then there exist three vertices x_1, x_2, x_3 such that*

$$N_2[x_1] \cap N_2[x_2] \cap N_2[x_3] = \emptyset.$$

Proof. Suppose there is a covering $\overline{P} = \{P_1, P_2\}$ of P satisfying condition A such that for $\iota = 1$ the triple $(P, \overline{P}, \overline{Q})$ satisfies conditions A″ and B″, where $\overline{Q} = \{Q_0, Q_1, P_2\}$ with $P_1 = Q_0 \cup Q_1$.

Since P is connected, without loss of generality we may assume that $d(P_2, Q_1) \leq 1$. Then A″(2b) is not satisfied for $l = 1$ and so, from A″(2a), there exists a $p_1 \in P_2$ such that $d(Q_1, p_1) \geq 3$. For $l = 0$ there are two cases for condition A″(2), either part 2a is met or part 2b is met.

For $l = 0$, either there exists a $p_0 \in P_2$ such that $d(Q_0, p_0) \geq 3$ or $d(P_2, Q_0) \geq 2$ by condition A″(2b).

Suppose $d(P_2, Q_0) \geq 2$, by connectivity of P we know $d(P_2, Q_1) \leq 1$ and $d(Q_0, Q_1) \leq 1$. Since $d(p_1, Q_1) \geq 3$, $N_2[p_1] \cap Q_1 = \emptyset$ and so

$$N_2[p_2] \subset (P_2 \cup Q_0) - Q_1.$$

Since $p_1 \in P_2$ and $d(Q_0, P_2) \geq 2$, $N_1[p_1] \cap Q_0 = \emptyset$ and $N_1[p_1] \subset P_2$. From

$$2 \leq d(Q_0, P_2) \leq d(Q_0, N_1[p_1]),$$

it follows that $N_2[p_1] \subset P_2$. Therefore, $d(N_2[p_1], Q_0) \geq 2$ and $d(p_1, Q_0) \geq 4$. Since $\text{diam}(P) = 4$, $d(p_1, Q_0) = 4$.

Let $p_1 - x - y - z - q_0$ be a geodesic for some $q_0 \in Q_0$. Then $z \notin Q_0$ since $d(z, p_1) = 3$, and $z \notin P_2$ since $d(q_0, z) = 1$. Hence $z \in Q_1$. Furthermore, $y \in N_2[p_1] \subset P_2$, so $d(z, P_2) \leq 1$ and $d(z, Q_0) \leq 1$. This implies condition B″(2) is not met for $z \in Q_1$, a contradiction. Therefore $d(P_2, Q_0) \leq 1$.

Then, for $l = 0$ condition A″(2a) is satisfied and there is a $p_0 \in P_2$ such that $d(Q_0, p_0) \geq 3$. Therefore, $N_2[p_0] \cap Q_0 = \emptyset$. Since $N_2[p_1] \cap Q_1 = \emptyset$, it follows that $N_2[p_0] \cap N_2[p_1] \subset P_2$.

From condition A″(1) there exists a $q \in Q_0$ such that $d(q, P_2) \geq 3$. Then $N_2[q] \cap P_2 = \emptyset$ and

$$N_2[p_0] \cap N_2[p_1] \cap N_2[q] = \emptyset.$$

□

Proposition 5.9. *There exists a graph P with $r(P) = \text{diam}(P) = 4$ and $\text{cov}_{AA''B''}(P) \neq 2$.*

Proof. Let P be the graph of a hexagonal prism as in Figure 4. Then $r(P) = \text{diam}(P) = 4$. If $\text{cov}_{AA''B''}(P) = 2$ then by Lemma 5.1 there exist vertices x_1, x_2, x_3 such that

$$N_2[x_1] \cap N_2[x_2] \cap N_2[x_3] = \emptyset,$$

or equivalently,

$$N_2[x_1]^c \cup N_2[x_2]^c \cup N_2[x_3]^c = V(P).$$

For any vertex x of P, the complement of $N_2[x]$ is a star with three pendant vertices as in Figure 4. Because the prism has twelve vertices, a set of three stars would cover the prism with no overlap. However, one can check this is not possible and so $\text{cov}_{AA''B''}(P) \neq 2$

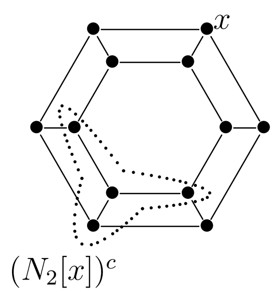

Figure 4: A hexagonal prism and the complement of a 2-neigborhood.

□

Proposition 5.10. *There exists a graph P with $r(P) = \text{diam}(P) = 4$ and $\text{cov}_{AA''B''}(P) = 2$.*

Proof. Let P be the graph of a heptagonal prism as in Figure 5. Then $r(P) = \text{diam}(P) = 4$. Consider the covering $\overline{P} = \{P_1, P_2\}$, where $P_1 = Q_0 \cup Q_1$ as in Figure 5. Here vertices of Q_0 are represented by the open circles, Q_1 by the open squares, and P_2 by the filled circles. One can verify this covering satisfies conditions A, A″ and B″, and so $\text{cov}_{AA''B''}(P) = 2$. □

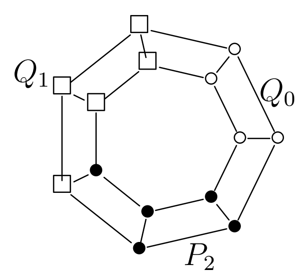

Figure 5: A graph P with $r(P) = \text{diam}(P) = 4$ and $\text{cov}_{AA''B''}(P) = 2$

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6. Appendage Numbers

In this section we determine $A_{\text{ucg}}(C, P)$ for most pairs of graphs (C, P) based on the structure of C, proving Theorems 6.1, 6.2, and 6.3, given in the introduction. Conjectures are also given for the two remaining cases.

Theorem 6.1. *Let $C = \{v\}$ and P be any graph with $r(P) \geq 2$. Then $A_{\text{ucg}}(C, P) = 0$.*

Proof. Let H be the graph with vertex set

$$V(H) = V(P) \cup \{v\}$$

and edge set

$$E(H) = E(P) \cup \{vp | p \in V(P)\}.$$

We claim H is a UCG with $\langle \mathcal{Z}(H) \rangle = C$, $\langle \mathcal{CP}(H) \rangle = P$ and $|\mathcal{I}(H)| = 0$.

Since v is adjacent to all other vertices in H, $e(v) = 1$. If p is a vertex in P there is a $p' \in V(P)$ satisfying $d_P(p, p') = 2$. Since p and p' are not adjacent in P, they are not adjacent in H and $e(p) \geq 2$. This implies $\langle \mathcal{Z}(H) \rangle = \{v\}$ and H is a UCG with $\langle \mathcal{CP}(H) \rangle = P$. □

Theorem 6.2. *When* $n \geq 2$

$$A_{\text{ucg}}(K_n, P) = \begin{cases} 2 & \textit{if } \operatorname{diam}(P) \geq 4 \textit{ and } r(P) \geq 3 \\ 3 & \textit{if } \operatorname{diam}(P) \geq 3 \textit{ and } r(P) = 2 \end{cases}$$

Furthermore, for all $t \in \mathbb{N}$ *there is a graph* P *with* $\operatorname{diam}(P) = r(P) = 2$ *and* $A_{\text{ucg}}(K_n, P) \geq t$.

Proof. This follows directly from Proposition 3.1, Theorem 3.1 and Propositions 5.1, 5.2, 5.3, and 5.4. □

Conjecture 1. *If* $\operatorname{diam}(P) = r(P) = 3$*, then* $A_{\text{ucg}}(K_n, P) = 2$*.*

When $\operatorname{diam}(P) = r(P) = 3$, $A_{\text{ucg}}(C, P) = 2$ or 3. However, all our examples show $A_{\text{ucg}}(K_n, P) = 2$, but we have not been able to prove this is always true.

Theorem 6.3. *If* C *is a graph with* $\operatorname{diam}(C) > 1$ *then*

$$A_{\text{ucg}}(C, P) = \begin{cases} 4 & \textit{if } \operatorname{diam}(P) = \infty \\ 5 & \textit{if } 5 \leq \operatorname{diam}(P) < \infty \\ 5 \textit{ or } 6 & \textit{if } \operatorname{diam}(P) = 4 \textit{ and } r(P) = 4 \\ 6 & \textit{if } \operatorname{diam}(P) = 4 \textit{ and } r(P) = 2 \\ 6 & \textit{if } \operatorname{diam}(P) = 3 \end{cases}$$

Furthermore, for all $t \in \mathbb{Z}$ *there is a graph* P *with* $\operatorname{diam}(P) = r(P) = 2$ *and* $A_{\text{ucg}}(C, P) \geq t$.

Proof. This follows directly from Theorem 4.1, Proposition 5.1 and Propositions 5.4, through 5.10. □

The only case not accounted for is when $\operatorname{diam}(P) = 4$ and $r(P) = 3$. In this case we know $\operatorname{cov}_{A'B'}(P) \neq 2$, so $A_{\text{ucg}}(C, P) = 5$ or 6. We also know $\operatorname{cov}_{A'}(P) \neq 2$. Therefore $A_{\text{ucg}}(C, P) = 5$ if and only if $\operatorname{cov}_{AA''B''}(P) = 2$. However, we have not found this to be the case for any such P. We also have been unable to show it is impossible, so we are left with the following conjecture.

Conjecture 2. *If* C *is a graph with* $\operatorname{diam}(C) > 1$ *and* P *is a graph with* $\operatorname{diam}(P) = 4$ *and* $r(P) = 3$*, then* $A_{\text{ucg}}(C, P) = 6$*.*

The $\operatorname{diam}(P) = r(P) = 4$ case also warrants further discussion. Propositions 5.9 and 5.10 show there are examples of P when $A_{\text{ucg}}(C, P) = 6$ and with $A_{\text{ucg}}(C, P) = 5$. It will be necessary to find another metric invariant other than diameter and radius to refine the results of this case. At this point we are unsure what a suitable invariant may be.

Finally, independent of C, there is a major difference between possible appendage numbers when $\operatorname{diam}(P) > 2$ and when $\operatorname{diam}(P) = 2$. When $\operatorname{diam}(P) > 2$, Theorems 6.2 and 6.3 show there are only finitely many possible appendage numbers, and which are independent of the size of $V(P)$. On the other hand, for $\operatorname{diam}(P) = r(P) = 2$ the graph P^α in Proposition 5.4 gives

$$A_{\text{ucg}}(K_n, P^\alpha) = |V(P^\alpha)| = 2\alpha.$$

This may suggest that appendage numbers are related to $|V(P)|$ when $\operatorname{diam}(P) = r(P) = 2$, however the following proposition shows this is not the case.

Proposition 6.1. *For every $\alpha, \beta \in \mathbb{N}$, there is a graph P such that $V(P) = 2\alpha+\beta$ and $A_{\text{ucg}}(K_n, P) = 2\alpha$.*

Proof. In this proof we modify the construction of P^α from Proposition 5.4. For $\alpha, \beta \in \mathbb{N}$ define $P^{\alpha,\beta} = (V^{\alpha,\beta}, E^{\alpha,\beta})$ as follows. Let

$$V^{\alpha,\beta} = \{e_i, f_i, g_j : 1 \le i \le \alpha, 1 \le j \le \beta\}$$

and

$$E^{\alpha,\beta} = \{e_ie_j, f_if_j, e_if_j, e_kg_l, f_kg_l : 1 \le i, j \le \alpha, i \ne j, 2 \le k \le \alpha, 1 \le l \le \beta\}.$$

That is, $P^{\alpha,\beta}$ is the graph from Proposition 5.4 with β new vertices, $g_i's$, that are adjacent to every vertex except themselves, e_1 and f_1. Observe that $\operatorname{diam}(P^{\alpha,\beta}) = r(P^{\alpha,\beta}) = 2$ if $\alpha \ge 2$.

We now show $\operatorname{cov}_A(P^{\alpha,\beta}) = \operatorname{cov}_{AB}(P^{\alpha,\beta}) = 2\alpha$. Let $\overline{P^{\alpha,\beta}} = \{P_1, \dots, P_k\}$ be a covering of $P^{\alpha,\beta}$ satisfying condition A. For $2 \le i \le \alpha$, $\{e_i\}, \{f_i\} \in \overline{P^{\alpha,\beta}}$ as in Proposition 5.4. If

$$\{e_1, f_1, g_1, \dots, g_\beta\} \in \overline{P^{\alpha,\beta}},$$

then $\overline{P^{\alpha,\beta}}$ fails to satisfy condition A. Therefore e_1, f_1 and the g_ls must be contained in least two elements of $\overline{P^{\alpha,\beta}}$, and so $\operatorname{cov}_A(P^{\alpha,\beta}) \ge 2\alpha$.

Let $P_1^{\alpha,\beta} = \{e_1\}$ and $Q_1^{\alpha,\beta} = \{f_1, g_1 \dots, g_\beta\}$, and for $2 \le i \le \alpha$ let $P_i^{\alpha,\beta} = \{e_i\}$ and $Q_i^{\alpha,\beta} = \{f_i\}$. Then $\{P_1^{\alpha,\beta}, Q_1^{\alpha,\beta}, \dots, P_\alpha^{\alpha,\beta}, Q_\alpha^{\alpha,\beta}\}$ is a covering of $P^{\alpha,\beta}$ that satisfies both conditions A and B. So $A_{\text{ucg}}(K_n, P^{\alpha,\beta}) = 2\alpha$ and $V(P^{\alpha,\beta}) = 2\alpha + \beta$.

$\square$

7. Other Appendage Numbers

In the paper [4] Gu defines $A_{\text{ucg}}(C)$ to be the minimum number of vertices needed to be added to C in order to create a uniform central graph G with $\langle \mathcal{Z}(G) \rangle = C$. To match notation we let $A_{\text{ucg}}(C, -) = A_{\text{ucg}}(C)$. Gu's main theorem is the following.

Theorem 7.1 (Gu). *If C is a connected graph, then*

$$A_{\mathrm{ucg}}(C,-) = \begin{cases} 2 & \text{if } C = \{v\} \\ 4 & \text{if } C = K_n, n \geq 2 \\ 6 & \text{otherwise} \end{cases} .$$

We use our results to give an alternative proof of Gu's result which is also true without the condition that C is connected.

Proof. Observe that $A_{\mathrm{ucg}}(C,-) = \min\{A_{\mathrm{ucg}}(C,P) + |V(P)|\}$ where the minimum is taken over all graphs P. By Proposition 2.2, $r(P) > 1$ and so we may assume $|V(P)| \geq 2$. Since $\mathrm{cov}_A(P) \geq 2$ for any P, by Proposition 3.1 $A_{\mathrm{ucg}}(K_n, P) \geq 2$ for $n \geq 2$, and by Proposition 4.1 $A_{\mathrm{ucg}}(C,P) \geq 4$ for any non-complete graph C. Hence $A_{\mathrm{ucg}}(K_n,-) \geq 4$ and $A_{\mathrm{ucg}}(C,-) \geq 6$.

Let P^2 be a graph with two isolated vertices $\{u,v\}$ and $\overline{P^2} = \{\{u\},\{v\}\}$ a covering. Note $\overline{P^2}$ satisfies conditions A, B, A′ and B′. Let

$$G_1 = \mathcal{G}(K_n, P^2, \overline{P^2}, 1) - \{x_{0,1}\}$$

and

$$G_2 = \mathcal{G}(K_n, P^2, \overline{P_2}, 2) - \{x_{0,1}, x_{0,2}\}.$$

The graph G_1 is the UCG in the first half of the proof of Theorem 3.1 and G_2 is a UCG by Proposition 4.2. Furthermore, $|\mathcal{I}(G_1)| = 2$ and $|\mathcal{I}(G_1)| = 4$, so $A_{\mathrm{ucg}}(K_n,-) = 4$ and $A_{\mathrm{ucg}}(C,-) = 6$.

Finally, $A_{\mathrm{ucg}}(\{v\},-) = 2$ by Theorem 6.1. □

We can also consider $A_{\mathrm{ucg}}(-,P)$, the minimum number of vertices needed to be added to P in order to construct a uniform central graph G with $\langle \mathcal{CP}(G)\rangle = P$. From Propositions 2.2 and 6.1, it follows that $A_{\mathrm{ucg}}(-,P) = \infty$ if $r(P) = 1$, and $A_{\mathrm{ucg}}(-,P) = 1$ otherwise.

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