Electronic Journal of Graph Theory and Applications **2** (2) (2014), 110–114

Electronic Journal of
Graph Theory and Applications

A remark on star-C_4 and wheel-C_4 Ramsey numbers

Yanbo Zhang[a,b], Hajo Broersma[b], Yaojun Chen[a]

[a]*Department of Mathematics, Nanjing University, Nanjing 210093, P.R. China*
[b]*Faculty of Electrical Engineering, Mathematics and Computer Science,*
University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

ybzhang@163.com, h.j.broersma@utwente.nl, yaojunc@nju.edu.cn

Abstract

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G_1 is a subgraph of G, or G_2 is a subgraph of the complement of G. Let C_n denote a cycle of order n, W_n a wheel of order $n+1$ and S_n a star of order n. In this paper, it is shown that $R(W_n, C_4) = R(S_{n+1}, C_4)$ for $n \geq 6$. Based on this result and Parsons' results on $R(S_{n+1}, C_4)$, we establish the best possible general upper bound for $R(W_n, C_4)$ and determine some exact values for $R(W_n, C_4)$.

Keywords: Ramsey number; star; wheel; quadrilateral
Mathematics Subject Classification : 05C55, 05D10

1. Introduction

In this note we deal with finite simple graphs only. For a nonempty proper subset $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraph induced by S and $V(G) - S$, respectively. Let $N_S(v)$ be the set of all the neighbors of a vertex v that are contained in S, let $N_S[v] = N_S(v) \cup \{v\}$ and let $d_S(v) = |N_S(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For two vertex-disjoint graphs G_1 and G_2, $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2. A star, a cycle and a complete graph of order n are denoted by S_n, C_n and K_n, respectively. A wheel $W_n = K_1 + C_n$ is a graph of order $n+1$. We use

Received: 01 May 2014, Revised: 01 September 2014, Accepted: 07 September 2014.

$\Delta(G)$, $\delta(G)$ and $\alpha(G)$ to denote the maximum degree, the minimum degree and the independence number, respectively, of a graph G.

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. It is well-known that it is difficult to deal with some extremal problems involving C_4. In this note, we are interested in the relationship between two Ramsey numbers involving C_4, that is, $R(S_{n+1}, C_4)$ and $R(W_n, C_4)$. The former has been well-studied and the latter has received more attention recently.

Parsons [6] began to consider the Ramsey numbers $R(S_{n+1}, C_4)$ back in 1975. By using the existence of projective planes over Galois fields and the generalized friendship theorem, in [6] he established upper bounds for $R(S_{n+1}, C_4)$ and determined the exact values for several specific values of n, as expressed in the following two results.

Theorem 1.1. *(Parsons [6]).* $R(S_{n+1}, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 2$ *for all* $n \ge 2$*, and if* $n = q^2 + 1$ *and* $q \ge 1$*, then* $R(S_{n+1}, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 1$.

Theorem 1.2. *(Parsons [6]). If* q *is a prime power, then* $R(S_{q^2+1}, C_4) = q^2+q+1$ *and* $R(S_{q^2+2}, C_4) = q^2 + q + 2$.

Noting that if $n = q^2$, then $n + \lfloor\sqrt{n-1}\rfloor + 2 = q^2 + q + 1$, we see that the general bound for $R(S_{n+1}, C_4)$ in Theorem 1.1 is best possible.

Obviously, S_{n+1} is a (spanning) subgraph of W_n and so $R(W_n, C_4) \ge R(S_{n+1}, C_4)$. By using an exhaustive computer search, Tse [10] was able to calculate the value of $R(W_n, C_4)$ for $3 \le n \le 12$. An interesting question in this respect is: what is the best possible upper bound for $R(W_n, C_4)$? Surahmat et al. [9] showed that $R(W_n, C_4) \le n + \lceil n/3\rceil + 1$ for $n \ge 6$. Clearly, this upper bound is not tight in general. Because $R(W_n, C_4) \ge R(S_{n+1}, C_4)$ showing that the best bound for $R(W_n, C_4)$ is at least $n + \lfloor\sqrt{n-1}\rfloor + 2$, one may ask whether $R(W_n, C_4) - R(S_{n+1}, C_4)$ is a constant or a function depending on n. Recently, by using Reiman's theorem [8] on the Turán number $t(n, C_4)$, Ore's theorem [5] on Hamiltonicity, a result of Faudree and Schelp [3] on $R(C_n, C_4)$ and the Erdős-Rényi graph, Dybizbański and Dzido [2] established a general upper bound for $R(W_n, C_4)$ for $n \ge 10$ and determined some exact values of $R(W_n, C_4)$. We summarized some of their results in the following theorem.

Theorem 1.3. *(Dybizbański and Dzido [2]).* $R(W_n, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 2$ *for all* $n \ge 10$*, and if* $q \ge 4$ *is a prime power, then* $R(W_{q^2}, C_4) = q^2 + q + 1$.

In the same paper, with the help of computers, they determined the exact values of some Ramsey numbers for a small wheel versus a C_4.

Theorem 1.4. *(Dybizbański and Dzido [2]).* $R(W_n, C_4) = n + 5$ *for* $13 \le n \le 16$.

Clearly, Theorem 1.3 implies that Parsons' bound for $R(S_{n+1}, C_4)$ is also a best possible upper bound for $R(W_n, C_4)$ if $n \ge 10$. In an unpublished paper, Wu et al. [11] obtained nine new values for $R(W_n, C_4)$; as in the other cases their calculations have been performed with the aid of computer search.

Theorem 1.5. *(Wu et al. [11])* $R(W_n, C_4) = n + 5$ *for* $17 \leq n \leq 20$*;* $R(W_{26}, C_4) = 32$*;* $R(W_n, C_4) = n + 7$ *for* $34 \leq n \leq 36$*;* $R(W_{43}, C_4) = 51$.

The exact values of the Ramsey numbers $R(S_{n+1}, C_4)$ for $n \leq 6$ can be found in [7]. For the value of $R(S_8, C_4)$, we get $R(S_8, C_4) \leq 11$ by Theorem 1.1. Since the Petersen graph contains no C_4 and its complement has no S_8, we get $R(S_8, C_4) \geq 11$ and so we obtain that $R(S_8, C_4) = 11$. Using Theorem 1.2, we can get the exact values of $R(S_{n+1}, C_4)$ for $n = 9, 10, 16, 17$. By considering Theorems 1.1, 1.2 and 1.3, and these known values of $R(S_n, C_4)$ and $R(W_n, C_4)$ for small $n \geq 6$, we observe that there is an infinite number of values of n for which $R(W_n, C_4) = R(S_{n+1}, C_4)$. Motivated by this observation, a natural question is whether this equality holds in general. In this note, we give an affirmative answer to this question. Our main result is as follows.

Theorem 1.6. $R(W_n, C_4) = R(S_{n+1}, C_4)$ *for* $n \geq 6$.

We postpone our proof of this result to the next section.

By Theorem 1.6, we see that the two functions $R(W_n, C_4)$ and $R(S_{n+1}, C_4)$ are in fact the same when $n \geq 6$. Because the Ramsey numbers $R(S_{n+1}, C_4)$ are well-studied, we can use Theorem 1.6 and known results on $R(S_{n+1}, C_4)$ to establish new results on $R(W_n, C_4)$. Of course, we can do that in reverse as well. Up to now, most known values of $R(W_n, C_4)$ for small n are obtained with the help of computers. Because finding an S_{n+1} is much easier than finding a W_n in a graph using computers, we can focus our calculation on $R(S_{n+1}, C_4)$ by computers instead of $R(W_n, C_4)$ if we want to determine some values of $R(W_n, C_4)$ with the help of computers.

Combining Theorems 1.1, 1.2 and 1.6, we obtain the following.

Theorem 1.7. $R(W_n, C_4) \leq n + \lfloor\sqrt{n-1}\rfloor + 2$ *for* $n \geq 6$*, and if* $n = q^2 + 1$ *and* $q \geq 3$*, then* $R(W_n, C_4) \leq n + \lfloor\sqrt{n-1}\rfloor + 1$*. Furthermore, if* $q \geq 3$ *is a prime power, then we have* $R(W_{q^2}, C_4) = q^2 + q + 1$ *and* $R(W_{q^2+1}, C_4) = q^2 + q + 2$.

Clearly, Theorem 1.7 is stronger than Theorem 1.3. Furthermore, by Theorems 1.1-1.7 and some other known results on $R(S_{n+1}, C_4)$, we can summarize several exact values (see Table 1) for $R(W_n, C_4)$ and $R(S_{n+1}, C_4)$ when $n \geq 6$ is small. Here the numbers marked with $*$ are obtained from the results in this paper, and the numbers marked with $\star$ can be obtained by Theorem 1.7 avoiding computer search.

n	6	7-8	9-10	11-15	16-17	18-20	25	26	34-36	43
$R(W_n, C_4)$	9	$n+4$	$n+4^\star$	$n+5$	$n+5^\star$	$n+5$	31	$32^\star$	$n+7$	51
$R(S_{n+1}, C_4)$	9	$n+4$	$n+4$	$n+5^*$	$n+5$	$n+5^*$	31	32	$n+7^*$	51^*

Table 1: Exact values of $R(W_n, C_4)$ and $R(S_{n+1}, C_4)$ for $6 \leq n \leq 43$.

As for the lower bounds of $R(S_{n+1}, C_4)$, Burr et al. [1] showed that $R(S_{n+1}, C_4) > n + \sqrt{n} - 6n^{11/40}$. In the same paper, they proposed the following conjecture, for which Erdős, one of the authors, offered \$100 for a proof or disproof.

Conjecture 1. *(Burr et al. [1]).* $R(S_{n+1}, C_4) < n + \sqrt{n} - c$ *holds infinitely often, where* c *is an arbitrary constant.*

After an easy calculation, we find that all exact values of $R(S_{n+1}, C_4)$ listed in Table 1 satisfy $R(S_{n+1}, C_4) \geq n + \lceil\sqrt{n}\rceil$. Thus we finish this section by posing the following intriguing problem.

Question. *Is it true that* $R(W_n, C_4) = R(S_{n+1}, C_4) \geq n + \lceil\sqrt{n}\rceil$ *for all* $n \geq 6$*?*

2. Proof of Theorem 1.6

In order to prove Theorem 1.6, we need the following four lemmas.

Lemma 2.1. *(Faudree and Schelp [3]).* $R(C_n, C_4) = n + 1$ *for* $n \geq 6$.

Lemma 2.2. *(Tse [10]).* $R(W_6, C_4) = 9$*,* $R(W_n, C_4) = n + 4$ *for* $7 \leq n \leq 10$.

Lemma 2.3. *(Faudree et al. [4]).* $R(S_7, C_4) = 9$.

Lemma 2.4. *(Zhang et al. [12]) Let C be a longest cycle in a graph G and $u \in V(G) - V(C)$. Then $\alpha(G) \geq d_C(u) + 1$.*

Proof of Theorem 1.6. We first prove that $R(S_{n+1}, C_4) \geq n+4$ for $n \geq 7$. Let $k = \lfloor (n+1)/4 \rfloor$ and $C = x_1x_2...x_{4k}x_1$ be a cycle of length $4k$. Set $X_1 = \{x_1, x_2\}$, $X_2 = \{x_3, x_4\}$, $X_3 = \{x_i \mid i \equiv 1, 2 \pmod 4$ and $i \geq 5\}$ and $X_4 = \{x_i \mid i \equiv 0, 3 \pmod 4$ and $i \geq 5\}$. We now construct a graph F of order $n + 3$ from C as follows: $V(F) = V(C) \cup \{z_i \mid 1 \leq i \leq l\}$, where $4k + l = n + 3$. If $n \equiv 3 \pmod 4$, then let $N(z_1) = X_1 \cup X_3$ and $N(z_2) = X_2 \cup X_4$; if $n \equiv 0 \pmod 4$, then let $N(z_1) = X_1 \cup \{z_2\}$, $N(z_2) = X_3 \cup \{z_1\}$ and $N(z_3) = X_2 \cup X_4$; if $n \equiv 1 \pmod 4$, then let $N(z_1) = X_1 \cup \{z_2\}$, $N(z_2) = X_3 \cup \{z_1\}$, $N(z_3) = X_2 \cup \{z_4\}$ and $N(z_4) = X_4 \cup \{z_2\}$; if $n \equiv 2 \pmod 4$, then let $N(z_1) = X_1 \cup \{z_2\}$, $N(z_2) = X_3 \cup \{z_1\}$, $N(z_3) = X_2 \cup \{z_4\}$, $N(z_4) = X_4 \cup \{z_2\}$ and $N(z_5) = \{z_1, z_2, z_3, z_4\}$. It is easy to check that F has no C_4 and $\delta(F) \geq 3$. Therefore, $R(S_{n+1}, C_4) \geq n + 4$ for $n \geq 7$.

Since $S_{n+1} \subseteq W_n$, we have $R(W_n, C_4) \geq R(S_{n+1}, C_4)$. By Lemmas 2.2 and 2.3, we see that $R(W_6, C_4) = R(S_7, C_4)$ and $R(W_n, C_4) = n + 4$ for $7 \leq n \leq 10$. Since $R(S_{n+1}, C_4) \geq n + 4$ for $n \geq 7$, we get that $R(W_n, C_4) = R(S_{n+1}, C_4)$ for $7 \leq n \leq 10$. Now it remains to show that $R(W_n, C_4) \leq R(S_{n+1}, C_4)$ for $n \geq 11$. Let G be a graph of order $N = R(S_{n+1}, C_4) \geq n + 4$. Set $v \in V(G)$ with $d(v) = \Delta(G)$, $Z = V(G) - N[v]$. Suppose to the contrary that neither G contains a W_n nor $\overline{G}$ contains a C_4. Thus, noting that $N = R(S_{n+1}, C_4)$, we have $d(v) \geq n$. If $d(v) \geq n+1$, then by Lemma 2.1, $G[N(v)]$ contains a C_n, which together with v forms a W_n in G, a contradiction. Hence we have $d(v) = n$. By Theorem 1.1, $|Z| = N - (n + 1) \leq \lfloor \sqrt{n-1} \rfloor + 1$. Let C be a longest cycle in $G[N(v)]$. By Lemma 2.1, we have $|C| \geq n - 1$, and so $|C| = n - 1$. Set $u = N(v) - V(C)$. If $d_C(u) \geq 3$, then by Lemma 2.4, $\alpha(G[N(v)]) \geq 4$, which implies that $\overline{G}$ contains a C_4, and hence $d_C(u) \leq 2$. If there exists some vertex $y \in V(G) - \{u\}$ such that y has two nonadjacent vertices $y_1, y_2 \in V(C) - N_C(u)$, then uy_1yy_2u is a C_4 in $\overline{G}$, and hence y has at most one nonadjacent vertex in $V(C) - N_C(u)$ for each $y \in V(G) - \{u\}$. Since $n \geq 11$, $|Z| \leq \lfloor \sqrt{n-1} \rfloor + 1$ and $d_C(u) \leq 2$, we have

$$\begin{aligned}|V(C) - N_C(u)| - |N_C(u) \cup Z| &= |C| - d_C(u) - |Z| - d_C(u) \\ &\geq (n-1) - 2 - (\lfloor \sqrt{n-1} \rfloor + 1) - 2 \geq 2.\end{aligned}$$

Because every vertex of $N_C(u) \cup Z$ has at least $|V(C) - N_C(u)| - 1$ adjacent vertices in $V(C) - N_C(u)$, by the Pigeonhole Principle, there exists some vertex $w \in V(C) - N_C(u)$ such that $N_C(u) \cup Z \subseteq N(w)$. Noting that w has at most one nonadjacent vertex in $V(C) - N_C(u)$ and $wv \in E(G)$, we have

$$d(w) \geq |V(C) - N_C(u)| - 2 + |N_C(u) \cup Z| + 1 = |C| + |Z| - 1 = N - 3 \geq n + 1,$$

which contradicts the fact that $d(v) = \Delta(G) = n$.

This completes the proof of Theorem 1.6. □

Acknowledgement

This research was supported by NSFC under grant numbers 11071115, 11371193 and 11101207.

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