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Harary index of bipartite graphs

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Abstract

The sum of reciprocals of distance between any two vertices in a graph G is called the Harary index. We determine the n-vertex extremal graphs with the maximum Harary index for all bipartite graphs, a given matching number, a given vertex-connectivity, and with a given edge-connectivity, respectively.

Keywords: Harary index, bipartite graph, matching number, vertex-connectivity, edge-connectivity
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1. Introduction

Throughout the paper let G be a connected graph with vertex set $V(G)$ and the edge set $E(G)$. We denote the degree of a vertex x in G by $d_G(x)$. We denote the distance of the shortest path between $x, y \in V(G)$ by $d_G(x, y)$.

A simple bipartite graph $G = (V_1, V_2; E)$, is the union of disjoint vertex partitions V_1 and V_2, such that none of the edges in G have both the end vertices in one partition. For every chosen two vertices from different partition in a bipartite graph are adjacent, then G is complete, denoted by $K_{a,b}$, where $a = |V_1|$ and $b = |V_2|$.

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A graph G is called *k-connected* if removing any set of k vertices from G, the result is a disconnected graph. In this context, the *connectivity* of G, denoted by $\kappa(G)$. Similarly, a graph G is called G *k'-edge-connected* if removing any k' edges from G, the result is a disconnected graph. Here, the *edge-connectivity* of G, denoted by $\kappa'(G)$.

Let $\mathcal{A}_n^k$, $\mathcal{C}_n^s$ and $\mathcal{D}_n^t$ denote the set of all $n-$vertex bipartite graph with matching number k (see below), connectivity s and edge-connectivity t, respectively.

Since 1947, the distance-based graph invariant *Wiener index* is received a lot of attention, it is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v).$$

In an analogous way, Harary index [4, 8] defined as

$$H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_G(u, v)}. \quad (1)$$

Xu [14] determined the extremal results of Harary indices on trees. Xu and Das [16] characterized the extremal bicyclic and unicyclic graphs for $H(G)$. Xu et al. [17] found the maximal $H(G)$ for a fixed matching number on unicyclic graphs (for other example, see [1, 2, 3, 6, 7, 9, 10, 11, 12, 13, 15, 18, 19, 20] and references cited therein).

Motivated by work of Li and Song [5], we determine the extremal graphs on n vertices with the maximum Harary index for all bipartite graphs with a given matching number, a given vertex-connectivity, and with a given edge-connectivity.

2. Harary index of bipartite graphs with a given matching number

We start by the following lemma, which holds immediately from the definitions.

Lemma 2.1. *Let G be a simple graph with $|V(G)| = n$ with $G \ncong K_n$. Then for every edge $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G, $H(G) < H(G + e)$.*

In the next result, we present the extremal graph having the maximum $H(G)$ for all bipartite graphs, for a fixed matching number.

Theorem 2.1. *Let G represents a bipartite graph with n vertices and matching number k. Then $K_{k,n-k}$ is the unique graph with the maximum Harary index.*

Proof. By choosing G in $\mathcal{A}_n^k$, such that its $H(G)$ is very large. If $k = \left\lfloor \frac{n}{2} \right\rfloor$, then using Lemma 2.1 we get, $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$ is maximum. So, we only consider $k < \left\lfloor \frac{n}{2} \right\rfloor$.

Let $G = (A, B; E)$ with $|B| \geq |A| \geq k$ and M is the maximal matching in G. If $|A| = k$ then we see that $G = K_{k,n-k}$ is the extremal graph in G. By Lemma 2.1, $H(G)$ increases by adding edges in G. So, we can assume that $|A| > k$.

Let A_M, B_M be the vertex subsets of A, B which are incident to M. Then $|A_M| = k$ and $|B_M| = k$. It is noted that there is no edge in G between the set $A - A_M$ of vertices and the set

$B - B_M$ of vertices. If so, then the edges may be together with M to increase the size of the matching greater than M, which violates the maximality of M.

By attaching all the possible edges to the vertices of A_M and B_M, A_M and $B - B_M$, $A - A_M$ and B_M, we achieve a new graph G'. By Lemma 2.1, $H(G) \le H(G')$. Also, G' has the matching number at least $k+1$. So, $G' \notin \mathcal{A}_n^k$ and $G \ncong G'$. Now, we construct an another new graph G'' based on G', by deleting all the edges between the set $A - A_M$ of vertices and the set B_M of vertices, and adding all the edges between $A - A_M$ and A_M in G'. It is easy to verify that $G'' \cong K_{k,n-k}$.

Let $|A| = n_1$, $|B| = n_2$, so $n_1 + n_2 = n$ and $n_2 \ge n_1 > k$ and using (1), we get

$$\begin{aligned} H(G') &= k^2 + k(n_2 - k) + k(n_1 - k) + \frac{C_{n_1}^2 + C_{n_2}^2}{2} + \frac{(n_1 - k)(n_2 - k)}{3} \\ &= \frac{n_1^2 + n_2^2}{4} + \frac{n_1 n_2}{3} + \frac{2kn}{3} - \frac{n}{4} - \frac{2k^2}{3}, \\ H(G'') &= k(n-k) + \frac{C_k^2 + C_{n-k}^2}{2} = \frac{n^2}{4} + \frac{kn}{2} - \frac{n}{4} - \frac{k^2}{2}. \end{aligned}$$

Therefore, by the fact that $k < n_1 \le n_2 = n - n_1 < n - k$, we have

$$H(G') - H(G'') = \frac{kn - k^2 - n_1 n_2}{6} = \frac{k(n-k) - n_1 n_2}{6} < 0,$$

as required. □

3. Harary index of bipartite graphs with a given vertex / edge-connectivity

In the current section, we determine the extremal graphs with the maximum Harary index among $\mathcal{C}_n^s$ and $\mathcal{D}_n^t$.

By $K_{p,0}$, $p \ge 1$, we mean pK_1 (p isolated vertices). Let $O_s \vee_1 (K_{n_1,n_2} \cup K_{m_1,m_2})$ be the graph obtained by adding all vertices of the empty graph O_s of order s ($s \ge 1$) to all vertices belonging to the part of cardinality n_1 in the bipartition of K_{n_1,n_2} and the part of cardinality m_1 in the bipartition of K_{m_1,m_2}, respectively.

Lemma 3.1. *If $s + q > p$ then*

$$H(O_s \vee_1 (K_{1,0} \cup K_{p,q})) < H(O_s \vee_1 (K_{1,0} \cup K_{p+1,q-1})).$$

Proof. Let $G = O_s \vee_1 (K_{1,0} \cup K_{p,q})$ and $G' = O_s \vee_1 (K_{1,0} \cup K_{p+1,q-1})$. By (1), we have

$$\begin{aligned} H(G) &= s + sp + pq + \frac{p + sq + C_s^2 + C_p^2 + C_q^2}{2} + \frac{q}{3} \\ &= \frac{3s}{4} + \frac{p}{4} + \frac{q}{12} + \frac{s^2 + p^2 + q^2}{4} + sp + pq + \frac{sq}{2} \end{aligned}$$

and

$$\begin{aligned} H(G') =& s + (p+1)s + (q-1)(p+1) \\ &+ \frac{(p+1) + s(q-1) + C_s^2 + C_{p+1}^2 + C_{q-1}^2}{2} + \frac{q-1}{3} \\ =& \frac{5s}{4} - \frac{p}{4} + \frac{7q}{12} + sp + pq + \frac{sq}{2} + \frac{s^2 + p^2 + q^2}{4} - \frac{1}{3}. \end{aligned}$$

So, by $s+q>p$, we have

$$H(G)-H(G')=\frac{p-(q+s)}{2}+\frac{1}{3}<0,$$

as claimed. □

Lemma 3.2. *If $s+q+1<p$ then*

$$H(O_s \vee_1 (K_{1,0}\cup K_{p,q}))<H(O_s \vee_1 (K_{1,0}\cup K_{p-1,q+1})).$$

Proof. Let $G=O_s \vee_1 (K_1\cup K_{p,q})$ and $G''=O_s \vee_1 (K_1\cup K_{p-1,q+1})$. By (1), we have

$$\begin{aligned} H(G) &= s+sp+pq+\frac{p+sq+C_s^2+C_p^2+C_q^2}{2}+\frac{q}{3} \\ &= \frac{3s}{4}+\frac{p}{4}+\frac{q}{12}+\frac{s^2+p^2+q^2}{4}+sp+pq+\frac{sq}{2} \end{aligned}$$

and

$$\begin{aligned} H(G'') =& s+(p-1)s+(q+1)(p-1)+\frac{p-1+s(q+1)+C_s^2+C_{p-1}^2+C_{q+1}^2}{2}+\frac{q+1}{3} \\ =& \frac{s}{4}+\frac{3p}{4}-\frac{5q}{12}+sp+pq+\frac{sq}{2}+\frac{s^2+p^2+q^2}{4}-\frac{2}{3}. \end{aligned}$$

Therefore, by $s+q+1<p$, we have

$$H(G)-H(G'')=\frac{s+q-p}{2}+\frac{2}{3}<0,$$

as claimed. □

Note that $K_{s,n-s}=O_s \vee_1 (K_{1,0}\cup K_{n-s-1,0})$, by Lemma 3.2, we have

Corollary 3.1. *If $1\le s\le \frac{n}{2}-1$ then*

$$H(K_{s,n-s})<H(O_s \vee_1 (K_1\cup K_{n-s-2,1})).$$

Lemma 3.3. *Let $G=(V_1,V_2;E)\in \mathcal{C}_n^s$ with a vertex-cut $I=I_1\cup I_2$ of order s such that $G-I$ has two components $G_1=(A,B;E_1)$ and $G_2=(C,D;E_2)$, where $V_1=A\cup I_1\cup C$ and $V_2=B\cup I_2\cup D$. If A,C,I_1 are non-empty sets, then G cannot be a graph with the maximum Harary index in $\mathcal{C}_n^s$.*

Proof. Assume that G has the maximum $H(G)$ in $\mathcal{C}_n^s$. By Lemma 2.1, G contains all edges between V_1 and V_2, except edges between A and D and between C and B. Let $|A|=m_1$, $|B|=m_2$, $|C|=n_1$, $|D|=n_2$, $|I_1|=k$ and $|I_2|=t$. Then $m_1\ge 1$, $n_1\ge 1$, $k\ge 1$ and $k+t=s$. So,

$$\begin{aligned} H(G) =& m_1(m_2+t)+k(m_2+t+n_2)+n_1(t+n_2) \\ &+\frac{m_1k+m_1n_1+kn_1+m_2n_2+m_2t+n_2t+C_{m_1}^2+C_k^2+C_{n_1}^2+C_{m_2}^2+C_{n_2}^2+C_t^2}{2} \\ &+\frac{m_1n_2+m_2n_1}{3}. \end{aligned}$$

Note that $G-(I_2\cup D)$ is not connected, we have $t+n_2\geq s=t+k$, and $n_2\geq k$. We partition D into D_1 and D_2 such that $D=D_1\cup D_2$ with $|D_1|=k$ and $|D_2|=n_2-k$. Let u_0 be any vertex of C, and $G'=G-\{u_0v|v\in D_2\}+\{ad|a\in A, d\in D\}+\{bc|b\in B, c\in C-\{u_0\}\}$. Then $G'\in\mathcal{C}_n^s$ with its bipartition (V_1,V_2) and a vertex-cut $I_2\cup D_1$ with s vertices. In fact, G' contains all edges between V_1 and V_2, except edges between u_0 and $B\cup I_2$, and

$$\begin{aligned}H(G')&=(m_1+k+n_1-1)(m_2+t+n_2)+(t+k)\\&+\frac{C^2_{m_1+k+n_1}+C^2_{m_2+t+n_2}}{2}+\frac{m_2+n_2-k}{3}.\end{aligned}$$

Thus,

$$H(G)-H(G')=-\frac{2}{3}(k+m_2(n_1-1)+n_2(m_1-1))<0,$$

a contradiction. □

Remark 3.1. By the symmetry, if B, D, I_2 are non-empty sets (see Lemma 3.3), then G fails to be a maximum $H(G)$ in $\mathcal{C}_n^s$.

Let U and V any two vertex sets of G. Denote by $E_G\,[U,V]$, edges of G with one of its end vertex in U and the other in V.

Lemma 3.4. *Let $n>4$ and $G=(V_1,V_2;E)\in\mathcal{C}_n^s$ with an edge-cut $E_t=E_1\cup E_2$ of size t such that $G-E_t$ has two components $G_1=(A,B;E')$ and $G_2=(C,D;E'')$, where $V_1=A\cup C$, $V_2=B\cup D$, $E_1=E_t\cap E_G[A,D]$ and $E_2=E_t\cap E_G[B,C]$. If A,B,C,D are non-empty sets, then G cannot be a graph with the maximum $H(G)$ in $\mathcal{D}_n^t$.*

Proof. Assume that G has the maximum $H(G)$ in $\mathcal{D}_n^t$. By Lemma 2.1, G contains all edges between A and B, edges between C and D and edges in E_t. Let $|A|=m_1$, $|B|=m_2$, $|C|=n_1$, $|D|=n_2$, $|E_1|=a$ and $|E_2|=b$. Then $a+b=t$ and $m_1+n_1+m_2+n_2=n>4$.

Suppose, we assume $m_1>1$ in the following. Let S_4, S_3, S_2 and S_1 denote the end-vertices of the edges of E_t in D, C, B and A, respectively. Let $|A-S_1|=a_1, |B-S_2|=a_2, |C-S_3|=a_3$ and $|D-S_4|=a_4$. Then G contains $m_1m_2+n_1n_2+t=|E(G)|$ vertex pairs at distance 1, $m_1n_2+m_2n_1-t$ vertex pairs at distance 3, and $a_1a_3+a_2a_4$ vertex pairs at distance 4. Remaining $C_n^2-|E_G|-(m_1n_1+m_2n_2-t)-(a_1a_4+a_2a_3)$ vertex pairs are at distance 2. Therefore,

$$\begin{aligned}H(G)&=|E(G)|+\frac{1}{3}(m_1n_2+m_2n_1-t)+\frac{1}{4}(a_1a_3+a_2a_4)\\&+\frac{1}{2}(C_n^2-|E(G)|-(m_1n_2+m_2n_1-t)-(a_1a_3+a_2a_4)).\end{aligned}$$

Let c_0 be a vertex and $d_G(c_0)=h+|D|=h+n_2$, where $h(\min\{b,m_2\}\geq h\geq 0)$ denotes the number of edges joining c_0 to B. It is easy to see that the set of edges incident to c_0 is an edge-cut of G, we have $h+n_2\geq t=a+b$ and $|D|=n_2\geq b\geq h$. We partition D into D_1 and D_2 such that $D=D_1\cup D_2$ with $|D_1|=t-h$ and $|D_2|=n_2-t+h$. Let $G'=G-\{c_0v|v\in D_2\}+\{ad|a\in A, d\in D\}+\{bc|b\in B, c\in C-\{c_0\}\}$. Then $G'\in\mathcal{D}_n^t$

with its bipartition (V_1, V_2) and an edge-cut of edges joining c_0 to the vertices in $B \cup D_1$ of size t. In fact, G' contains all edges between $A \cup C - \{c_0\}$ and $B \cup D$, edges between c_0 and D_1 and h edges joining c_0 to B. Then G' contains $(m_1 + n_1 - 1)(m_2 + n_2) + t = |E(G')|$ vertex pairs at distance 1, $(m_2 - h) + |D_2| = m_2 + n_2 - t$ pairs of vertices at distance 3, and all the other $C_n^2 - |E(G')| - (m_2 + n_2 - t)$ vertex pairs are at the distance 2. Therefore,

$$|E(G')| + \frac{C_n^2 - |E(G')| - (m_2 + n_2 - t)}{2} + \frac{m_2 + n_2 - t}{3} = H(G').$$

Since A, B, C, D are non-empty sets and $m_1 > 1$, then

$$H(G) - H(G') = -\frac{1}{4}(a_1 a_3 + a_2 a_4) - \frac{2}{3}m_2(n_1 - 1) - \frac{2}{3}n_2(m_1 - 1) < 0,$$

which is a contradiction. □

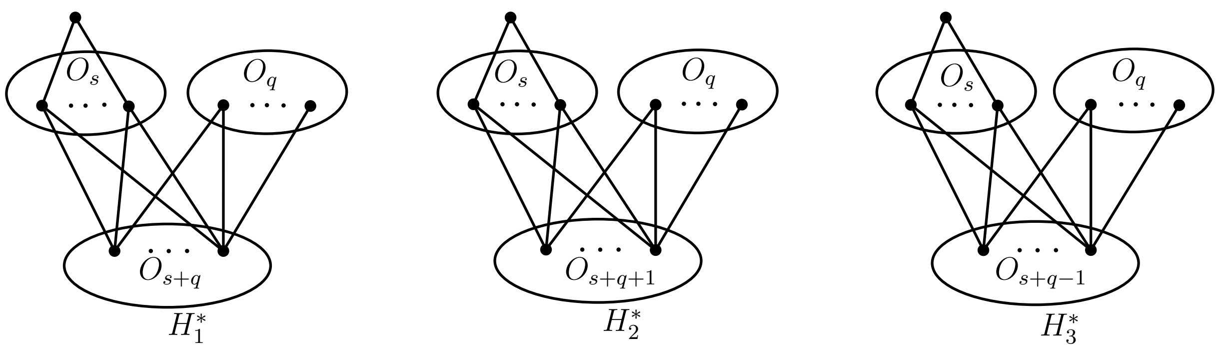

Figure 1. Graphs H_1^*, H_2^*, H_3^* in Theorems 3.1 and 3.2.

Theorem 3.1. *If $\mathcal{C}_n^s$ has the graph G with the maximum $H(G)$, where $1 \le s \le \frac{n}{2}$. Then $G \in \{H_1^*, H_2^*, H_3^*\}$, where H_1^*, H_2^* and H_3^* are depicted in Figure 1.*

Proof. Assume that, G has the maximum $H(G)$ in $\mathcal{C}_n^s$. Let I be a vertex-cut of G having s vertices, and $G_1, G_2, \cdots, G_t$ are the components of $G - I$, where $t \ge 2$.

If one of its components has a minimum of two vertices, then using Lemma 2.1 the component should be a complete bipartite .

If one of its components is a singleton i, then i must be adjacent to every vertex of I and the subgraph $G[I]$ induced by I has no edges; or else $\kappa(G) < s$. Therefore, I is contained in the same part of bipartition of G by Lemma 2.1.

Now, we consider the following cases:

- **Case 1.** If every component of $G - I$ is a singleton, then $G = K_{s,n-s}$. So $t \ge \frac{n}{2} - 1$ by Corollary 3.1. It is conventional to see that, for odd n, $K_{s,n-s} \cong H_1^*$ and for even n, $K_{s,n-s} \in \{H_2^*, H_3^*\}$.

- **Case 2.** If one of its component $G - I$ has a minimum of two vertices. Then $G - I$ has precisely two components; otherwise, we can get a graph $G' \in \mathcal{C}_n^s$ by adding some edges in G such that the subgraph induced by $V(G_1 \cup G_2 \cup \cdots \cup G_{t-1})$ is a complete bipartite graph, and $H(G) < H(G')$ by Lemma 2.1, which is a contradiction. If $G - I$ has two components G_1, G_2, then by Lemma 3.3 and Remark 3.1, either $G_1 = K_1$ or $G_2 = K_1$. Let us assume that $G_2 = K_1 = \{i\}$. Then, $G_1 \cong K_{p,q}$ and u is joined to all vertices of I. So, I is contained in the same part of the bipartition of G, and each vertex of I is joined to all vertices in the same part of the bipartition of G_1 by Lemma 2.1. Hence, $G = O_s \vee_1 (K_{1,0} \cup K_{p,q})$, where $s = |I|$. And $p \geq s$ since p vertices in the same part of the bipartition of $K_{p,q}$ is a vertex-cut of G. Since G is a graph in $\mathcal{C}_n^s$ with the maximum Harary index, and by Lemmas 3.1 and 3.2, we have $s + q - 1 \leq p \leq s + q + 1$ and $G \in \{H_1^*, H_2^*, H_3^*\}$.

□

Using Lemma 3.4 and utilizing proof of the previous theorem, we conclude the following result.

Theorem 3.2. *Let $\mathcal{D}_n^t$ has the graph G with the maximum $H(G)$ and $1 \leq t \leq \frac{n}{2}$. Then $G \in \{H_1^*, H_2^*, H_3^*\}$, where H_1^*, H_2^* and H_3^* are depicted in Figure 1.*

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