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Total coloring conjecture on certain classes of product graphs

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Abstract

A total coloring of a graph G is an assignment of colors to the elements of the graph G such that no adjacent vertices and edges receive the same color. The total chromatic number of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. Behzad and Vizing conjectured that for any graph G, $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. In this paper, we prove the Behzad and Vizing conjecture for Indu - Bala product graph, Skew and Converse Skew product graph, Cover product graph, Clique cover product graph and Comb product graph.

Keywords: total coloring, Indu-Bala product, skew and converse skew product, cover product, clique cover product, comb product
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1. Introduction

All the graph should be considered here are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with the sets of vertices $V(G)$ and edges $E(G)$ respectively. A *total coloring* of G is a mapping $f : V(G) \cup E(G) \to C$, where C is a set of colors, satisfying the following three conditions (a)-(c).

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(a) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$,
(b) $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$, and
(c) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The *total chromatic number* of a graph *G*, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of *G*. Behzad [2] and Vizing [23] conjectured (Total Coloring Conjecture (TCC)) that for every graph G, $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$. If a graph G is total colorable with $\Delta(G) + 1$ colors then the graph is called Type - I, and if it is total colorable with $\Delta(G) + 2$ colors but not $\Delta(G) + 1$ colors, then it is Type - II. A graph G is said to be total colorable if the elements of G are colored with at most $\Delta(G) + 2$ colors. This conjecture was verified by Rosenfeld [15] and Vijayaditya [22] for $\Delta(G) = 3$ and by Kostochka [11, 12] for $\Delta(G) \leq 5$. For planar graphs, the conjecture was verified by Borodin [3] for $\Delta(G) \geq 9$. In 1992, Yap and Chew [24] proved that any graph *G* has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq |V(G)| - 5$, where $|V(G)|$ is the number of vertices in G. The adjacent vertex distinguishing index by sums in total proper colorings[14]. In 1993, Hilton and Hind [6] proved that any graph *G* has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq \frac{3}{4}|V(G)|$. In particular, Mc Diarmid and Arroyo [4] proved that the problem of determining the total coloring of μ-regular bipartite graph is NP-hard, $\mu \geq 3$. Direct product, cartesian product, strong product and lexicographic product graphs given by Imrich[8] et la. Recently, Vignesh et al. [21, 16] verified TCC for certain classes of deleted lexicogaphic product graphs. In [20], they also proved that Vertex, Edge and Neighborhood corona products of graphs are type-I graphs. In [10] verified Maximum average degree of list edge-critical graphs and vizing conjecture. Recently [19] analysis On twin edge colorings in m-ary trees. The following theorem is due to Yap [25].

Theorem 1.1. *Let K_n be the complete graph. Then* $\chi''(K_n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ n+1, & \text{if } n \text{ is even.} \end{cases}$

2. Indu - Bala Product Graph

Let G and H be two connected graphs with m and n vertices, respectively. The join of G and H is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{i, j\} : i \in V(G), j \in V(H)\}$. It is denoted by $G \vee H$.

The *Indu-Bala product* of G and H [9], is denoted by $G \blacktriangledown H$ and is obtained from two disjoint copies of the join $G \vee H$ of G and H by joining the corresponding vertices in the two copies of H. The Indu-Bala product is not commutative. That is $G \blacktriangledown H \not\simeq H \blacktriangledown G$. If G and H are two connected graphs with m and n vertices, respectively then the maximum degree is $\Delta(G \blacktriangledown H) = max\{\Delta(G) + n, \Delta(H) + m + 1\}$.

In [9], they obtained the distance spectrum of $G \blacktriangledown H$ in terms of the adjacency spectra of G and H. Also they prove that the class of graphs $\overline{K_n} \blacktriangledown \overline{K_{n+1}}$ has integral distance spectrum. In this section, we prove the Behzad - Vizing conjecture for Indu - Bala product of some classes of the graphs.

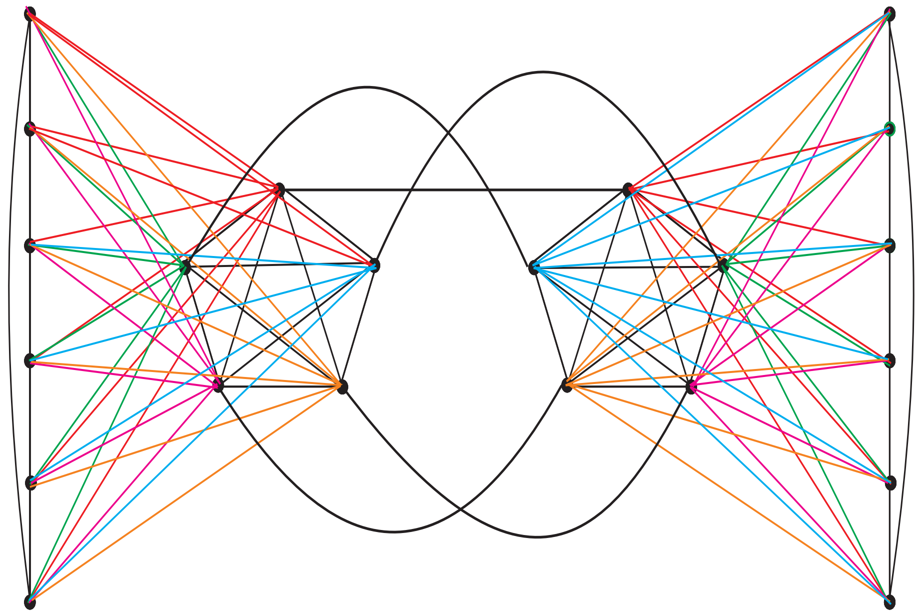

Figure 1. $C_6 \blacktriangledown K_5$

Theorem 2.1. *Let G be total colorable graph with m vertices and H be any graph with n vertices respectively. If $\Delta(G) > \Delta(H)$ and $n \geq m$ then $\chi''(G\blacktriangledown H) \leq \Delta(G\blacktriangledown H) + 2$.*

Proof. The maximum degree $\Delta\left(G\blacktriangledown H\right) = \{\Delta(G) + n\}$. Since G is total colorable, we color the elements of G with $\Delta(G) + 2$ colors. Assign colors to the edges of H using the same $\Delta(G) + 2$ colors. Color all the edges between G and H with n colors other than $\Delta(G) + 2$ colors.

Here, the vertices between G and H may have the same colors. To avoid this, we start recoloring the vertices in H and some join edges between G and H in the following way.

Since $\Delta(G) > \Delta(H)$, there will be at least one color at each vertex in H, which is common to the missing colors at the vertices of G. Remove the colors of n matching edges (having different colors) between G and H in $G \vee H$ and recolor these edges with the missing colors. Color all the vertices of H with the removed colors.

For the second copy of $G \vee H$ in $G\blacktriangledown H$, give the same color assignment as in the first copy for all the elements of G and edges of H. Now for the edges between G and H in the second copy of $G \vee H$, assign the color $c \to (c + 1)$ and take $n + 1$ as 1. Remove the colors of the same n edges with different colors in $G \vee H$ and give the removed colors to the vertices, missing colors to the edges. Note that the corresponding vertices of H in $G\blacktriangledown H$ will receive the different vertex coloring. Since $\Delta(G) > \Delta(H)$, there will be some more colors (at least one) that are not assigned to any of the edges incident with H in both the copies. Assign these missing colors to the edges between the corresponding vertices of H. We use only $\Delta(G\blacktriangledown H) + 2$ colors. Hence the graph $G\blacktriangledown H$ is total colorable. □

Theorem 2.2. *Let H be total colorable graph with n vertices and G be any graph with m vertices respectively. If $\Delta(H) > \Delta(G)$ and $m \geq n$ then $\chi''(G\blacktriangledown H) \leq \Delta(G\blacktriangledown H) + 2$.*

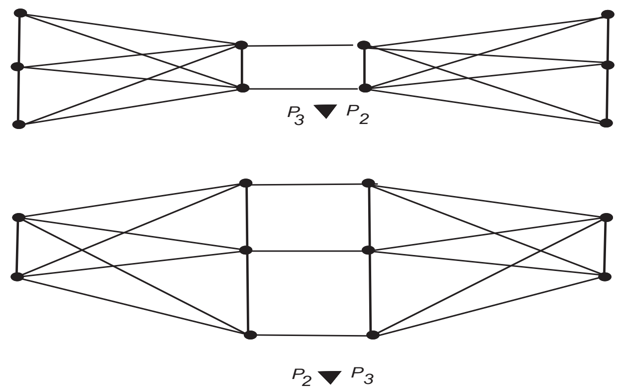

Figure 2. $P_3 \blacktriangledown P_2 \not\cong P_2 \blacktriangledown P_3$

Proof. The maximum degree is $\Delta(G\blacktriangledown H) = \Delta(H) + m + 1$. Since H is total colorable, we color the elements of H using $\Delta(H)+2$ colors. Color all the edges of G with the same $\Delta(H)+2$ colors. Color the join edges between G and H with m colors. Similar to the previous case, we remove the colors of m matching edges (having different colors) between G and H and assign these removed colors to the vertices of G, and assign the missing colors at the vertices of H to these m edges.

For the second copy of $G\blacktriangledown H$, assign the same coloring of H with $c \to (c + 1)$ and take $\Delta(H) + 3$ as 1 to the second copy of G and H. Now for the edges between G and H in $G \vee H$, assign the m colors other than $\Delta(H) + 2$ colors. Remove the colors of the m matching edges (having different colors) in $G \vee H$ and give these removed colors to the vertices of G, assign the missing colors to these m edges. Note that the corresponding vertices of H in $G\blacktriangledown H$ will receive the different vertex coloring.

Since $\Delta(H) > \Delta(G)$, there will be some more colors (at least one) that are not assigned to any of the edges incident with H in both the copies. Assign these missing colors to the edges between the corresponding vertices of H. We use only $\Delta(G\blacktriangledown H)+2$ colors. Hence the graph $G\blacktriangledown H$ is total colorable. □

We have verified the total coloring conjecture in the above theorem for some classes of $G\blacktriangledown H$. In the following theorems, we prove the tight bound of the total coloring conjecture for certain classes of $G\blacktriangledown H$.

Theorem 2.3. *Let G be any graph with m vertices and H be a Type -I graph with n vertices. If $\Delta(G) \leq \Delta(H)$ and $m > n$ then $\chi''(G\blacktriangledown H) = \Delta(G\blacktriangledown H) + 1$.*

Proof. The maximum degree $\Delta(G\blacktriangledown H) = \Delta(H) + m + 1$.
Color all the elements of H using $\Delta(H) + 1$ colors. Assign the edge coloring of H to the edges of G with the same $\Delta(H) + 1$ colors. Let $C = \{c_1, c_2, \ldots, c_m\}$ be a color set with m colors. Color the vertices of G from the color set C. Consider a vertex in G, one color from C is already assigned to that vertex and there are $m - 1$ colors available. Similarly, at each vertex in G, there are $m - 1$ different available colors. Color the join edges between the vertices G and H with these $m - 1$ available colors in a cyclic way.

For the second copy of $G \vee H$ in $G\blacktriangledown H$, assign the color $c \to (c+1)$ and take $\Delta(H)+2$ as 1 and c_{m+1} as c_1. In this coloring assignment, the corresponding vertices of H in both the copies will receive different colors. Assign a new color to the edges between the two copies of H. Therefore, $\chi''(G\blacktriangledown H) = \Delta(G\blacktriangledown H)+1$. □

Theorem 2.4. *Let K_m be a complete graph and H be any graph with n vertices. If $\Delta(H) < m-1 < n$ then $\chi''(K_m\blacktriangledown H) = \Delta(K_m\blacktriangledown H)+1$.*

Proof. Here, $\Delta(K_m\blacktriangledown H) = m+n-1$.

Case 1. m is odd.

From the Theorem 1.1, we know that K_m requires m colors. Color the elements of K_m with m colors. Let $C = \{c_1, c_2, \ldots, c_{n-1}, c_n\}$ be a set of n new colors. Color the vertices in H using all the n colors. Now, based on the vertex coloring of H, color the edges between K_m and H from C with a cyclic way. Color the edges in H using the same m colors such that the adjacent edges receive different colors.

For the second copy of $K_m \vee H$ in $K_m\blacktriangledown H$, we use the same $m+n$ colors from the first copy to color the elements of K_m and the join edges between K_m and H. Change the color c to $c+1$ to color the elements of K_m and the join edges between K_m and H, here, the color $m+1$ is taken as 1 and the color c_{n+1} is taken as c_1. According to this coloring assignment, color the vertices of H and the edges of H in such a way that there is a common missing color between the corresponding vertices of the two copies of H. Now, we give these missing colors to the edges between the two copies of H.

Case 2. m is even.

From the Theorem 1.1, we know that K_m requires $m+1$ colors. Color the elements of K_m with $m+1$ colors. Let $C = \{x_i, c_1, c_2, \ldots, c_{n-1}\}$ be a set of colors, where x_i is the missing color at the i^{th} vertex in K_m and $c_1, \ldots, c_{n-1}$ are the new colors. Consider a set of matching edges between K_m and H and assign the missing colors x_i to the matching edges, which are incident with i^{th} vertex in K_m. Color the remaining join edges between K_m and H from C with a cyclic way. Color the vertices in H using the colors $n-1$ new colors and a color that is not assigned to any of the vertices in K_m. Now, based on the vertex coloring of H, color the edges in H using the same $m+1$ colors such that the adjacent edges receive the different colors.

For the second copy of $K_m \vee H$ in $K_m\blacktriangledown H$, similar to the previous case, we use the colors from the first copy to color the elements of K_m and the join edges between K_m and H. Change the color c to $c+1$ to color the elements of K_m and the join edges between K_m and H, here, the color $m+2$ is taken as 1 and the color c_n is taken as c_1. According to this coloring assignment, color the vertices of H and the edges of H in such a way that there is a common missing color between the corresponding vertices of two copies of H. Now, we give these missing colors to the edges between the two copies of H. Hence, in both the cases, we used $\Delta(K_m\blacktriangledown H)+1$ colors for a total coloring of $K_m\blacktriangledown H$. □

Theorem 2.5. *Let K_n be a complete graph and* **G** *be any graph with m vertices. If $\Delta(G) < n-1 < m$ then $\chi''(G\blacktriangledown K_n) = \Delta(G\blacktriangledown K_n)+1$.*

Proof. Here, $\Delta(G\blacktriangledown K_n) = (n-1)+m+1$.

Case 1. n is odd.

Color the elements of K_n with n colors. Let $C = \{c_1, c_2, \ldots, c_m\}$ be a set of new m colors. Color the vertices of G using m colors from the color set C. Now, based on the vertex coloring, color the join edges between K_n and G from C with a cyclic way. Using the colors of K_n, we assign the colors to the edges of G such that there is no same coloring assignment to the adjacent edges.

For the second copy of $G \vee K_n$ in $G \blacktriangledown K_n$, assign the color $c \to (c+1)$ to the elements, where c is a color in the first copy and take $n+1$ as 1 and c_{m+1} as c_1. In this coloring assignment, the corresponding vertices of two copies of K_n will receive different colors. We give a new color to the edges between the two copies of K_n.

Case 2. n is even.

Color the elements of K_n with $n+1$ colors. Let $C = \{x_i, c_1, c_2, \ldots, c_{m-1}\}$ be a set of colors, where x_i is the missing color at the i^{th} vertex in K_n and $c_1, \ldots, c_{m-1}$ are new colors. Consider a set of matching edges between K_n and G and assign the missing colors x_i to the matching edges, which are incident with i^{th} vertex in K_n. Color the remaining join edges between K_n and G from C with a cyclic way. Color the vertices in G using the colors in C colors and a color that is not assigned to any of the vertices in K_n. Now, based on the vertex coloring of G, color the edges in G using the same $n+1$ colors such that the adjacent edges receive the different colors.

For the second copy of $G \vee K_n$ in $G \blacktriangledown K_n$, assign the color $c \to (c+1)$ to the elements of $G \vee K_n$, where $n+2$ is taken as 1 and c_m is taken as c_1. In this coloring assignment, the corresponding vertices of the two copies of K_n will receive different colors. We give a new color to the edges between the two copies of K_n.

Therefore, $\chi''(G \blacktriangledown K_n) = \Delta(G \blacktriangledown K_n) + 1$. $\square$

3. Skew Product and Converse Skew Product

The Skew product and the converse skew product graphs were introduced by Shibata and Kikuchi [18].

Let G and H be two connected graphs. The *skew product* of G and H, denoted by $G \Delta H$, has the vertex set $V(G) \times V(H)$ and the edge set $E(G \Delta H) = \{((u_1, v_1), (u_2, v_2)) \mid u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)\}$.

The converse skew product of G and H, denoted by $G \nabla H$, has the vertex set $V(G) \times V(H)$ and the edge set $E(G \nabla H) = \{((u_1, v_1), (u_2, v_2)) \mid v_1 = v_2$ and $u_1 u_2 \in E(G)$ or $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)\}$.

In [5] Ziming Duan, et al. considered the skew product and the converse skew product for $L(2,1)$ - labeling. They obtained upper bounds for the $L(2,1)$ - labeling number, which improves the upper bound obtained by Shao and Zhang [17] in many cases.

In this section, we study the total coloring of skew and converse skew product graphs. Figure 3 shows the graph $P_3 \Delta C_4$.

Theorem 3.1. *If H is a total colorable graph then $P_m \Delta H$ is also total colorable.*

Proof. Let H be any total colorable graph with n vertices. The graph $P_m \Delta H$ can be viewed as m copies $H_1, H_2, \ldots, H_m$ of H with direct product edges $E(P_m \times H)$. We know that $\Delta(P_m \Delta H)$

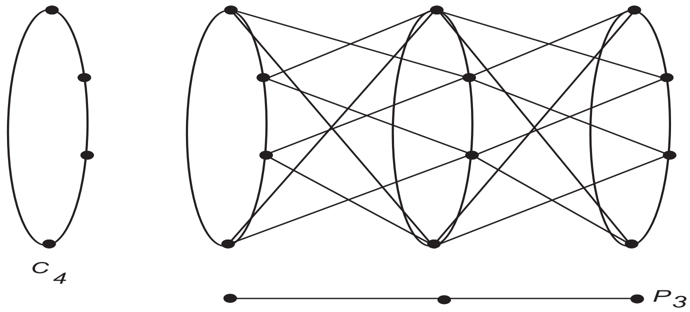

Figure 3. $P_3 \Delta C_4$

= $\Delta(P_m) \times \Delta(H) + \Delta(H)$ = $3\Delta(H)$. First, we color all the copies of H with $\Delta(H) + 2$. Each vertex v in H_i is adjacent to $d(v)$ vertices in H_{i+1} and H_{i-1}. Note that there is no edges between the corresponding vertices in H_{i+1} and H_{i-1}. Now, assign the $\Delta(H)$ colors to edges between the odd and even copies of H and assign another $\Delta(H)$ colors to the even and odd copies H colors. Therefore, $P_m \Delta H$ satisfies TCC. □

The above theorem gives only upper bound. In the following corollary and theorem, we prove the tight bound of the total coloring conjecture.

Corollary 3.1. *If H is any Type - 1 graph then $\chi''(P_m \Delta H) = \Delta(P_m \Delta H) + 1$.*

If H is a Type - 2 graph then $P_m \Delta K_n$ may be Type - 1 or Type - 2. For example, $P_2 \Delta P_2 \simeq C_4$ is a Type-2 graph. In the following theorem, we prove that $P_3 \Delta K_n$ is always Type - 1 for all n.

Theorem 3.2. $\chi''(P_3 \Delta K_n) = \Delta(P_3 \Delta K_n) + 1$.

Proof. Let K_n be a complete graph with n vertices. If n is odd then from the above corollary, it is easy to see that $P_3 \Delta K_n$ is a Type - 1 graph. Let $n = 2k$, $k \geq 2$. Here $\Delta\left(P_3 \Delta K_n\right) = 3(n-1)$. We give a total coloring of K_n as in [7]. $c''_n(i,j) \equiv (\tau_i(j) + \tau_j(i) + 2) \mod (n+1)$, $i \neq j$, $i, j \in [n]_0$ defines a special $(n+1)$-edge coloring of K_n with p colors and color $(p+1) \mod p$ are missing in the line $p \in [n]_0$, where τ_p is the transposition of p and $n-1$. The vertices are colored by the canonical vertex-coloring to obtain a special total coloring of K_n. We give this total coloring of the first copy of K_n in this way. For the second copy we give the same edge coloring as in the first copy of K_n. In this edge coloring, the color p and the color $(p+1) \mod p$ are missing in the line $p \in [n]_0$ (at the vertex $p \in [n]_0$). Assign one of these missing colors to the edge between the first and second copy. Color the vertices in the second copy of K_n with new $n-1$ colors and color $n+1$ (which is not assigned to any of the vertices in the first copy). Now for the edges between the first and second copy, we need $n-2$ colors. At each of the vertices in the second copy there will be $n-2$ colors, we assign these colors to the edges between the two copies. For the edges between the second and third copy, we use the another set of missing colors at the vertices in each of the copies of K_n and new $n-2$ colors. Color K_n in the third layer with same color as first layer.

This gives a total coloring of $P_3 \Delta K_n$ as $3n-2$. □

The above theorems are also true for converse skew product.

4. Cover, Clique Cover and Comb Products

The cover product of two graphs G and H (introduced by Llamas and Bernal [13]) with fixed vertex covers $C(G)$ and $C(H)$ is a graph $G \circledast H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{i,j\} : i \in C(G), j \in C(H)\}$. The cover product is commutative but not symmetric. Figure 4 shows the graph $G \circledast H$. In [13], Llamas and Bernal described the Betti polynomial of $G \circledast H$ in terms of those of G and H. The cover product of two graphs is a generalization of the join of two graphs.

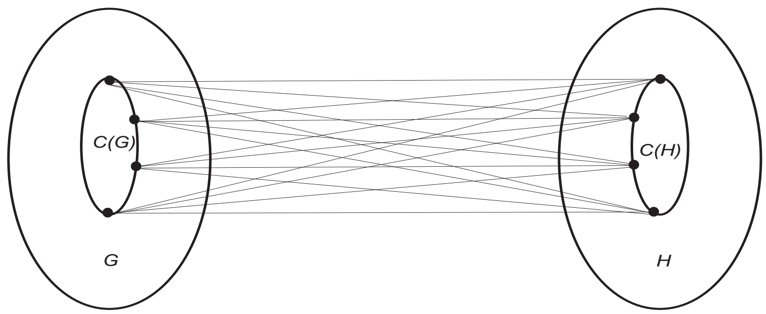

Figure 4. $G \circledast H$

Theorem 4.1. *Let G and H be two total colorable graphs. Let k_1 and k_2 be the vertex covering numbers of G and H, respectively. If either $\Delta(H) \leq \Delta(G)$ and $k_1 \leq k_2$ or $\Delta(G) \leq \Delta(H)$ and $k_2 \leq k_1$ then $\chi''(G \circledast H) \leq \Delta(G \circledast H) + 2$.*

Proof. Let G and H be two total colorable graphs. Let $C(G)$ and $C(H)$ be the minimal vertex cover sets of G and H, respectively and assume that the vertices with maximum degrees of G and H are in $C(G)$ and $C(H)$ respectively. Let $\#C(G) = k_1$ and $\#C(H) = k_2$. The maximum degree $\Delta(G \circledast H)$ = max $\{\Delta(G) + k_2, \Delta(H) + k_1\}$.

Case 1. Suppose $\Delta(H) \leq \Delta(G)$ and $k_1 \leq k_2$.

In this case, $\Delta(G \circledast H) = \Delta(G) + k_2$. Assign the $\Delta(G) + 2$ colors to the elements of G and H. Remove the colors of the vertices in $C(H)$ and assign the k_2 colors to the k_2 vertices in $C(H)$. Each vertex in $C(H)$ is incident with k_1 vertices in $C(G)$. Since $k_1 \leq k_2$, take the k_2 colors and assign to the edges between $C(G)$ and $C(H)$ with a cyclic way.

Case 2. Suppose $\Delta(G) \leq \Delta(H)$ and $k_2 \leq k_1$.

In this case, $\Delta(G \circledast H) = \Delta(H) + k_1$. Assign the $\Delta(H) + 2$ colors to the elements of G and H. Remove the colors of the vertices in $C(G)$ and assign the k_1 colors to the k_1 vertices in $C(G)$. Each vertex in $C(G)$ is incident with k_2 vertices in $C(H)$. Since $k_2 \leq k_1$, take the k_1 colors and assign to the edges between $C(G)$ and $C(H)$ with a cyclic way.

In both cases, we use only $\Delta(G \circledast H) + 2$ colors. Hence the theorem. $\square$

Corollary 4.1. *Let G and H be two Type-I graphs. Let k_1 and k_2 be the vertex covering numbers of G and H, respectively. If either $\Delta(H) \leq \Delta(G)$ and $k_1 \leq k_2$ or $\Delta(G) \leq \Delta(H)$ and $k_2 \leq k_1$ then $\chi''(G \circledast H) = \Delta(G \circledast H) + 1$.*

Let G and H be two graphs. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_q\}$ be a clique cover of G and U be a subset of $V(H)$. A new graph operation called clique cover product (introduced by Bao-Xuan Zhu) [26], denoted by $G^{\mathscr{C}} \star H^U$, as follows: for each clique $C_i \in \mathscr{C}$, add a copy of the graph H and join every vertex of C_i to every vertex of U.

For example, consider the two graphs G and H with $V(G) = \{u_1, \ldots, u_6\}$ and $V(H) = \{v_1, \ldots v_5\}$. Let $\mathscr{C} = \{K_4 = \{u_1, u_2, u_5, u_6\}, K_3 = \{u_5, u_3, u_4\}, K_2 = \{u_2, u_3\}\}$ be a clique cover of G and $U = \{v_2, v_3, v_4\}$ be a subset of $V(H)$. Figure 5 shows an example of $G^{\mathscr{C}} \star H^U$. In [26], Bao-Xuan Zhu showed the clique cover product of some graphs preserves symmetry, unimodality, log - concavity or reality of zeros of independence polynomials.

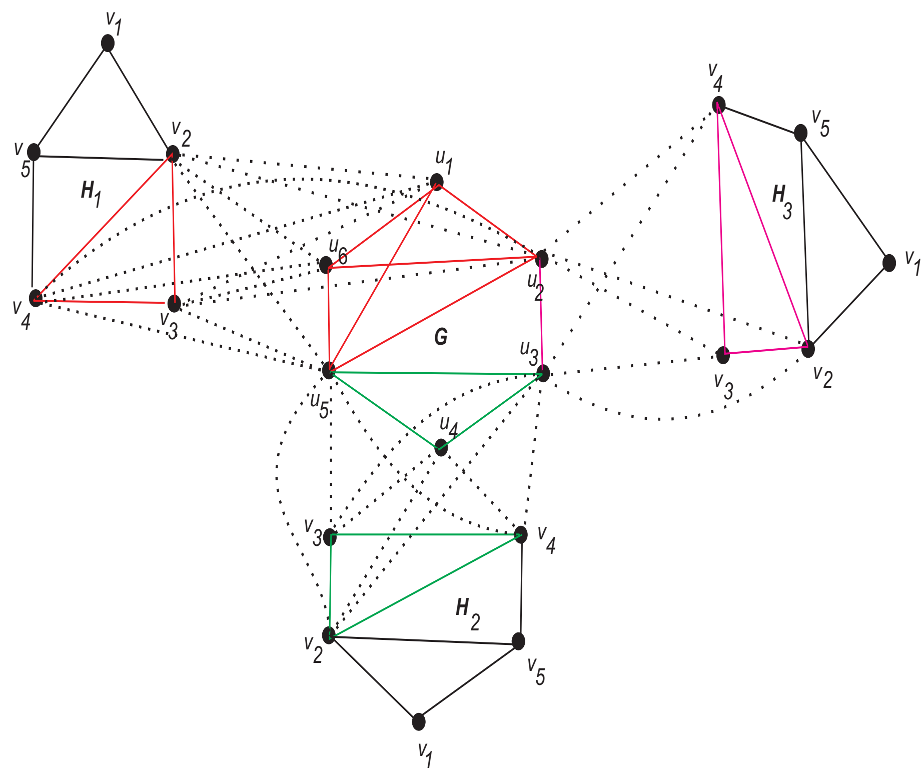

Figure 5. $G^{\mathscr{C}} \star H^U$

Theorem 4.2. *Let G and H be two total colorable graphs with $\Delta(H) \leq \Delta(G)$. Then $G^{\mathscr{C}} \star H^U$ is also total colorable.*

Proof. Let G and H be two total colorable graphs with $\Delta(H) \leq \Delta(G)$. Let k be the clique number of G. Choose a subset U in $V(H)$ such that $|U| = r \geq k$.

Here , $\Delta(G^{\mathscr{C}} \star H^U) = \Delta(G) + r$. Let $H_1, H_2, \ldots, H_k$ be the k copies of H corresponding to the k cliques in G. Assign the $\Delta(G)+2$ colors to the elements of G and $H_1, H_2, \ldots, H_k$. Consider the first clique and H_1. Remove the colors of the vertices in U of H_1 and assign the r colors to the vertices in U of H_1. Each vertex in U of H_1 is incident with the first clique in G. Since the clique size $k \leq r$, take the r colors and assign to the edges between U of H_1 and the first clique in G with a cyclic way. The same procedure can be applied to color the vertices of U in $H_2, \dots, H_k$ and the join edges between the cliques and U in $H_2, \dots, H_k$. Therefore $\chi''(G^{\mathscr{C}} \star H^U) \leq \Delta(G)+r+2$. $\square$

The *comb product* graph was introduced by Accardi, Ghorbal and Obata [1]. Let G and H be two graphs. The comb product of G and H with a distinguished vertex $o \in V(H)$ is by definition a graph obtained by grafting a copy of H at vertex o into each vertex of G. This comb product is denoted by $G \rhd_o H$.

In other words, $G \rhd_o H$ is a graph with $V(G \rhd_o H) = \{(g,h) | \, g \in V(G)$ and $h \in V(H)\}$ and $E(G \times H) = \{((g,h),(g',h')) | \, gg' \in E(G)$ and $h = h' = o$; or $g = g'$ and $hh' \in E(H)\}$, where $o \in V(H)$ is the distinguished vertex $V(H)$. Figure 4 shows the graph $G \rhd_o H$.

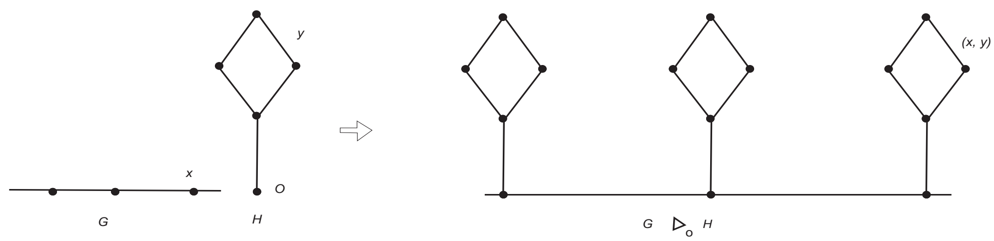

Figure 6. $G \rhd_o H$

Theorem 4.3. *If G and H are two total colorable graphs then $G \rhd_o H$ is also total colorable.*

Proof. Let G be a total colorable graph with n vertices and $H_1, H_2, \dots, H_n$ be the n copies of H. Let $o \in V(H)$ be the distinguished vertex in H. The maximum degree $\Delta(G \rhd_o H)$ = $\max\{\Delta(G)+\Delta(H), \Delta(G)+deg(o), \Delta(H)\}$.

Case 1. Suppose $\Delta(G \rhd_o H) = \Delta(G) + \Delta(H)$.

In this case, the distinguished vertex becomes the vertex of maximum degree. Since G is total colorable, we give $\Delta(G) + 2$ colors to the elements of G. Now, we have to the color the elements of $H_i, i = 1, 2, \dots, n$. Since the vertex $o \in H_i$ is merged with i^{th} vertex in G, the vertex o is colored and there will be at least one missing color at o. As H is total colorable, the i^{th} copy H_i requires $\Delta(H) + 2$ colors. Now, we use the missing colors at o and the color of o with extra $\Delta(H)$ colors to color the elements of $H_i, i = 1, 2, \dots, n$.

Case 2. Suppose $\Delta(G \rhd_o H) = \Delta(G) + deg(o)$.

Color the elements of G with $\Delta(G)+2$ colors. At the vertex o, there will be at least one missing color. We use the color of o, the missing colors at o, the colors that are not used to the edges of G incident at o and deg(o) colors to color the elements of $H_i, i = 1, 2, \dots, n$.

Case 3. Suppose $\Delta(G \rhd_o H) = \Delta(H)$.

In this case $\Delta(G) < \Delta(H)$. First color the elements of G with $\Delta(H) + 2$ colors. At the vertex o, we have used at most $\Delta(G) + 1$ colors to color the vertex o and edges incident at o in G. Now, we color the elements of H_i with the vertex color o, missing colors at o and the remaining colors.

In all cases, we use only $\Delta(G \rhd_o H) + 2$ colors to color the elements of $G \rhd_o H$. Hence $G \rhd_o H$ is total colorable. $\square$

Corollary 4.2. *If G and H are Type-I graphs then $G \rhd_o H$ is also a Type-I graph.*

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