Electronic Journal of Graph Theory and Applications **8** (1) (2020), 9–15

Electronic Journal of Graph Theory and Applications

C_4-decomposition of the tensor product of complete graphs

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Abstract

Let G be a simple and finite graph. A graph is said to be *decomposed* into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if G is the edge disjoint union of H_1 and H_2. If $G = H_1 \oplus H_2 \oplus H_3 \oplus \cdots \oplus H_k$, where H_1,H_2,H_3, ..., H_k are all isomorphic to H, then G is said to be H-decomposable. Futhermore, if H is a cycle of length m then we say that G is C_m-decomposable and this can be written as $C_m|G$. Where $G \times H$ denotes the tensor product of graphs G and H, in this paper, we prove the necessary and sufficient conditions for the existence of C_4-decomposition of $K_m \times K_n$. Using these conditions it can be shown that every even regular complete multipartite graph G is C_4-decomposable if the number of edges of G is divisible by 4.

Keywords: cycle decomposition, tensor product, complete graph
Mathematics Subject Classification: 05C70
DOI: 10.5614/ejgta.2020.8.1.2

1. Introduction

Let C_m, K_m and $K_m - I$ denote cycle of length m, complete graph on m vertices and complete graph on m vertices minus a 1-factor respectively. By an m-cycle we mean a cycle of length m. Let $K_{n,n}$ denote the complete bipartite graph with n vertices in each bipartition set and $K_{n,n} - I$ denote $K_{n,n}$, with a 1-factor removed. All graphs considered in this paper are simple and finite. A graph is said to be *decomposed* into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if

Received: 7 February 2019, Revised: 25 May 2019, Accepted: 14 June 2019.

G is the edge disjoint union of H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where H_1,H_2, ..., H_k are all isomorphic to H, then G is said to be H-decomposable. Futhermore, if H is a cycle of length m then we say that G is C_m-decomposable and this can be written as $C_m|G$. A k-factor of G is a k-regular spanning subgraph. A k-factorization of a graph G is a partition of the edge set of G into k-factors. A C_k-factor of a graph is a 2-factor in which each component is a cycle of length k. A *resolvable* k-cycle decomposition (for short k-RCD) of G denoted by $C_k||G$, is a 2-factorization of G in which each 2-factor is a C_k-factor.

For two graphs G and H their tensor product $G \times H$ has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. From this, note that the tensor product of graphs is distributive over edge disjoint union of graphs, that is if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$. Now, for $h \in V(H)$, $V(G) \times h = \{(v, h)|v \in V(G)\}$ is called a *column* of vertices of $G \times H$ corresponding to h. Further, for $y \in V(G)$, $y \times V(H) = \{(y, v)|v \in V(H)\}$ is called a *layer* of vertices of $G \times H$ corresponding to y. It is true that $K_m \times K_2$ is isomorphic to the complete bipartite graph $K_{m,m}$ with the edges of a perfect matching removed, i.e. $K_m \times K_2 \cong K_{m,m} - I$, where I is a 1-factor of $K_{m,m}$.

The lexicographic product $G * H$ of two graphs G and H is the graph having the vertex set $V(G) \times V(H)$, in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1, g_2 \in E(G)$; or $g_1 = g_2$ and $h_1, h_2 \in E(H)$.

For very recent works on decomposition of graphs, see [6, 8]. The problem of finding C_k-decomposition of K_{2n+1} or $K_{2n} - I$ where I is a 1-factor of K_{2n}, is completely settled by Alspach, Gavlas and Šajna in two different papers (see [2, 17]). A generalization to the above complete graph decomposition problem is to find a C_k-decomposition of $K_m * \overline{K}_n$, which is the complete m-partite graph in which each partite set has n vertices. The study of cycle decompositions of $K_m * \overline{K}_n$ was initiated by Hoffman et al. [7]. In the case when p is a prime, the necessary and sufficient conditions for the existence of C_p-decomposition of $K_m * \overline{K}_n$, $p \geq 5$ is obtained by Manikandan and Paulraja in [11, 12, 14]. Billington [3] has studied the decomposition of complete tripartite graphs into cycles of length 3 and 4. Furthermore, Cavenagh and Billington [5] have studied 4-cycle, 6-cycle and 8-cycle decomposition of complete multipartite graphs. Billington et al. [4] have solved the problem of decomposing $(K_m * \overline{K}_n)$ into 5-cycles. Similarly, when $p \geq 3$ is a prime, the necessary and sufficient conditions for the existence of C_{2p}-decomposition of $K_m * \overline{K}_n$ is obtained by Smith (see [19]). For a prime $p \geq 3$, it was proved in [20] that C_{3p}-decomposition of $K_m * \overline{K}_n$ exists if the obvious necessary conditions are satisfied. As the graph $K_m \times K_n \cong K_m * \overline{K}_n - E(nK_m)$ is a proper regular spanning subgraph of $K_m * \overline{K}_n$. It is therefore natural to think about the cycle decomposition of $K_m \times K_n$.

The results in [11, 12, 14] also gives the necessary and sufficient conditions for the existence of a p-cycle decomposition, (where $p \geq 5$ is a prime number) of the graph $K_m \times K_n$. In [13] it was shown that the tensor product of two regular complete multipartite graph is Hamilton cycle decomposable. Muthusamy and Paulraja in [15] proved the existence of C_{kn}-factorization of the graph $C_k \times K_{mn}$, where $mn \neq 2(\text{mod } 4)$ and k is odd. Paulraja and Kumar [16] showed that the necessary conditions for the existence of a resolvable k-cycle decomposition of tensor product of complete graphs are sufficient when k is even. In a recent work by the present authors, it was proven that the necessary and sufficient conditions for the decomposition of the graph $K_m \times K_n$

into cycles of length six is that m *or* $n \equiv 1$ *or* $3 \pmod 6$ (see [1]).
In this paper, we prove the necessary and sufficient conditions for $K_m \times K_n$, where $m, n \geq 2$, to have a C_4-decomposition. Among other results, here we prove the following main result.

Theorem 1.1. *For $m, n \geq 2$, $C_4 | K_m \times K_n$ if and only if either*

1. *$n \equiv 0$ (mod 4) and m is odd,*
2. *$m \equiv 0$ (mod 4) and n is odd or*
3. *m or $n \equiv 1$ (mod 4)*

Let ρ be a permutation of the vertex set V of a graph G. For any subset U of V, ρ acts as a function from U to V by considering the restriction of ρ to U. If H is a subgraph of G with vertex set U, then $\rho(H)$ is a subgraph of G provided that for each edge $xy \in E(H)$, $\rho(x)\rho(y) \in E(G)$. In this case, $\rho(H)$ has vertex set $\rho(U)$ and edge set $\{\rho(x)\rho(y) : xy \in E(H)\}$.

Next, we give some existing results on cycle decomposition of complete graphs.

Theorem 1.2. *[9] Let m be an odd integer and $m \geq 3$. If $m \equiv 1$ or 3 (mod 6) then $C_3 | K_m$.*

Theorem 1.3. *[17] Let n be an odd integer and m be an even integer with $3 \leq m \leq n$. The graph K_n can be decomposed into cycles of length m whenever m divides the number of edges in K_n.*

Now we have the following lemma, this lemma gives the cycle decomposition of the complete graph K_m into cycles of length 3 and 4.

Lemma 1.1. *For $m \equiv 5$(mod 6), there exist positive integers p and q such that K_m is decomposable into p 3-cycles and q 4-cycles.*

Proof. Let the vertices of K_m be $0, 1, ..., m-1$. The 4-cycles are $(i, i+1+2s, i-1, i+2+2s)$, $s = 0, 1, ..., (m-i)/2-2$, $i = 1, 3, ..., m-4$. The 3-cycles are $(m-1, i-1, i)$, $i = 1, 3, ..., m-2$. Hence the proof. □

The following theorem is on the complete bipartite graph minus a 1-factor, it was obtained by Ma et. al [10].

Theorem 1.4. *[10] Let m and n be positive integers. Then there exist an m cycle system of $K_{n,n} - I$ if and only if $n \equiv 1$ (mod 2), $m \equiv 0$ (mod 2), $4 \leq m \leq 2n$ and $n(n-1) \equiv 0$ (mod m).*

From the theorem above we have the following corollary.

Corollary 1.1. *The graph $K_{n,n} - I$, where I is a 1-factor of $K_{n,n} - I$ admits a C_4 decomposition if and only if $n \equiv 1$ (mod 4).*

The following result is on the complete bipartite graphs.

Theorem 1.5. *[18] The complete bipartite graph $K_{a,b}$ can be decomposed into cycles of length $2k$ if and only if a and b are even, $a \geq k$, $b \geq k$ and $2k$ divides ab.*

2. C_4 Decomposition of $C_m \times K_n$

We begin this section with the following lemma.

Lemma 2.1. $C_4|C_3 \times K_4$.

Proof. Following from the definition of the tensor product of graphs, let $U^1 = \{u_1, v_1, w_1\}$, $U^2 = \{u_2, v_2, w_2\}$,..., $U^4 = \{u_4, v_4, w_4\}$ form the partite sets of vertices in the product $C_3 \times K_4$. For $1 \leq i, j \leq 4$, surely $U^i \cup U^j$, $i \neq j$ induces a $K_{3,3} - I$, where I is a 1-factor of $K_{3,3}$.
A C_4 decomposition of $C_3 \times K_4$ is given below:
$\{u_1, v_4, u_2, w_3\}$,$\{u_1, v_3, u_4, w_2\}$,$\{u_1, v_2, u_3, w_4\}$,$\{u_2, v_3, w_2, v_1\}$,$\{u_3, v_1, w_3, v_4\}$,
$\{u_2, w_1, v_3, w_4\}$,$\{u_3, w_2, v_4, w_1\}$,$\{u_4, v_1, w_4, v_2\}$ and $\{u_4, w_1, v_2, w_3\}$ □

Next, we have the following lemma which follows from Lemma 2.1.

Lemma 2.2. $C_4|C_3 \times K_5$.

Proof. Suppose we fix the 4-cycles already given in Lemma 2.1, clearly the graph which remains after removing the edges of $C_3 \times K_4$ from $C_3 \times K_5$ can be decomposed into 3 copies of $K_{2,4}$. Now, by Theorem 1.5 the graph $K_{2,4}$ can be decomposed into cycles of length 4. Hence $C_4|C_3 \times K_5$. □

The following theorem is an extension of Lemma 2.1 and Lemma 2.2.

Theorem 2.1. *$C_4|C_3 \times K_n$ if and only if $n \equiv 0$ or 1 (mod 4).*

Proof. Suppose that $C_4|C_3 \times K_n$. The graph $C_3 \times K_n$ has $3n(n-1)$ edges. For $C_4|C_3 \times K_n$ it implies that $n(n-1) \equiv 0 \pmod 4$. Hence $n \equiv 0$ *or* $1 \pmod 4$.
Following the definition of tensor product of graphs, let $U^1 = \{u_1, v_1, w_1\}$, $U^2 = \{u_2, v_2, w_2\}$,..., $U^n = \{u_n, v_n, w_n\}$ form the partite sets of vertices in the product $C_3 \times K_n$. For $1 \leq i, j \leq n$, surely $U^i \cup U^j$, $i \neq j$ induces a $K_{3,3} - I$, where I is a 1-factor of $K_{3,3}$.
Next, we prove the sufficiency in two cases.
Case 1. Whenever $n \equiv 0(\text{mod } 4)$. Let $n = 4t$ where $t \geq 1$.
Next we note that $C_3 \times K_n \cong (C_3 \times K_4) + (C_3 \times K_4) + (C_3 \times K_4) + \cdots + (C_3 \times K_4) + H^*$, H^* is the graph containing the edges of $C_3 \times K_n$ which are not covered by these t copies of $C_3 \times K_4$. By Lemma 2.1 the product $C_3 \times K_4$ admits a C_4-decomposition.
Furthermore, we define the set $U = \{u^1, u^2, ..., u^p\}$,$V = \{v^1, v^2, ..., v^p\}$ and $W = \{w^1, w^2, ..., w^p\}$ where $p = n/4$ and for $j = 1, 2, ..., p$, $u^j = \{u_i | 4j - 3 \leq i \leq 4j\}$, $v^j = \{v_i | 4j - 3 \leq i \leq 4j\}$ and $w^j = \{w_i | 4j - 3 \leq i \leq 4j\}$.
Now, H^* is decomposable into graphs isomorphic to $K_{4,4n-4}$. Indeed, the $K_{4,4n-4}$ graphs in the decomposition of H^* are induced by $(u^i \cup v^1 \cup v^2 \cup \cdots \cup v^p) \setminus v^i$, $(u^i \cup w^1 \cup w^2 \cup \cdots \cup w^p) \setminus w^i$ and $(v^i \cup w^1 \cup w^2 \cup \cdots \cup w^p) \setminus w^i$, $i = 1, 2, ..., p$. By Theorem 1.5 $C_4|K_{4,4n-4}$. Therefore we have decomposed $C_3 \times K_n$ into 4-cycles when $n \equiv 0(\text{mod } 4)$.

Case 2. Whenever $n \equiv 1(\text{mod } 4)$. Let $n = 4t + 1$ where $t \geq 1$.
By removing U^1, we obtain a copy of $C_3 \times K_{n-1}$, so we may apply Case 1. The remaining structure can be decomposed into $3K_{2,4t}$ and by Theorem 1.5 $C_4|K_{2,4t}$. Therefore $C_4|C_3 \times K_n$ when $n \equiv 1(\text{mod } 4)$. □

Next, we establish the following lemma.

Lemma 2.3. *For all* $n \geq 3$, $C_4 | C_4 \times K_n$.

Proof. From the definition of tensor product of graphs, let $U^1 = \{u_1, v_1, w_1, x_1\}$, $U^2 = \{u_2, v_2, w_2, x_2\}, \cdots, U^n = \{u_n, v_n, w_n, x_n\}$ form the partite sets of vertices in the product $C_4 \times K_n$. Also, for $1 \leq i, j \leq n$ and $i \neq j$, $U^i \cup U^j$ induces $K_{4,4} - 2I$, where I is a 1-factor of $K_{4,4}$. Now, each set $U^i \cup U^j$ is isomorphic to $K_{4,4} - 2I$. But $K_{4,4} - 2I$ admits a 4-cycle decomposition. Hence the proof. □

Furthermore, we quote the following result on decomposition of the tensor product of graphs into cycles of odd length.

Lemma 2.4. *[12] For* $k \geq 1$ *and* $m \geq 3$, $C_{2k+1} | C_{2k+1} \times K_m$

The next lemma is an extension of Lemma 2.3 and Lemma 2.4.

Lemma 2.5. *For* $m \geq 3$ *and* $n \geq 2$, $C_m | C_m \times K_n$

Proof. We shall split the proof of this lemma into two cases.

Case 1. When $m = 2k + 1$, $k \geq 1$
The proof of this case is immediate from Lemma 2.4.

Case 2. When $m = 2k$, $k \geq 2$
Following from the definition of tensor product of graphs, let $U_1 = \{u_1^1, u_1^2, u_1^3, ..., u_1^m\}$, $U_2 = \{u_2^1, u_2^2, u_2^3, ..., u_2^m\}$,..., $U_n = \{u_n^1, u_n^2, u_n^3, ..., u_n^m\}$ form the partite sets of vertices in the product $C_m \times K_n$. Now, for $1 \leq i, j \leq n$ and $i \neq j$, the subgraph induced by $U_i \cup U_j$ is isomorphic to $K_{m,m} - (m-2)I$, where I is a 1-factor of $K_{m,m}$. But $K_{m,m} - (m-2)I$ admits an m-cycle decomposition. Hence the proof. □

3. Proof of the Main Theorem

Proof of Theorem 1.1. Assume that $C_4 | K_m \times K_n$, for some m and n with $2 \leq m, n$. Then every vertex of $K_m \times K_n$ has even degree and 4 divides the number of edges of $K_m \times K_n$. These two conditions translates to $(m-1)(n-1)$ being even and $8 | mn(m-1)(n-1)$ respectively. Hence by the first fact, m or n has to be odd, i.e. has to be congruent to 1 *or* 3 *or* $5 \pmod 6$. The second condition is satisfied precisely when one of the following holds.

1. $n \equiv 0 \pmod 4$ and m is odd,
2. $m \equiv 0 \pmod 4$ and n is odd, or
3. m *or* $n \equiv 1 \pmod 4$.

Next we proceed to prove the sufficiency in two cases.
Case 1. Since the tensor product is commutative, we may assume that m is odd and so $m \equiv 1$ *or* 3 *or* $5 \pmod 6$. Suppose that $n \equiv 0 \pmod 4$.
Subcase 1. Let $m \equiv 1$ *or* $3 \pmod 6$
Now since $m \equiv 1$ *or* $3 \pmod 6$ it implies that by Theorem 1.2 $C_3|K_m$. Therefore, the graph $K_m \times K_n = ((C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n))$. But $n \equiv 0 \pmod 4$ therefore by Theorem 2.1 we have that $C_4|C_3 \times K_n$. Hence $C_4|K_m \times K_n$.
Subcase 2. Let $m \equiv 5 (\text{mod } 6)$
By Lemma 1.1, there exist positive integers p and q such that K_m is decomposable into p 3-cycles and q 4-cycles. Hence $K_m \times K_n$ has a decomposition into p copies of $C_3 \times K_n$ and q copies of $C_4 \times K_n$. By Theorem 2.1 $C_4|C_3 \times K_n$ and also Lemma 2.3 shows that $C_4|C_4 \times K_n$. Hence $C_4|K_m \times K_n$.
Case 2. By commutativity of the tensor product we assume that $m \equiv 1 \pmod 4$. The graph $K_m \times K_n = ((K_m \times K_2) \oplus \cdots \oplus (K_m \times K_2))$. Since $m \equiv 1 \pmod 4$, by Corollary 1.1, $C_4|K_{m,m} - I$, and $K_m \times K_2 \cong K_{m,m} - I$. Hence $C_4|K_m \times K_n$. This completes the proof. $\square$

Lastly, we draw our conclusion by the following remark.

Remark 3.1. The product $K_m \times K_n$ can also be viewed as an even regular complete multipartite graph. So by the conditions given in Theorem 1.1 we have that every even regular complete multipartite graph G is C_4-decomposable if the number of edges of G is divisible by 4.

Acknowledgement

The authors are grateful to the anonymous referees for their helpful comments to improve the paper.

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