Electronic Journal of Graph Theory and Applications **7** (2) (2019), 235–250

Electronic Journal of Graph Theory and Applications

Decomposition of complete graphs into connected unicyclic bipartite graphs with eight edges

John Fahnenstiel, Dalibor Froncek

Department of Mathematics and Statistics,
University of Minnesota Duluth,
Duluth, MN 55812, U.S.A.

fahne006@d.umn.edu, dalibor@d.umn.edu

Abstract

We prove that each of the 34 non-isomorphic connected unicyclic bipartite graphs with eight edges decomposes the complete graph K_n whenever the necessary conditions are satisfied.

Keywords: graph decomposition, Rosa type labeling, α-labeling, σ^+-labeling
Mathematics Subject Classification: 05C51, 05C78
DOI: 10.5614/ejgta.2019.7.2.4

1. Introduction

Let U be a graph on m vertices. A *decomposition* of the graph U is a family of pairwise edge disjoint subgraphs $\mathcal{D} = \{G_0, G_1, \dots, G_s\}$ such that every edge of U belongs to exactly one member of $\mathcal{D}$. If each subgraph G_i is isomorphic to a given graph G we speak about G-*decomposition* of U (or we say that G forms a decomposition of U), or a G-*design* when $U \cong K_m$. The decomposition of K_m is *cyclic* if there exists an ordering $(x_1, x_2, \dots, x_m)$ of the vertices of K_m and isomorphisms $\phi_i : G_0 \to G_i$, $i = 0, 1, 2, \dots, n-1$, such that $\phi_i(x_j) = x_{i+j}$ for each $j = 1, 2, \dots, m$. Subscripts are taken modulo m.

A graph G is *unicyclic* if it contains exactly one cycle. In this paper, we provide the necessary and sufficient conditions for decompositions of complete graphs into each of the 34 connected unicyclic bipartite graphs with eight edges.

Received: 24 October 2018, Revised: 12 April 2019, Accepted: 28 April 2019.

We will use standard decomposition methods based on α- and σ^+-labelings, introduced by Rosa [9] and El-Zanati and Vanden Eynden [4], respectively.

2. Related Results

Based on available literature beginning with Rosa's [9] paper on graph labelings, there has been no attempt to classify graphs with 8 vertices and 8 edges decomposing complete graphs. This section will summarize what is known about classifications of smaller graphs, that is, graphs where $|E(G)| \leq 8$ or $|V(G)| \leq 8$ that form decompositions of complete graphs.

Kang and Wang [6] investigated complements of P_5, that is, graphs $K_5 - P_5$, denoted $\overline{P_5}$. They found necessary and sufficient conditions for the existence of $\overline{P_5}$-designs.

Theorem 2.1 (Kang and Wang 2004). *There exists a $\overline{P_5}$-decomposition of K_n if and only if $n \equiv 0, 1, 4, 9 \pmod{12}$ and $n \geq 9$.*

Yin and Gong [11] found necessary and sufficient conditions for the existence of a G-design for graphs with six vertices and $3 \leq |E(G)| \leq 6$. There are 28 non-isomorphic graphs of this type.

Theorem 2.2 (Yin and Gong 1990). *There exists a G-decomposition of K_n for G on six vertices and $3 \leq |E(G)| \leq 6$ if and only if the necessary conditions are satisfied except in five cases.*

Cui [3], Blinco [1], Kang, Zuo, and Zhang [8], and Tian, Du, and Kang [10] studied graphs with six vertices and seven edges.

Theorem 2.3 (Cui 2002, Blinco 2003, Kang et al. 2004, Tian et al. 2006). *There exists a G-decomposition of K_n into connected graphs G on six vertices and seven edges if and only if the necessary conditions are met except for eight exceptions when $n = 7$ or $n = 8$.*

Graphs with five vertices and eight edges were examined by Colbourn, Ge, and Ling in 2008, due to their applicability with respect to the problem of grooming traffic in optical networks [2]. There are only two non-isomorphic graphs with five vertices and eight edges, shown if Figure 1.

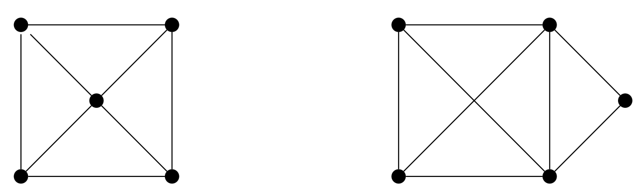


Figure 1: Graphs $G_{5,1}$ and $G_{5,2}$ (from left to right).

Colbourn, Ge and Ling proved the following results for the graphs $G_{5,1}$ and $G_{5,2}$.

Theorem 2.4 (Colbourn, Ge, Ling 2008). *There exists a $G_{5,1}$-decomposition of K_n if and only if $n \equiv 0 \pmod{16}$ except possibly when $n = 32$ or $n = 48$.*

Theorem 2.5 (Colbourn, Ge, Ling 2008). *There exists a $G_{5,2}$-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ except when $n = 16$ and possibly when $n = 48$.*

www.ejgta.org

Kang, Yuan, and Liu researched graphs with six vertices and eight edges in 2005 [7]. There are 22 non-isomorphic graphs of this type, and they proved the following theorem with respect to decompositions of complete graphs.

Theorem 2.6 (Kang, Yuan, Liu 2005). *Let G be a connected graph with six vertices and eight edges. There exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$ with two possible exceptions for $n = 32$.*

3. Definitions and tools

First we define the labelings that will be used in our work. We start with a ρ-labeling, first defined by Rosa [9]. Note that Rosa originally used the term *valuation* instead of labeling.

Definition 3.1. Let G be a graph with n edges. A ρ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, \dots, 2n\}$ inducing a function $\ell : E(G) \to \{1, 2, \dots, n\}$ defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\}$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = \{1, 2, \dots, n\} .$$

A more restrictive version is the σ-labeling, also introduced by Rosa in [9].

Definition 3.2. Let G be a graph. A σ-labeling of G is a ρ-labeling such that $\ell(uv) = |f(u)-f(v)|$.

An even more restrictive labeling is the well-known graceful labeling (which was originally called a β-valuation by Rosa [9]).

Definition 3.3. Let G be a graph with n edges. A *graceful labeling* is a σ-labeling f with the additional restriction that $f : V(G) \to \{0, 1, \dots, n\}$, that is, the labels are assigned integers from the set $\{0, 1, \dots, n\}$ instead of $\{0, 1, \dots, 2n\}$.

Rosa [9] proved that if a graph G with n edges has one of the above labelings, then a decomposition of the complete graph K_{2n+1} exists.

Theorem 3.4 (Rosa 1967). *A cyclic decomposition of the complete graph K_{2n+1} into subgraphs isomorphic to a given graph G with n edges exists if and only if there exists a ρ-labeling of the graph G.*

A useful restriction of a graceful labeling is the following.

Definition 3.5. An α-labeling is a graceful labeling with the additional property that there exists an integer λ such that for each edge xy either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$.

While the α-labeling (defined also by Rosa in [9]) is a more restrictive form of graceful labeling, it facilitates decompositions of a much larger set of complete graphs, that is, decompositions of K_{2nk+1} instead of just K_{2n+1}, where k is an arbitrary positive integer. An α-labeled graph must be bipartite, and when V_1 and V_2 the partite sets of the graph G, then without loss of generality if $v_1 \in V_1$, then $f(v_1) \leq \lambda$ and if $v_2 \in V_2$, then $f(v_2) > \lambda$.

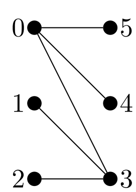
In 1967 Rosa proved the following [9]:

Theorem 3.6 (Rosa 1967)**.** *If a graph G with n edges has an α-labeling, then there exists a cyclic decomposition of the complete graph K_{2nk+1} into subgraphs isomorphic to G, where k is an arbitrary positive integer.*

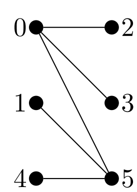
El-Zanati and Vanden Eynden extended the results of Rosa by defining several new labelings (see, e.g., [4]), which are less restrictive than α-labeling but also give decompositions of larger complete graphs.

Definition 3.7. A graceful labeling f of a graph G with vertex bipartition (V_1, V_2) is called a *near α-labeling* if it has the property that for each edge v_1v_2 with $v_1 \in V_1$ and $v_2 \in V_2$, $f(v_1) < f(v_2)$.

The only difference between α- and near α-labelings is that while in α-labeling all vertices in V_1 have labels smaller than all vertices in V_2, in near α-labeling for each $v_1 \in V_1$, all neighbors v_2 of v_1 must satisfy $f(v_1) < f(v_2)$. Thus, a near α-labelings can be seen as a "locally α-labeling" as some labels of vertices in V_1 can exceed labels of some vertices in V_2 when they are not neighbors.



(a) α-labeling



(b) Near-α-labeling

Figure 2: Examples of α- and near-α-labeling.

El-Zanati, Kenig, and Vanden Eynden [5] proved that near α-labelings also give decompositions of complete graphs.

Theorem 3.8 (El-Zanati, Kenig, Vanden Eynden 2000)**.** *If a bipartite graph G with n edges has a near α-labeling, then there exists a G-decomposition of K_{2nk+1} for any positive integer k.*

In fact, we will use an even less restrictive labeling mentioned by El-Zanati and Vanden Eynden in [4] and called *σ^+-labeling*.

Definition 3.9. A σ-labeling f of a bipartite graph G with vertex bipartition (V_1, V_2) is called a *σ^+-labeling* if for each edge v_1v_2, where $v_1 \in V_1$ and $v_2 \in V_2$, it holds that $f(v_1) < f(v_2)$.

www.ejgta.org

Notice that every near α-labeling is also a σ^+-labeling with the additional restriction that the highest label is n. Although the following theorem was never formally proved in [4], we state it nevertheless. The proof is identical to the proof of Theorem 3.8, since the property that all vertex labels are at most n is never used there.

Theorem 3.10. *If a bipartite graph G with n edges has a σ^+-labeling, then there exists a G-decomposition of K_{2nk+1} for any positive integer k.*

The above labelings enable isomorphic decompositions of complete graphs K_m of order $m \equiv 1 \pmod{2nk}$, but similar methods exist for order $m \equiv 0 \pmod{2nk}$ under certain conditions. It is well known that ρ-labelings can be also used for decompositions of K_{2nk}.

Theorem 3.11 (see, e.g., [4]). *Let G be a graph with n edges and let v be a vertex of degree 1 in G. If $G - v$ has a ρ-labeling, then there exists a G-decomposition of K_{2n}.*

We will prove a similar result for α- and σ^+-labelings to obtain our main result.

4. Necessary conditions and catalog of graphs

Let G be a graph with 8 edges. Since the number of edges in a G-decomposable complete graph K_m must be a multiple of the number of edges of G, we have

$$m(m-1)/2 = 8n,$$

which yields

$$m = 16n \text{ or } m = 16n + 1.$$

Therefore, we only need to investigate graphs K_{16n} and K_{16n+1} for $n \geq 1$. We also notice that because C_8 is regular of degree 2, it cannot decompose K_{16n}, since the vertices in the complete graph K_{16n} are of an odd degree $16n - 1$. Hence, we have the following.

Proposition 4.1. *Let G be a connected unicyclic graph with 8 edges and K_m be a G-decomposable complete graph. Then $m = 16n$ or $n = 16n + 1$ for some positive integer n except for $G \cong C_8$, when $m = 16n + 1$ only.*

Now we list all connected unicyclic graphs with 8 edges by providing their drawings. There are 34 non-isomorphic unicyclic graphs with an even cycle on 8 vertices (including C_8). We first introduce some notation. The *type* of such graph G will be determined by an l-tuple $i_1, i_2, \ldots, i_l$, where l is the length of the only cycle C_l in G with vertices $v_1, v_2, \ldots, v_l$ and i_j is the number of edges in the tree attached to v_j. We will always attach the largest tree to v_1 and of course list only one of the two symmetric options.

So for instance, $1, 0, 0, 1, 0, 0$ denotes a six-cycle graph with pendant vertices having two vertices attached to v_1 and v_4, respectively, and $3, 1, 0, 0$ denotes four-cycle graphs with an attached tree with 3 edges containing v_1 and a pendant vertex connected to v_2. Notice that there are four non-isomorphic graphs of this type. We present the graphs in an order that minimizes the space used for the figures and skip the obvious figure of C_8.

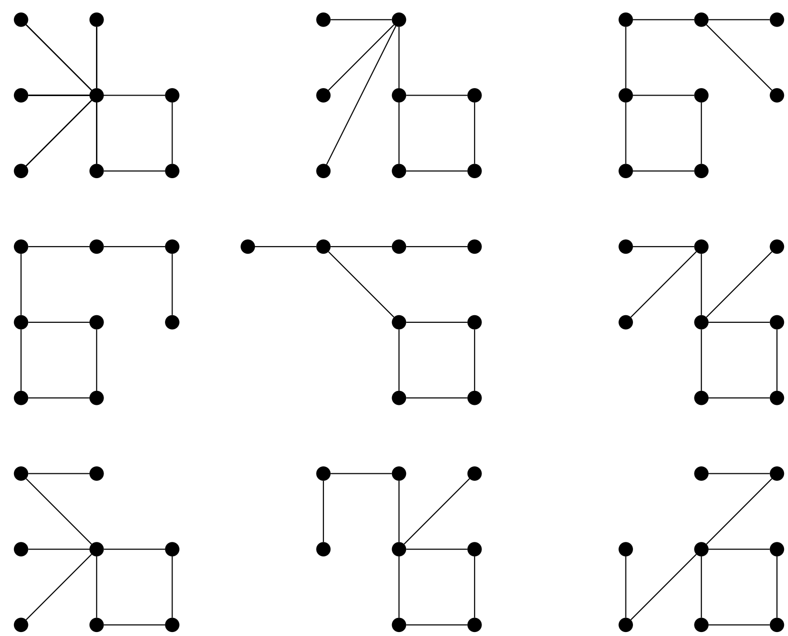


Figure 3: Type 4,0,0,0

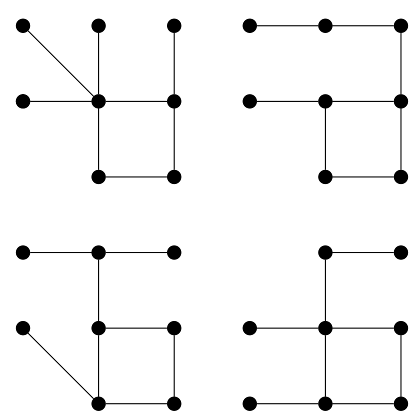


Figure 4: Type 3,1,0,0

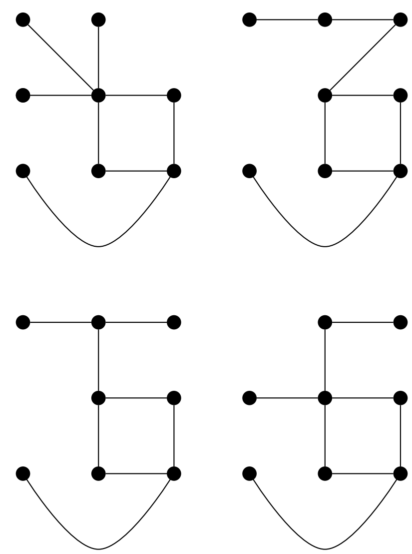


Figure 5: Type 3,0,1,0

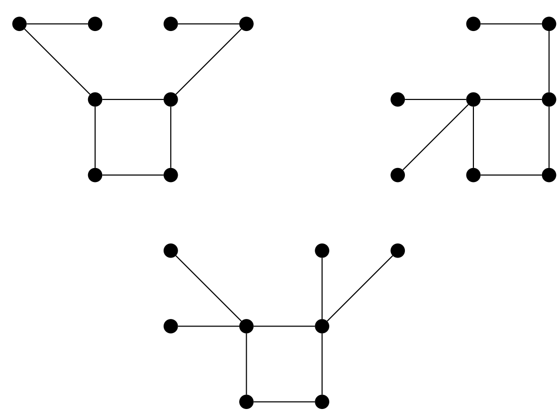


Figure 6: Type 2,2,0,0

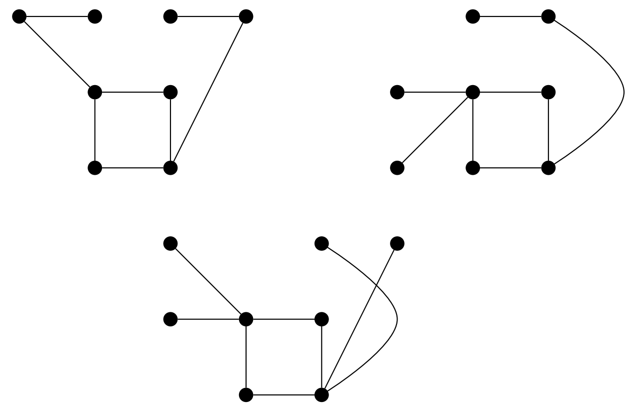


Figure 7: Type 2,0,2,0

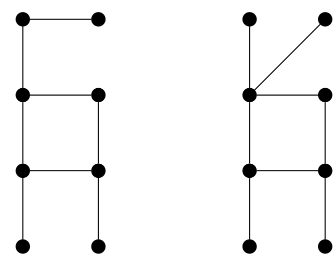


Figure 8: Type 2,1,1,0

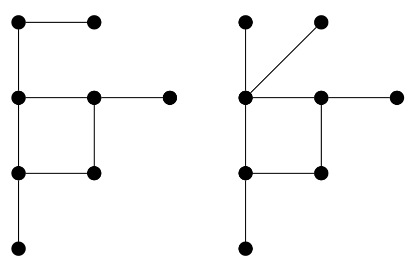


Figure 9: Type 2,1,0,1

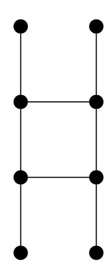


Figure 10: Type 1,1,1,1

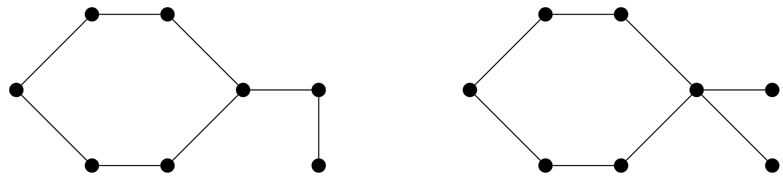


Figure 11: Type 2,0,0,0,0,0

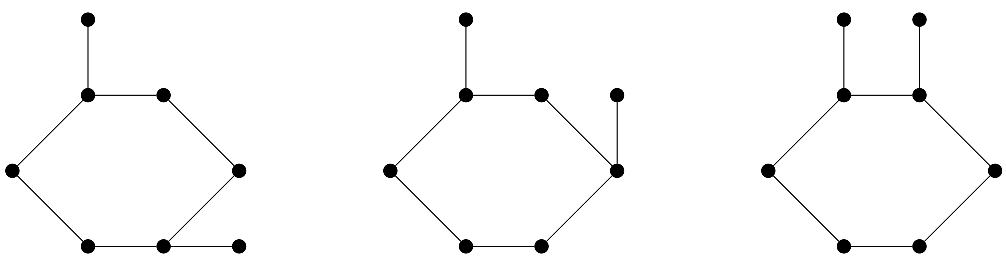


Figure 12: Types 1,0,0,1,0,0; 1,0,1,0,0,0; 1,1,0,0,0,0

5. Labelings

In this section we present α- and σ^+-labelings for all graphs in our class. For decompositions of complete graphs K_{16n+1}, no additional properties would be needed. However, to apply Theorem 3.11, we need a pendant edge of length 8. Once we have it, we can apply the following theorem to justify our main result.

Theorem 5.1. *Let G be a bipartite graph on r edges with an α- or σ^+-labeling such that the longest edge of length r is a pendant edge e. Then there exists a graph H on rk edges that has a ρ-labeling and can be decomposed into k copies of G and a graph H^- on $rk-1$ edges that has a ρ-labeling and can be decomposed into $k-1$ copies of G and one copy of $G-e$.*

Proof. Let G be a graph as above with bipartition V_0 and V_1. Without loss of generality, let V_0 comprise $\{x_1^0, x_2^0, \ldots, x_a^0\}$ and V_1 comprise $\{x_1^1, x_2^1, \ldots, x_b^1\}$. From the Definitions 3.7 and 3.9 it follows that $f(x_i^0) < f(x_j^1)$ whenever $x_i^0 x_j^1$ is an edge of G and the lengths of all edges $x_i^0 x_j^1 \in E(G)$ form a bijection with $\{1, 2, \ldots, r\}$. Denote this copy of G by G_1 and assume that the longest edge e of length $\ell(e) = r$ is $x_1^0 x_1^1$ where x_1^1 is of degree 1.

Now for $t = 2, 3, \ldots, k$ take a graph G_t, isomorphic to G, with bipartition V_0 and $V_t = \{x_1^t, x_2^t, \ldots, x_b^t\}$. Add $(t-1)r$ to each vertex label in V_1 of G_1 to label vertices in V_t and keep the labels in V_0 the same as before. More precisely, let $f(x_j^t) = f(x_j^1) + (t-1)r$.

Now the length of an edge $x_i^0 x_j^t \in E(G_t)$ will be

$$\ell(x_i^0 x_j^t) = \ell(x_i^0 x_j^1) + (t-1)r$$

and the edge lengths of G_t form the set $\{(t-1)r+1, (t-1)r+2, \ldots, tr\}$. The resulting graph H now can be decomposed into k copies of G and contains edges of lengths $1, 2, \ldots, rk$. To obtain the graph H^-, we simply remove the pendant, longest edge $x_1^0 x_1^k$ of length rk. It should be clear from the construction that H^- can also be decomposed as required. Because all edge lengths were calculated as $f(x_j^t) - f(x_i^0)$, it is irrelevant whether the total number of vertices of the complete graph that we are decomposing is $2rk-1$ or $2rk+1$. □

The following theorem is a direct consequence of Theorems 3.11 and 5.1.

Theorem 5.2. *Let G be a graph with an α- or σ^+-labeling on r edges such that the edge of length r is a pendant edge. Then there exists a G-decomposition of K_{2rk} for any positive integer k.*

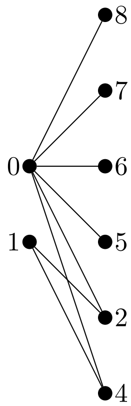
Proof. By Theorem 5.1, we can construct a ρ-labeled graph H^- consisting of $k-1$ edge-disjoint copies of G and one copy of $G-e$, where $e = uv$ and v is of degree one. Let H be the graph arising from H^- by adding back the edge $e = uv$. Then the graph H with rk edges satisfies the assumptions of Theorem 3.11 and therefore decomposes K_{2rk}. □

Rosa [9] proved that C_{4m} has an α-labeling for any $m \geq 1$. Hence, the following special case holds by Theorem 3.6.

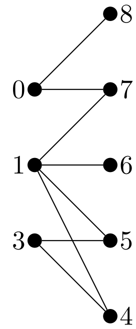
Theorem 5.3 (Rosa 1967). *The cycle C_8 has an α-labeling and thus decomposes K_{16n+1} for every positive integer n.*

Next we present α- or σ^+-labelings of all connected bipartite unicyclic graphs with 8 edges except C_8, which we do not need because of the result above.

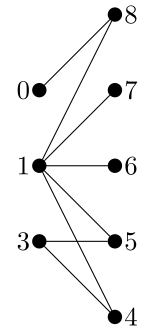
Notice that the graphs in Figure 18 have a σ^+-labeling, while the remaining ones have an α-labeling. The number in brackets is just a counter for different non-isomorphic graphs of the same type. For simplicity, in sub-captions we state just the type of the graph and the counter (where needed), omitting the word "type."



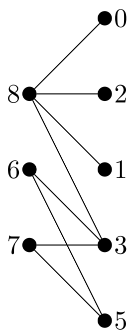
(a) 4,0,0,0 [1]



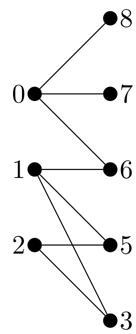
(b) 4,0,0,0 [2]



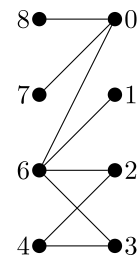
(c) 4,0,0,0 [3]



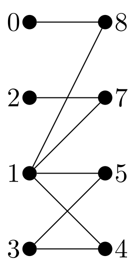
(d) 4,0,0,0 [4]



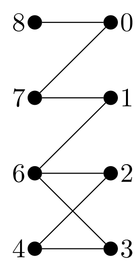
(e) 4,0,0,0 [5]



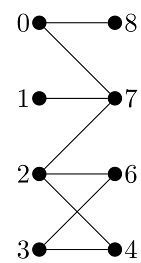
(f) 4,0,0,0 [6]



(g) 4,0,0,0 [7]



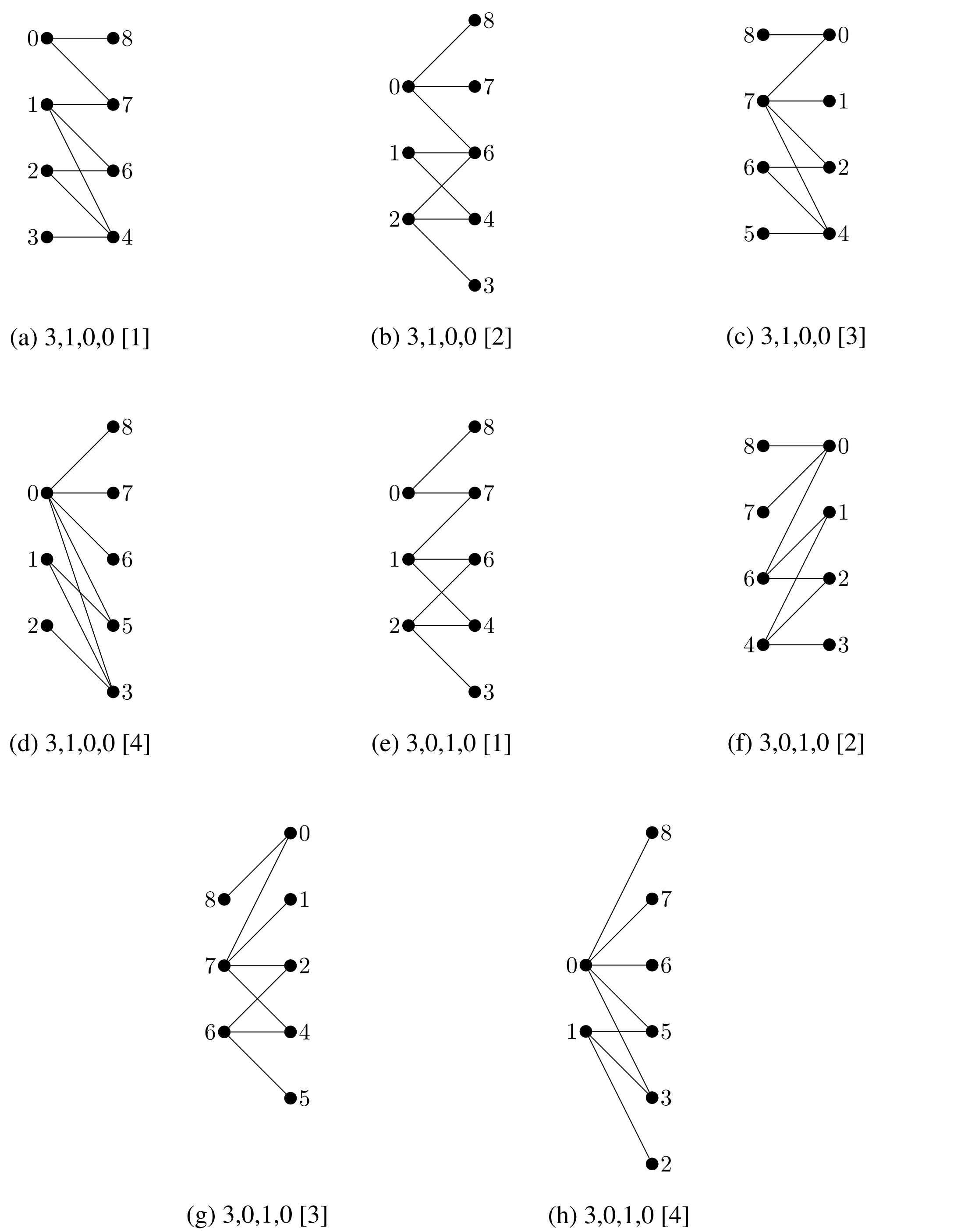
(h) 4,0,0,0 [8]



(i) 4,0,0,0 [9]

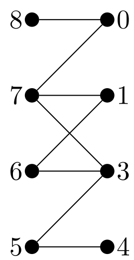
Figure 13: Graphs of type $4, 0, 0, 0$ with α-labeling

www.ejgta.org

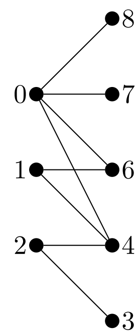


(a) 3,1,0,0 [1]
(b) 3,1,0,0 [2]
(c) 3,1,0,0 [3]
(d) 3,1,0,0 [4]
(e) 3,0,1,0 [1]
(f) 3,0,1,0 [2]
(g) 3,0,1,0 [3]
(h) 3,0,1,0 [4]

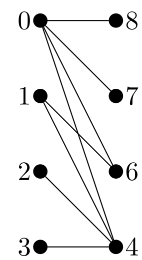
Figure 14: Graphs of type $3, 1, 0, 0$ and $3, 0, 1, 0$ with α-labeling



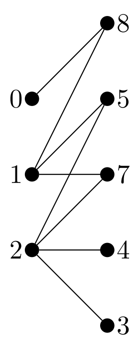
(a) 2,2,0,0 [1]



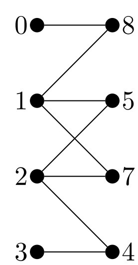
(b) 2,2,0,0 [2]



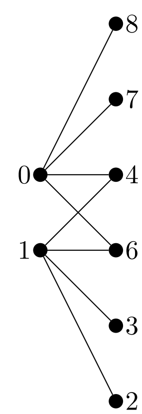
(c) 2,2,0,0 [3]



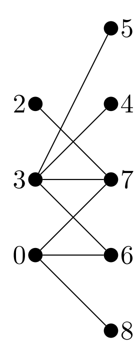
(d) 2,0,2,0 [1]



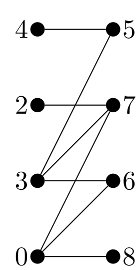
(e) 2,0,2,0 [2]



(f) 2,0,2,0 [3]

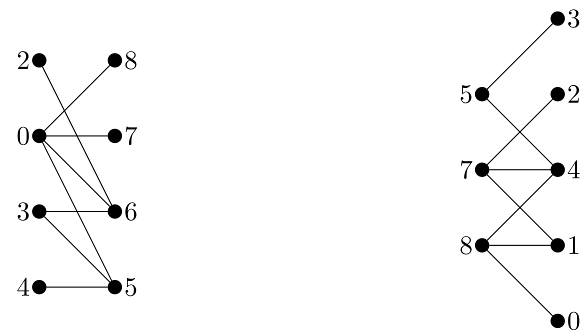


(g) 2,1,1,0 [1]



(h) 2,1,1,0 [2]

Figure 15: Graphs of type $2, 2, 0, 0$; $2, 0, 2, 0$; and $2, 1, 1, 0$ with α-labeling



(a) 2,1,0,1 [1] (b) 2,1,0,1 [2]

Figure 16: Graphs of type $2, 1, 0, 1$ with α-labeling

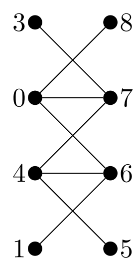
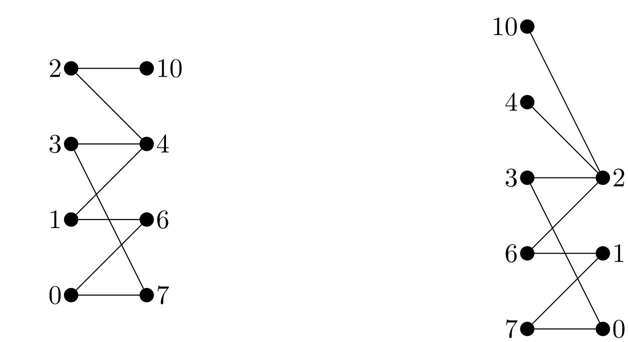


Figure 17: Graph of type $1, 1, 1, 1$ with α-labeling



(a) 2,0,0,0,0,0 [1] (b) 2,0,0,0,0,0 [2]

Figure 18: Graphs of type $2, 0, 0, 0, 0, 0$ with σ^+-labeling

www.ejgta.org

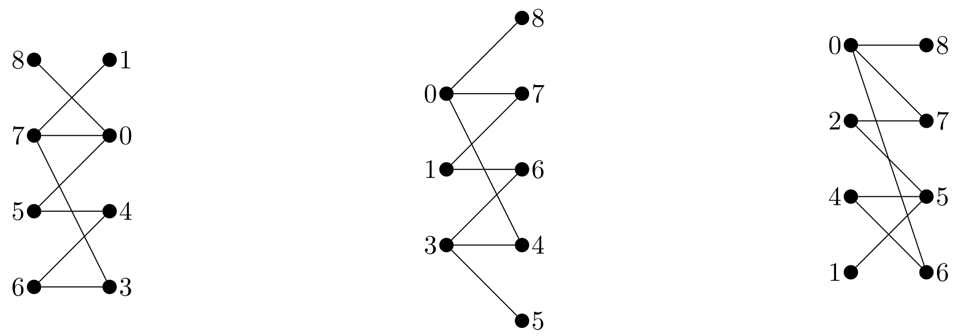


Figure 19: Graphs of type $1, \dots, 0, 0$ with α-labeling

We are now ready to prove our main result.

Theorem 5.4. *Every connected bipartite unicyclic graph G on 8 edges other than C_8 decomposes the complete graph K_m if and only if $m \equiv 0, 1 \pmod{16}$.*

Proof. The necessity follows from Proposition 4.1. Figures 13 to 19 show that all graphs in question have α- or σ^+-labeling with a pendant edge of length 8. Therefore, they satisfy assumptions of Theorems 3.6 and 5.2 and by these theorems, they decompose every K_{16n} and K_{16n+1}. □

Acknowledgement. We would like to thank the anonymous referee for the thoughtful and detailed comments and suggestion, which significantly helped to improve the paper.

References

[1] A. Blinco, Decompositions of complete graphs into theta graphs with fewer than ten edges, *Util. Math.* **64** (2003), 197–212.

[2] C. Colbourn, G. Ge, A. Ling, Graph designs for the eight-edge five-vertex graphs, *Discrete Math.* **309** (2009), 6440–6445.

[3] Y.G. Cui, *Decompositions and packings of λK_v into K_{2s2} with a pendent edge*, Hebei Normal University, Master Thesis, 2002.

[4] S. El-Zanati, C. Vanden Eynden, On Rosa-type labelings and cyclic graph decompositions, *Math. Slovaca* **59** (1) (2009), 1–18.

[5] S. El-Zanati, M.J. Kenig, C.V. Eynden, Near α-labelings of bipartite graphs, *Australas. J. Combin.* **21** (2000), 275–285.

[6] Q. Kang, Z. Wang, Optimal packings and coverings of λK_v with graphs $K_5 - P_5$, *Bull. Inst. Combin. Appl.* **41** (2004), 22–41.

[7] Q. Kang, L. Yuan, S. Liu, Graph Designs for all Graphs with Six Vertices and Eight Edges, *Acta Math. Appl. Sin. (Engl. Ser.)* **21** (3) (2005), 469–484.

[8] Q. Kang, H. Zuo, Y. Zhang, Decompositions of λK_v into k-circuits with one chord, *Australas. J. Combin.* **30** (2004), 229–246.

[9] A. Rosa, *On certain valuations of the vertices of a graph*, In: Theory of Graphs (Intl. Symp. Rome 1966), Gordon and Breach, Dunod, Paris, 1967, 349–355.

[10] Z. Tian, Y. Du, Q. Kang, Decomposing complete graphs into graphs with six vertices and seven edges, *Ars Combin.* **81** (2006), 257–279.

[11] J. Yin, B. Gong, Existence of G-designs with $|V(G)| = 6$, In: W.D. Wallis et al (eds). Combinatorial designs and applications, Marcel Dekker, New York, 1990, 201–218.