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Odd facial colorings of acyclic plane graphs

Július Czap and Peter Šugerek

Department of Applied Mathematics and Business Informatics, Faculty of Economics
Technical University of Košice, Němcovej 32, 04001 Košice, Slovakia

julius.czap@tuke.sk, peter.sugerek@tuke.sk

Abstract

Let G be a connected plane graph with vertex set V and edge set E. For $X \in \{V, E, V \cup E\}$, two elements of X are facially adjacent in G if they are incident elements, adjacent vertices, or facially adjacent edges (edges that are consecutive on the boundary walk of a face of G). A coloring of G is facial with respect to X if there is a coloring of elements of X such that facially adjacent elements of X receive different colors. A facial coloring of G is odd if for every face f and every color c, either no element or an odd number of elements incident with f is colored by c. In this paper we investigate odd facial colorings of trees. The main results of this paper are the following: (i) Every tree admits an odd facial vertex-coloring with at most 4 colors; (ii) Only one tree needs 6 colors, the other trees admit an odd facial edge-coloring with at most 5 colors; and (iii) Every tree admits an odd facial total-coloring with at most 5 colors. Moreover, all these bounds are tight.

Keywords: facial coloring, odd facial coloring, plane graph
Mathematics Subject Classification: 05C10, 05C15
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1. Introduction and Notations

All graphs considered in this paper are simple connected plane graphs provided that it is not stated otherwise. We use standard graph theory terminology according to [2]. However, the most frequent notions of the paper are defined through it. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane such that no edges intersect except at their endvertices. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The

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boundary of a face f is the boundary in the usual topological sense. It is the collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of the initial vertex and the direction, and is called the *boundary walk* of the face f (see [8], p. 101).

Two vertices (two edges) are *adjacent* if they are connected by an edge (have a common endvertex). A vertex and an edge are *incident* if the vertex is an endvertex of the edge. A vertex (or an edge) and a face are incident if the vertex (or the edge) lies on the boundary of the face. Two edges of a plane graph G are *facially adjacent* if they are consecutive on the boundary walk of a face of G.

An *edge-*, *vertex-*, *total-coloring* of G is an assignment of colors to the edges, vertices, edges and vertices of G, respectively, one color to each element. An edge-coloring (vertex-coloring) c of a graph G is *proper* if for any two adjacent edges (vertices) x_1 and x_2 of G, $c(x_1) \neq c(x_2)$ holds.

A *facial edge-coloring* c of a plane graph G is an edge-coloring such that for any two facially adjacent edges e_1 and e_2 of G, $c(e_1) \neq c(e_2)$ holds. Observe that this coloring need not to be proper in a usual sense. We require only that facially adjacent edges must receive different colors. On the other hand proper edge-coloring and facial edge-coloring coincide in the class of subcubic plane graphs. Facial edge-coloring was first studied for the family of cubic bridgeless plane graphs and for the family of plane triangulations. Already Tait [13] observed that the Four Color Problem is equivalent to the problem of facial 3-edge-coloring of any plane triangulation and to the problem of facial 3-edge-coloring of cubic bridgeless plane graphs (see e.g. [12]).

The concept of *facial total-coloring* of plane graphs was introduced by Fabrici, Jendroľ, and Vrbjarová [6]. A facial total-coloring of a plane graph G is a coloring of the vertices and edges such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color.

An *odd facial vertex-coloring* of a plane graph is a proper vertex-coloring such that for every face f and every color c, either no vertex or an odd number of vertices incident with f is colored by c. In [5] it was proved that every 2-connected plane graph admits an odd facial vertex-coloring with at most 118 colors. The bound 118 was improved to 97 by Kaiser et al. [9]. Czap [3] proved that any 2-connected outerplane graph has an odd facial vertex-coloring with at most 12 colors, moreover, if a 2-connected outerplane graph is bipartite, then 8 colors suffice. He presented an outerplane graph on 10 vertices which require 10 colors for such a coloring. Wang, Finbow, and Wang [14] proved that only two 2-connected outerplane graphs need 10 colors, the other outerplane graphs admit an odd facial vertex-coloring with at most 9 colors.

An *odd facial edge-coloring* of a plane graph is a facial edge-coloring such that for every face f and every color c, either no edge or an odd number of edges incident with f is colored by c. Czap et al. [4] proved that every 2-edge-connected plane graph G has an odd facial edge-coloring with at most 20 colors, this bound was later improved to 16 by Lužar and Škrekovski [11]. In the case when G is a 3-edge-connected (resp. 4-edge-connected) plane graph, then 12 (resp. 9) colors are sufficient, see [4]. In [1] it is proved that every 2-edge-connected outerplane graph admits an odd facial 9-edge-coloring with one exception.

In this paper we introduce the concept of odd facial total-coloring of plane graphs, which strengthens the requirement for the facial total-coloring. An *odd facial total-coloring* of a plane graph is a facial total-coloring such that for every face f and every color c, either no element or an odd number of elements incident with f is colored by c.

The main results of this paper are the following: (i) Every tree admits an odd facial vertex-coloring with at most 4 colors; (ii) Only one tree needs 6 colors, the other trees admit an odd facial edge-coloring with at most 5 colors; and (iii) Every tree admits an odd facial total-coloring with at most 5 colors. Moreover, all these bounds are tight.

2. Odd facial colorings of trees

A set is *odd* if it has an odd number of elements, otherwise it is *even*. Vertices of degree one are *leaves*. An edge incident to a leaf is a *pendant edge*.

2.1. Odd facial vertex-coloring of trees

A set of vertices is called *independent*, if no two of its members are adjacent. The vertex set of every tree T on at least two vertices can be decomposed into two independent sets, called *partite sets*.

Let $\chi_o(G)$ denote the minimum number of colors required in an odd facial vertex-coloring of a plane graph G.

Theorem 2.1. *Let T be a tree on at least two vertices. Then*

(i) $\chi_o(T) = 2$ *if and only if both partite sets of T are odd,*

(ii) $\chi_o(T) = 3$ *if and only if one partite set of T is odd,*

(iii) $\chi_o(T) = 4$ *if and only if no partite set of T is odd.*

Proof. (i) Clearly, if a partite set of T is even, then no proper 2-vertex-coloring is an odd facial vertex-coloring. Consequently, $\chi_o(T) = 2$ implies that both partite sets of T are odd. On the other hand, if both partite sets of T are odd, then trivially $\chi_o(T) = 2$.

(ii) If $\chi_o(T) = 3$, then T has an odd number of vertices, hence exactly one partite set is odd. On the other hand, if one partite set of T is odd, then by (i) $\chi_o(T) \geq 3$. If we color the vertices of T in the odd partite set with color 1, color one vertex from the even partite set with color 2, and color all other vertices with color 3, we obtain an odd facial 3-vertex-coloring of T.

(iii) If $\chi_o(T) = 4$, then T has an even number of vertices. From (i) it follows that both partite sets are even. On the other hand, if both partite sets are even, then $\chi_o(T) \leq 4$ (since every partite set can be decomposed into two odd sets). By (i) and (ii) we have $\chi_o(T) = 4$. □

2.2. Odd facial edge-coloring of trees

Lemma 2.1. *Let T be a tree and T' its subtree. Every facial 3-edge-coloring of T' can be extended to a facial 3-edge-coloring of T.*

Proof. Let c be a facial 3-edge-coloring of T'. We extend the coloring c step by step. In each step we color one uncolored edge of T.

First we choose an uncolored edge uv (i.e. an edge from $E(T) - E(T'))$ which is incident with a vertex of T'. W.l.o.g., assume that $v \in V(T')$. Observe that u is incident only with uncolored edges (otherwise T' with uv contains a cycle). This implies that uv has at most two facially adjacent edges in T'. Consequently, there is an admissible color for uv.

In the next step $T' \cup \{uv\}$ plays the role of T'. □

Let $\chi'_o(G)$ denote the minimum number of colors required in an odd facial edge-coloring of a plane graph G.

Lemma 2.2. *If T is a tree, then $\chi'_o(T) \leq 6$. Moreover, this bound is tight.*

Proof. Let $\widetilde{T}$ be the tree depicted in Figure 1.

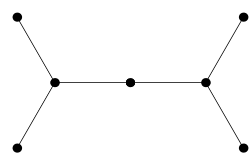


Figure 1. The tree $\widetilde{T}$.

It is easy to see that $\chi'_o(\widetilde{T}) = 6$.

Lemma 2.1 implies that every tree T has a facial 3-edge-coloring. Every facial 3-edge-coloring can be modified to an odd facial edge-coloring using at most 6 colors. If in a facial 3-edge-coloring a color appears on an even number of edges, then we recolor one of them with a new color. Since we recolor at most three edges, the new coloring uses at most six colors. □

Any tree in this paper is embedded in the plane. The particular embedding is very important. The tree depicted in Figure 2 with the embedding on the left has an odd facial 2-edge-coloring, and with the embedding on the right, its facial 2-edge-coloring is not odd.

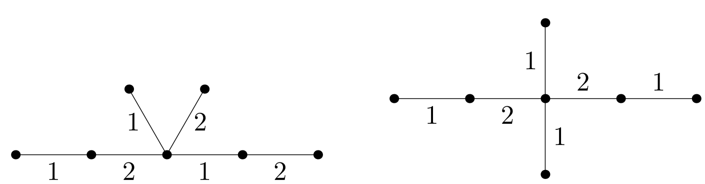


Figure 2. Two different embeddings of the same tree.

Lemma 2.3. *Let c be a facial 3-edge-coloring of a tree T. If a color appears on an odd number of edges (under the coloring c), then T has an odd facial 5-edge-coloring.*

Proof. Assume that c uses the colors $1, 2, 3$ and the color 1 appears an odd number of times in T. If the color 2 (resp. 3) appears on an even number of edges, then we recolor one edge of color 2 (resp. 3) with a new color 4 (resp. 5). □

Theorem 2.2. *Every tree T distinct from $\widetilde{T}$ (depicted in Figure 1) has an odd facial 5-edge-coloring.*

Proof. Let $P = v_1v_2\dots v_n$ be a longest path in T, where $v_i \in V(T)$ for $i = 1, 2, \dots, n$. We distinguish some cases according to the length of P.

Case 1. P has at least five edges.

Let T' be the subtree of T consisting of the first five edges of P. Color the edges v_1v_2, v_4v_5 with color A; the edge v_3v_4 with color B; and the edges v_2v_3, v_5v_6 with color C. By Lemma 2.1 this coloring of T' can be extended to a facial 3-edge-coloring of T. If all colors A, B, C appear an even number of times in T, then we recolor the edges v_1v_2, v_3v_4, v_5v_6 with a new color E and we obtain an odd facial 4-edge-coloring. Otherwise we apply Lemma 2.3.

Case 2. P has exactly four edges, i.e. $P = v_1v_2v_3v_4v_5$.

Case 2.1 The degree of v_3 is at least 3 in T.

Let v be a vertex adjacent to v_3 distinct from v_2 and v_4. Let T' be the subtree of T consisting of the edges of P and the edge v_3v. Color the edges v_1v_2, v_3v_4 with color A; the edge v_3v with color B; and the edges v_2v_3, v_4v_5 with color C. By Lemma 2.1 this coloring of T' can be extended to a facial 3-edge-coloring of T. If all colors A, B, C appear an even number of times in T, then we recolor the edges v_1v_2, v_3v, v_4v_5 with a new color E and we obtain an odd facial 4-edge-coloring. Otherwise we apply Lemma 2.3.

Case 2.2 The degree of v_3 is 2 in T.

Since P is a longest path in T, the vertices v_2 and v_4 are incident only with leaves except for v_3.

Case 2.2.1 At least one of the vertices v_2, v_4 has degree at least 4.

Without loss of generality, assume that $\deg_T(v_2) \geq 4$. Let vv_2 be an edge of T not facially adjacent to v_2v_3. Let T' be the subtree of T consisting of the edges $vv_2, v_2v_3, v_3v_4, v_4v_5$. Color the edges vv_2, v_3v_4 with color A; the edge v_2v_3 with color B; and the edge v_4v_5 with color C. By Lemma 2.1 this coloring of T' can be extended to a facial 3-edge-coloring of T. If all colors A, B, C appear an even number of times in T, then we recolor the edges vv_2, v_2v_3, v_4v_5 with a new color E and we obtain an odd facial 4-edge-coloring. Otherwise we apply Lemma 2.3.

Case 2.2.2 The vertices v_2 and v_4 have degree at most 3.

Since $T \neq \widetilde{T}$, v_2 or v_4 has degree 2. Consequently, T has at most five edges. So it has an odd facial 5-edge-coloring.

Case 3. P has exactly three edges, i.e. $P = v_1v_2v_3v_4$.

In this case, color the edge v_2v_3 with A and all other edges with B and C so that facially adjacent edges receive different colors. If the color B (resp. C) appears on an even number of edges, then we recolor one edge of color B (resp. C) with a new color D (resp. E).

Case 4. The length of P is at most 2.

In this case T is a star. It is easy to see that every star has an odd facial 5-edge-coloring. □

Corollary 2.1. *Let T be a tree on an even number of edges. If $T \neq \widetilde{T}$, then $\chi'_o(T) \in \{2, 4\}$. Moreover, it is easy to check whether $\chi'_o(T) = 2$ or $\chi'_o(T) = 4$.*

Proof. Since $T \neq \widetilde{T}$ has an even number of edges, every its odd facial edge-coloring uses an even number of colors. By Theorem 2.2 we have $\chi'_o(T) \leq 5$. Consequently, $\chi'_o(T) \in \{2, 4\}$.

Clearly, if a tree has a facial 2-edge-coloring, then this coloring is unique. If each color class has an odd number of elements, then $\chi'_o(T) = 2$, otherwise $\chi'_o(T) = 4$. □

Note that a tree T has a facial 2-edge-coloring if and only if every internal vertex of T is even.

Corollary 2.2. *If T is a tree on an odd number of edges, then $\chi'_o(T) \in \{1, 3, 5\}$.*

A challenging open problem in this direction is the following.

Problem 1. *Characterize all trees that admit an odd facial 3-edge-coloring.*

Note that there are infinitely many trees with $\chi'_o(T) = 3$ and also infinitely many trees with $\chi'_o(T) = 5$. Let G_k be a tree obtained from a path $P = v_1 v_2 \dots v_{4k+3}$ on $4k+3$ vertices, $k \geq 0$, so that we add $2k+1$ new vertices and join each vertex v_{2i}, $i = 1, 3, \dots, 2k+1$, with one of them, see Figure 3 for illustration. Since the vertices $v_2, v_4, \dots, v_{4k+2}$ have degree three and they cover all edges of G_k, every color appears on $2k+1$ edges in any facial 3-edge-coloring of G_k, so $\chi'_o(G_k) = 3$. Let H_k be a tree obtained from G_k so that we add two new vertices and join both of them with v_{4k+3}, see Figure 3 for illustration. It is not hard to see that H_k has no odd facial 3-edge-coloring. Since H_k has an odd number of edges, Corollary 2.2 implies that $\chi'_o(H_k) = 5$.

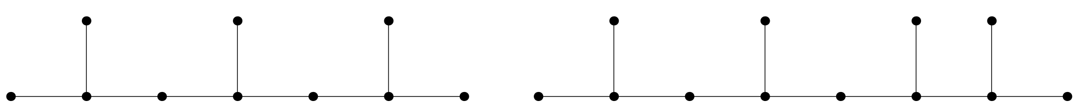


Figure 3. The graphs G_1 and H_1.

2.3. Odd facial total-coloring of trees

Let $\chi''_o(G)$ denote the minimum number of colors required in an odd facial total-coloring of a plane graph G.

Theorem 2.3. *Every tree T on at least three vertices admits an odd facial total-coloring with exactly five colors, i.e. $\chi''_o(T) \leq 5$. Moreover, this bound is tight.*

Proof. Suppose there is a counterexample to Theorem 2.3. Let T be a counterexample with the minimum number of vertices.

The only tree on three vertices is a path on three vertices. Clearly, it has an odd facial 5-total-coloring. There are two trees on four vertices. They are depicted in Figure 4 and they also have an odd facial 5-total-coloring.

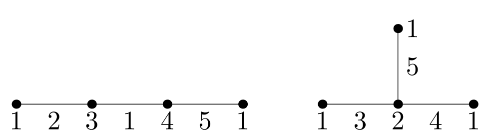


Figure 4. Trees on four vertices and their odd facial total-colorings.

So we can assume that T has at least five vertices. Let $P = v_1 v_2 \dots v_{n-1} v_n$ be a longest path in T. There are two possibilities: either the vertices v_2 and v_{n-1} have degree two or at least one of them has degree at least three in T.

Case 1. Both vertices v_2 and v_{n-1} have degree two.

Let $T' = T - \{v_1, v_n\}$ be the tree obtained from T by removing the vertices v_1 and v_n. The tree T' admits an odd facial total-coloring with five colors, since it has fewer vertices than T. This coloring can be extended to an odd facial 5-total-coloring of T in the following way: First we color the edges $v_1v_2, v_{n-1}v_n$ with the same color distinct from the colors of $v_2, v_2v_3, v_{n-2}v_{n-1}, v_{n-1}$. Thereafter we color the vertices v_1, v_n with the same color distinct from the colors of v_2, v_{n-1}, v_1v_2.

Case 2. v_2 or v_{n-1} has degree at least three.

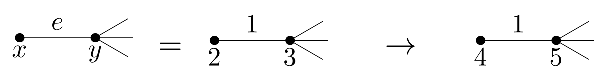
Every tree on at least three vertices admits a facial total-coloring with exactly four colors, see [7]. Let c be such a coloring of T with colors $1, 2, 3, 4$. In the following we show that c can be modified to an odd facial 5-total-coloring.

First observe that T has an odd number of elements (vertices and edges). Therefore in c one or three colors are used an odd number of times. If three colors are used an odd number of times in c, say $1, 2, 3$, then it is sufficient to recolor one element of color 4 with (a new) color 5.

Now assume that only one color, say 1, is used an odd number of times.

Case 2.1 T has a pendant edge $e = xy$ of color 1, where x is a leaf.

Since c is a facial total-coloring we have $c(x) \neq 1$, $c(y) \neq 1$, and $c(x) \neq c(y)$. Without loss of generality, we can assume that $c(x) = 2$ and $c(y) = 3$. In this case it suffices to recolor x with 4 and recolor y with (a new) color 5.



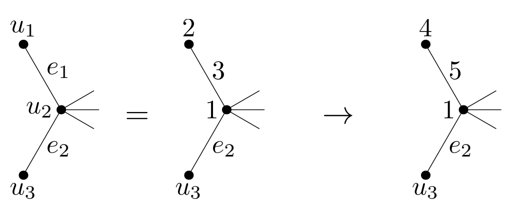
Case 2.2 No pendant edge of T has color 1.

Let $e_1 = u_1u_2$ and $e_2 = u_2u_3$ be two facially adjacent pendant edges in T (such two edges exist because v_2 or v_{n-1} has degree at least three).

We distinguish four cases.

Case 2.2.1 $c(u_2) = 1$

In this case, the colors $c(u_1), c(e_1), c(e_2)$, and $c(u_3)$ are distinct from 1. W.l.o.g., assume that $c(u_1) = 2$ and $c(e_1) = 3$. It suffices to recolor u_1 with 4 and recolor e_1 with (a new) color 5.

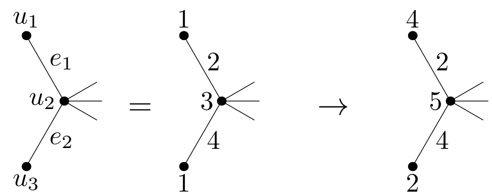


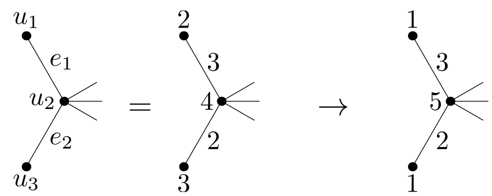
Case 2.2.2 $c(u_1) = c(u_3) = 1$

In this case, $\{c(e_1), c(u_2), c(e_2)\} = \{2, 3, 4\}$. W.l.o.g., assume that $c(e_1) = 2$, $c(u_2) = 3$, and $c(e_2) = 4$. It suffices to recolor u_1 with 4, u_2 with 5, and u_3 with 2.

Case 2.2.3 $1 \notin \{c(u_1), c(u_2), c(u_3)\}$

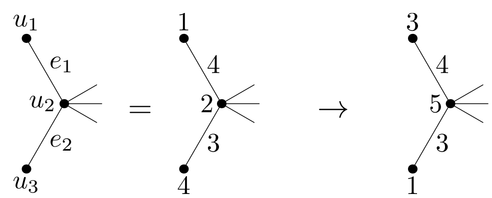
In this case, $\{c(u_1), c(e_1), c(u_2)\} = \{2, 3, 4\}$. W.l.o.g., assume that $c(u_1) = 2$, $c(e_1) = 3$, and $c(u_2) = 4$. Then necessarily, $c(e_2) = 2$ and $c(u_3) = 3$. It suffices to recolor u_2 with 5, u_1 and u_3 with 1.





Case 2.2.4 $1 \in \{c(u_1), c(u_3)\}$ and $c(u_1) \neq c(u_3)$

W.l.o.g., assume that $c(u_1) = 1$. Then $\{c(u_2), c(e_2), c(u_3)\} = \{2, 3, 4\}$. W.l.o.g., assume that $c(u_2) = 2$, $c(e_2) = 3$, and $c(u_3) = 4$. Then necessarily, $c(e_1) = 4$. It suffices to recolor u_2 with 5, u_1 with 3, and u_3 with 1.



So T is not a counterexample. □

Corollary 2.3. *If T is a tree on at least two vertices, then $\chi''_o(T) \in \{3, 5\}$. Moreover, it is easy to check whether $\chi''_o(T) = 3$ or $\chi''_o(T) = 5$.*

Proof. Since T has an odd number of elements (vertices and edges), every its odd facial total-coloring uses an odd number of colors. Hence, Theorem 2.3 implies $\chi''_o(T) \in \{3, 5\}$.

Clearly, if a tree has a facial total-coloring with 3 colors, then this coloring is unique. If each color class has an odd number of elements, then $\chi''_o(T) = 3$, otherwise $\chi''_o(T) = 5$. □

3. Remarks

Note that odd facial edge-coloring of a plane graph G is closely related to edge decomposition of the dual graph G^* into odd subgraphs (subgraphs with all vertices having odd degree), see e.g. [4]. Edge decompositions of graphs into odd (even) subgraphs or characterization of odd (even) factors of graphs have recently drawn a substantial amount of attention, see e.g. [10].

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