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Embedding partial 3-star designs

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Abstract

Define a 3-star decomposition of K_n as being a collection of subgraphs, each isomorphic to $K_{1,3}$, with the property that each edge of K_n appears in exactly one of the subgraphs. A partial 3-star decomposition is similarly defined except each edge appears in at most one of the subgraphs. In this work, it is shown that any partial 3-star decomposition of K_n can be embedded into a decomposition of K_{n+s} where $s \leq 4$. Furthermore, we determine, for any maximal partial 3-star decomposition $\mathscr{P}$ of K_n, the minimum $s \in \{1, 2, 3, 4\}$ such that $\mathscr{P}$ can be embedded into a decomposition of K_{n+s}.

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1. Introduction

All graphs will be simple and finite. Define a k-star to be the complete bipartite graph $K_{1,k}$. A k-star decomposition of the complete graph K_m consists of a collection $\mathscr{B}$ of subgraphs, each isomorphic to $K_{1,k}$, with the property that each edge of K_m appears in exactly one of the subgraphs. A partial k-star decomposition $\mathscr{P}$ of K_n is similarly defined except each edge appears in at most one of the subgraphs. If $\mathscr{P} \subseteq \mathscr{B}$, say that $\mathscr{P}$ is *embeddable* in $\mathscr{B}$, or alternately that $\mathscr{B}$ is

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an *embedding* of $\mathscr{P}$. We will designate by G_0 the graph formed by deleting from K_n all edges found in any subgraph of $\mathscr{P}$, where in various literature G_0 is termed the *leave* of the partial decomposition. Furthermore, it is free to assume in our work that $\mathscr{P}$ is maximal – that is, all vertices of the corresponding G_0 have degree less than k. We will be primarily concerned with the case $k = 3$, and although the graph $K_{1,3}$ is commonly called a "claw", we will refer to it as being a 3-star to keep with the terminology used in [2].

In [2], Hoffman and Roberts prove that, regardless of the value of n, any partial k-star decomposition of K_n can be embedded into a k-star decomposition of K_{n+s} where s is at most $7k-4$ if k is odd, and $8k-4$ if k is even. These bounds are lowered by the authors in [3], where it is shown that such an s can be selected which is at most $3k-2$ for k odd and $4k-2$ for k even. Bounds on s are lowered even further in [1], where De Vas Gunasekara and Horsley prove that s can be chosen satisfying $s < \frac{9}{4}k$ when k is odd and $s < (6-2\sqrt{2})k$ when k is even. Furthermore, it is shown in [1] that those bounds are optimal in the sense that they cannot be improved for general k.

In our work, we consider the specific case of $k = 3$, showing that $s = 4$ is a sharp upper bound. We conclude by determining, for each maximal partial 3-star decomposition $\mathscr{P}$, the minimum $s \in \{1, 2, 3, 4\}$ such that $\mathscr{P}$ is embeddable into some decomposition of K_{n+s}.

2. Preliminaries

Our method of proof will be constructive and we illustrate it with the following example. Consider the partial decomposition $\mathscr{P}$ of K_6 given in Figure 1, where G_0 has edge set $\{v_1v_5, v_1v_6, v_3v_4\}$.

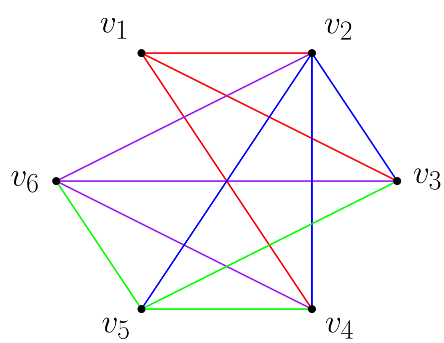

Figure 1. A Partial 3-Star Decomposition of K_6

Consider the adjacency matrix of G_0, looking only at the triangular portion of the matrix below the main diagonal. The edges of the original copy of K_6 are in bijective correspondence with the cells of this portion of the matrix. Form a "staircase" diagram by coloring any cell containing a 0 and making blank any cell containing a 1, as is done in Figure 2.

Three cells in such a diagram correspond to the edges of a 3-star if and only if they appear in the same row, or they appear in the same column, or they appear in rows and columns that intersect in a cell on the main diagonal of the adjacency matrix. In particular, since any partial 3-star decomposition we are considering can be assumed maximal, each column of a staircase diagram for a potential G_0 can have zero, one, or two blank cells only.

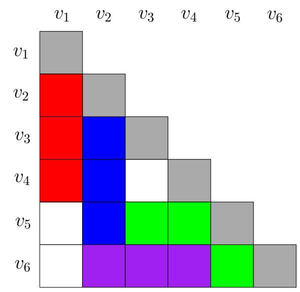

Figure 2. The Staircase Diagram for the Partial Decomposition Given in Figure 1

In forming an embedding of $\mathscr{P}$, we extend the diagram by affixing s additional rows, for some $s \in \{1, 2, 3, 4\}$, which represent the additional vertices $x_1, \ldots, x_s$. For our example, it turns out that $s = 3$ does the trick. For each additional row, we take note of the number of cells in that row computed modulo 3. In the staircase diagram for G_0, for each column containing one blank cell, we may then extend our partial decomposition by coloring that blank cell along with two blank cells simultaneously lying in that column and in the affixed rows. We call such a maneuver a *2-drop* as we are "dropping" two colored cells into the additional rows. Similarly, for each column in the G_0 diagram containing two blank cells, we may color those cells along with one cell lying in that column and in an additional row. Refer to this move as a *1-drop*. Returning to our example, these moves are executed in Figure 3.

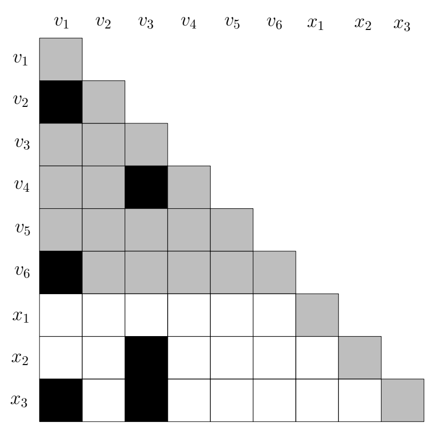

Figure 3. An Extended Staircase Diagram for the Partial Decomposition Given in Figure 1

Note that, in Figure 3, after the described 1- and 2-drops have been made, the remaining numbers of uncolored cells in the rows corresponding to vertices x_1, x_2, and x_3 are each congruent to 0 modulo 3. What remains to be seen (and what we intend to show in the next section) is that this process of designating 1-drops and 2-drops can *always* be performed with a suitable $s \in \{1, 2, 3, 4\}$ so that all cells are colored in the original diagram for G_0, and the number of uncolored cells in each additional row is congruent to 0 modulo 3. This circumstance allows us to complete our decomposition of K_{n+s} by iteratively coloring sets of three cells, with each set consisting of three cells lying in the same additional row.

3. Results

Throughout this section, we will use the fact that $|E(K_n)| \equiv 0 \pmod 3$ if $n \equiv 0$ or $1 \pmod 3$ and $|E(K_n)| \equiv 1 \pmod 3$ if $n \equiv 2 \pmod 3$.

Theorem 3.1. *Let $n \equiv 0$ or $1 \pmod 3$. A partial 3-star decomposition $\mathscr{P}$ of K_n is embeddable in a 3-star decomposition of K_{n+3}.*

Proof. Assume $\mathscr{P}$ maximal, and consider a staircase diagram of G_0. Let α_1 designate the number of columns in the diagram having one blank cell, and denote by α_2 the number of columns having two blank cells. Since $\mathscr{P}$ is only a partial decomposition, we have $\alpha_1 + \alpha_2 > 0$. Also, since $|E(K_n)| \equiv 0 \pmod 3$, we must have $\alpha_1 + 2\alpha_2 \equiv 0 \pmod 3$, which implies $\alpha_1 \equiv \alpha_2 \pmod 3$. Affix three additional rows to the staircase diagram and note that the number of cells in these three rows will be congruent to $0, 1,$ and $2 \pmod 3$, respectively, with the order of those depending on whether $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$. Regardless, labeling the additional rows R_1, R_2, R_3, and letting r_i be the total number of empty cells in row R_i, we may assume $r_1 \equiv 1 \pmod 3$, $r_2 \equiv 2 \pmod 3$, and $r_3 \equiv 0 \pmod 3$. We now consider cases.

Case 1. $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 3$
We have α_1 2-drops to make and α_2 1-drops to make. Place all 2-drops in rows R_1 and R_2, along with placing all 1-drops in row R_2.

Case 2. $\alpha_1 \equiv \alpha_2 \equiv 2 \pmod 3$
Place all 2-drops in rows R_1 and R_2. Place all 1-drops in row R_1.

Case 3. $\alpha_1 \equiv \alpha_2 \equiv 0 \pmod 3$
Since $\alpha_1 + \alpha_2 > 0$, here we must have α_1 and α_2 at least 3. Place one 1-drop in row R_1 and the rest in R_2. Place all 2-drops in rows R_1 and R_2. □

Theorem 3.2. *Let $n \equiv 2 \pmod 3$. A partial 3-star decomposition $\mathscr{P}$ of K_n is embeddable in a 3-star decomposition of K_{n+4}.*

Proof. Assume $\mathscr{P}$ maximal, and define α_1 and α_2 as in the proof of Theorem 3.1. Consider a staircase diagram of G_0. As $|E(K_n)| \equiv 1 \pmod 3$, we have $\alpha_1 + 2\alpha_2 \equiv 1 \pmod 3$, which can be expressed as $\alpha_1 \equiv \alpha_2 + 1 \pmod 3$. Affix four additional rows R_1, R_2, R_3, R_4 to the staircase diagram, and for $i \in \{1, 2, 3, 4\}$, let r_i designate the number of empty cells in row R_i before any "drops" have been made. Note that $r_1 \equiv 2 \pmod 3$, $r_2 \equiv 0 \pmod 3$, $r_3 \equiv 1 \pmod 3$, and $r_4 \equiv 2 \pmod 3$. Also, when it is advantageous, we may select a column which has no blank cells in the G_0 diagram (the column corresponding to v_n, for example) and color three cells that simultaneously lie in that column and in three of the additional rows. This maneuver, a "3-drop" using the terminology we have introduced, will play a factor in the cases that follow.

Case 1. $\alpha_1 \equiv 1 \pmod 3$, $\alpha_2 \equiv 0 \pmod 3$
Here we have α_1 2-drops, α_2 1-drops, and optional access to at least one 3-drop. Our plan will then be to place the 3-drop in rows R_1, R_3, and R_4. Place all 2-drops in rows R_1 and R_4. Place all 1-drops in row R_1.

Case 2. $\alpha_1 \equiv 2 \pmod 3$, $\alpha_2 \equiv 1 \pmod 3$
Place all 2-drops in rows R_1 and R_4. Place all 1-drops in row R_3.

Case 3. $\alpha_1 \equiv 0 \pmod 3$, $\alpha_2 \equiv 2 \pmod 3$
Again, place a 3-drop in rows R_1, R_3, and R_4. Place all 2-drops in rows R_1 and R_2. Conclude by placing one 1-drop in row R_1 and the rest in row R_4. □

The above two theorems establish an upper bound for the minimum s such that any partial 3-star decomposition of K_n can be embedded into a decomposition of K_{n+s}. To see that $s = 4$ cannot be lowered, we need only observe the theorem below, originally proved by Yamamoto, et al. [6], and also seen as a consequence of more general results of Tarsi in [4] and [5].

Theorem 3.3. *The complete graph K_n admits a k-star decomposition if and only if $|E(K_n)| \equiv 0 \pmod k$ and $n \geq 2k$.*

It then stands that a partial k-star decomposition of K_2, wherein $\mathscr{P}$ would be empty, cannot be embedded into a k-star decomposition of K_n for any $n < 2k$. For the general problem studied in [2], this gives a lower bound on the minimum s such that $\mathscr{P}$ can be embedded into a k-star decomposition of K_{n+s}, with the bound being $s = 2k - 2$. However, for our case of $k = 3$, we were able to determine, for any given $\mathscr{P}$, exactly the minimum $s \in \{1, 2, 3, 4\}$ needed. We do this in the following two theorems.

Theorem 3.4. *Let $\mathscr{P}$ be a maximal partial 3-star decomposition of K_n. Then $\mathscr{P}$ can be embedded into a decomposition of K_{n+1} if and only if the following hold.*

(i) $n \equiv 0$ *or* $2 \pmod 3$

(ii) Each component of G_0 is an isolated vertex, an even cycle, or a path with an odd number of vertices.

Proof. The first condition is obvious as it ensures $|E(K_{n+1})|$ is a multiple of 3. To see that the second condition is sufficient, place a new vertex x, making it adjacent to every vertex of G_0. Let C_{2r} be a cycle of G_0 with vertices $c_1, \ldots, c_{2r}$, and let P_{2s+1} be a path of G_0 with vertices $p_1, \ldots, p_{2s+1}$. Consider every other vertex of C_{2r} – that is, $c_1, c_3, \ldots, c_{2r-1}$ and for each, identify a 3-star of $G_0 + \{x\}$ centered at that vertex. Consider vertices $p_2, p_4, \ldots, p_{2s}$ of the path and for each of these, identify a 3-star centered at that vertex. Repeat this process for any other even cycles or odd paths in G_0. The only edges of $G_0 + \{x\}$ not appearing in any of these identified 3-stars each have x as one of their endpoints. We can then complete the decomposition of $G_0 + \{x\}$ through use of 3-stars centered at x.

For necessity, suppose that G_0 contains an odd cycle C_{2r+1}. When attempting to decompose $G_0 + \{x\}$, note that any edge of C_{2r+1} must appear in a 3-star centered at a vertex of the cycle. However, any such 3-star contains two edges of C_{2r+1}, and it follows that all $2r + 1$ edges cannot appear in a decomposition of $G_0 + \{x\}$. Now suppose that G_0 contains a path P_{2s}. The same argument applies, as P_{2s} contains an odd number of edges, and any 3-star of $G_0 + \{x\}$ contains either zero or two edges of P_{2s}. □

Theorem 3.5. *Let $\mathscr{P}$ be a maximal partial 3-star decomposition of K_n. Then $\mathscr{P}$ can be embedded into a decomposition of K_{n+2} if and only if the following hold.*

(i) $n \equiv 1$ *or* $2 \pmod 3$

(ii) G_0 *has at least one vertex of degree 2.*

Proof. The first condition ensures $|E(K_{n+2})|$ is a multiple of 3. For necessity of the second condition, assume G_0 has only vertices of degree 0 or 1, and consider a potential decomposition of $G_0 + K_2$ where $V(K_2) = \{x, y\}$. Any edge $e = ab$ of G_0 must appear in a 3-star centered at a or b. Removing from $G_0 + K_2$ all such 3-stars centered at vertices of G_0, we are left with a graph G_1 in which $\deg(x) = \deg(y)$ and x, y are the only vertices of degree greater than 2. Consequently, any further 3-stars in G_1 must be centered at x or y. Without loss of generality, assume that in the decomposition, edge $e = xy$ appears in a 3-star centered at x. It then follows that, in G_1, $\deg(x) \equiv 0 \pmod 3$. This implies $\deg(y) - 1 \equiv 0 \pmod 3$, giving us a contradiction.

For sufficiency of the second condition, let $v_1 \in V(G_0)$ where $\deg(v_1) = 2$. We again consider a staircase diagram of G_0 and note that the leftmost column has two blank cells. Affix two additional rows to the diagram labeled R_1 and R_2 where again, r_1, r_2 denote the total number of empty cells in the corresponding row. Define α_1 and α_2 as is done in the proofs of Theorems 3.1 and 3.2.

Case 1. $n \equiv 1 \pmod 3$

We have $\alpha_1 \equiv \alpha_2 \pmod 3$ with $r_1 \equiv 1 \pmod 3$ and $r_2 \equiv 2 \pmod 3$. As we only have two additional rows, we are forced to place all 2-drops in rows R_1 and R_2. If $\alpha_2 \equiv 1 \pmod 3$, place all 1-drops in row R_2. If $\alpha_2 \equiv 2 \pmod 3$, instead place all 1-drops in row R_1. And if $\alpha_2 \equiv 0 \pmod 3$ (which necessarily implies $\alpha_2 \geq 3$), place one 1-drop in row R_1 and the rest in row R_2.

Case 2. $n \equiv 2 \pmod 3$

Here, $\alpha_1 \equiv \alpha_2 + 1 \pmod 3$, $r_1 \equiv 2 \pmod 3$, and $r_2 \equiv 0 \pmod 3$. Again, we are forced to place all 2-drops in rows R_1 and R_2. If $\alpha_2 \equiv 1 \pmod 3$, place all 1-drops in row R_2. If instead $\alpha_2 \equiv 2 \pmod 3$, place all 1-drops in row R_1. And if $\alpha_2 \equiv 0 \pmod 3$ (which again means $\alpha_2 \geq 3$), place one 1-drop in row R_1 and the rest in row R_2. □

Finally, we note that for any $n \equiv 2 \pmod 3$, there indeed exists a partial 3-star decomposition $\mathscr{P}$ of K_n that requires $s \geq 4$ to embed $\mathscr{P}$ into a 3-star decomposition of K_{n+s}. To see this, in light of the previous two theorems, we just need to find $\mathscr{P}$ in which G_0 has each vertex of degree less than 2. For $n = 2$, this is obvious. For $n = 5$, we have the partial 3-star decomposition given in Figure 4.

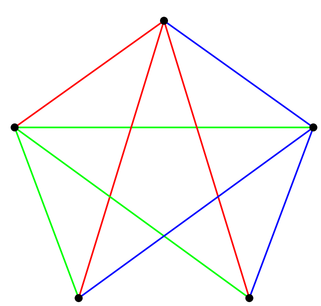

Figure 4. A Partial 3-star Decomposition of K_5

If $n \geq 8$, label the vertices of K_n as $v_1, \ldots, v_n$, and by Theorem 3.3, there exists a 3-star decomposition of the induced subgraph of K_n on vertices $v_1, \ldots, v_{n-1}$. We have $\deg(v_n) \equiv 1 \pmod{3}$ so we can remove 3-stars centered at v_n until only one edge remains.

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