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A unique and novel graph matrix for efficient extraction of structural information of networks

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Abstract

In this article, we propose a new type of square matrix associated with an undirected graph by trading off the natural embedded symmetry in them. The proposed matrix is defined using the neighbourhood sets of the vertices, called as neighbourhood matrix $\mathcal{NM}(G)$. The proposed matrix also exhibits a bijection between the product of the two graph matrices, namely the adjacency matrix and the graph Laplacian. Alternatively, we define this matrix by using the breadth-first search traversals from every vertex, and the subgraph induced by the first two levels in the level decomposition from that vertex. The two levels in the level decomposition of the graph give us more information about the neighbours along with the neighbours-of-neighbour of a vertex. This insight is required and is found useful in studying the impact of broadcasting on social networks, in particular, and complex networks, in general. We establish several properties of $\mathcal{NM}(G)$. Additionally, we also show how to reconstruct a graph G, given an $\mathcal{NM}(G)$. The proposed matrix also solves many graph-theoretic problems using less time complexity in comparison to the existing algorithms.[1]

Keywords: graph matrices, graph characterization, matrix product, graph properties, strongly regular graphs, C_4-free.
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[1] A preliminary version of this article namely the definition of the newly proposed matrix was presented in ICDM 2016(June 09-11), Siddaganga Institute of Technology, Tumkur-572102, Karnataka, INDIA and few of the characterisations presented in the Fifth India-Taiwan Conference on Discrete Mathematics(18-21 July 2017), Tamkang University, Taiwan.

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1. Introduction

In the study of complex and social networks, one of the interesting and challenging problems is to study the impact of a change that occurs to a node. In the literature, such studies are carried out to analyse the network's behavioural changes both locally as well as globally, [12]. One such problem is to reconstruct a graph when partial information is known and to predict the dynamical changes occurring in a network. To tackle this problem, we were determined to approach it by studying graphs through their matrices.

Matrices play a vital role in the study of graphs and their representations. Among all the graph matrices, adjacency matrix and Laplacian matrix has received extensive attention due to their symmetric nature and the ability to exhibit various properties [2, 6, 9]. In the literature, many other types of matrices have been associated with a graph [1, 3, 7, 9, 10]. The spectral studies on graph matrices have also received extensive attention in the literature [8, 11, 13]. For an undirected graph, most of the matrices are symmetric and not of help to solve our problem. Further, in [4], the authors discuss the product of two graphs and its representation using the product of the adjacency matrices of the graphs. Also, powers of adjacency matrix and square of distance matrix has also been studied in the literature [3]. However, there is no literature dealing with the product of two types of matrices of a graph.

In this paper, we handle one such problem involved in defining, analysing and correlating the product of graph matrices with the graph and several of its properties. To this end, we propose a novel representative matrix for a graph referred to as $\mathcal{NM}(G)$. We first define this matrix by using the notion of the neighbourhood of a vertex in a graph and then endorse its relationship with the product of two different types of graph matrices. We make sure that the matrix that we are defining in this paper is not always symmetric, and this helps us in proving many network properties quite easily.

The organisation of this paper is as follows: In section 2, we present all the basic definitions, notations and properties required. In section 3, we introduce the novel concept of $\mathcal{NM}(G)$ and discuss several of its properties. In section 4, we discover some interesting characterisations of the graph using the $\mathcal{NM}(G)$. We conclude the paper in section 5 with some insight on the future scope.

2. Definitions and Notations

Throughout this paper, we consider only undirected, unweighed simple graphs. For all basic notations and definitions of graph theory, we follow the books by J.A. Bondy and U.S.R. Murty [5] and D.B. West [14]. In this section, we present all the required notations and define the $\mathcal{NM}(G)$. Let $G(V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of all *neighbours* of v and $N_G[v] = \{v\} \cup N_G(v)$, denote the *closed neighbourhood* of v. The *degree* of a vertex v is given by $deg(v)$ or $|N_G(v)|$. Let A_G (or A) denote the *adjacency matrix* of the graph G. Let the *degree matrix* $D(G)$ (or D) be the diagonal matrix with the degree of the vertices as its diagonal elements. Let $C(G)$ be the *Laplacian matrix* obtained by $C(G) = D(G) - A_G$.

Definition 1. *Given a graph* G*, the product of the adjacency matrix and the degree matrix, denoted by* $AD = [ad_{ij}]$*, is defined as*

$$ad_{ij} = \begin{cases} |N_G(j)|, & \text{if } (i,j) \in E(G), \\ 0, & otherwise. \end{cases}$$

Similarly, the product of the degree matrix and the adjacency matrix, denoted by $DA = [da_{ij}]$*, is defined as*

$$da_{ij} = \begin{cases} |N_G(i)|, & \text{if } (i,j) \in E(G), \\ 0, & otherwise. \end{cases}$$

Remark 2.1. From the above definitions it follows immediately that $(AD)^T = DA$.

Remark 2.2. If G is regular or contains regular-components then by the definition, AD matrix is symmetric. Hence by above remark AD and DA becomes equal.

Definition 2. *Given a graph* G*, the square of the adjacency matrix* $A^2 = [a^2_{ij}]$*, is defined as*

$$a^2_{ij} = \begin{cases} |N_G(i)|, & \text{if } i = j, \\ |N_G(i) \cap N_G(j)|, & \text{if } i \neq j. \end{cases}$$

It is well known that the ij^{th} entries of the square of adjacency matrix denotes the number of walks of length 2 between i and j.

We now extend the above notion of product of graph matrices to obtain a new class of matrix and establish its properties.

3. $\mathcal{NM}(G)$ and its properties

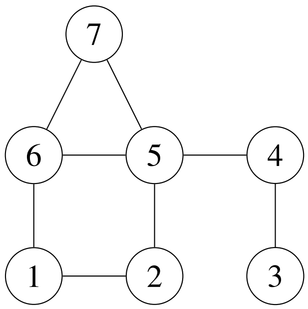
Now we introduce the idea of $\mathcal{NM}(G)$ and describe its properties

Definition 3. *Given a graph* G*, the neighbourhood matrix, denoted by* $\mathcal{NM}(G) = [nm_{ij}]$ *is defined as*

$$nm_{ij} = \begin{cases} -|N_G(i)|, & \text{if } i = j, \\ |N_G(j) \setminus N_G(i)|, & \text{if } (i,j) \in E(G), \\ -|N_G(i) \cap N_G(j)|, & \text{if } (i,j) \notin E(G). \end{cases}$$

Example 3.1. A graph G and its corresponding $\mathcal{NM}(G)$ representation are given in Figure 1. In this example, the neighbourhood set of each vertex of G is given by $N_G(1)$= {2,6}, $N_G(2)$={1,5}, $N_G(3)$={4}, $N_G(4)$={3,5}, $N_G(5)$={2,4,6,7}, $N_G(6)$={1,5,7}, $N_G(7)$={5,6}.

Proposition 3.1. *The* $\mathcal{NM}(G)$ *can also be defined by using the product of adjacency matrix and Laplacian matrix of a graph* G*.*



(a) a graph G

$$\mathcal{NM}(G) = [nm_{ij}] = \begin{pmatrix} -2 & 2 & 0 & 0 & -2 & 3 & -1 \\ 2 & -2 & 0 & -1 & 4 & -2 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 4 & -1 & -1 \\ -2 & 2 & -1 & 2 & -4 & 2 & 1 \\ 2 & -2 & 0 & -1 & 3 & -3 & 1 \\ -1 & -1 & 0 & -1 & 3 & 2 & -2 \end{pmatrix}$$

(b) $\mathcal{NM}$ corresponding to G

Figure 1. A graph G and its $\mathcal{NM}(G)$.

Proof. Consider the definition of product of two matrices

$$\begin{aligned} A \times C(G) &= A \times (D(G) - A) \\ &= AD - A^2 \\ &= [ad_{ij}] - [a^2_{ij}] \\ &= [ad_{ij}] - [a^2_{ij}] \\ &= \begin{cases} 0 - |N_G(i)|, & \text{if } i = j, \\ |N_G(j) \setminus N_G(i)|, & \text{if } (i,j) \in E(G), \\ 0 - |N_G(i) \cap N_G(j)|, & \text{if } (i,j) \notin E(G). \end{cases} \end{aligned}$$

Note that the last equality represents the $\mathcal{NM}(G)$. Hence the proof. □

Proposition 3.2. *Given a graph G, the $\mathcal{NM}(G)$ can be obtained from adjacency matrix and vice versa.*

Proof. By Proposition 3.1, it is immediate that the matrix $\mathcal{NM}(G)$ can be constructed from the adjacency matrix.

Given $\mathcal{NM}(G)$, if $i \neq j, nm_{ij} > 0$ implies that by the definition, $nm_{ij} = |N_G(j) \setminus N_G(i)|$ and that $(i,j) \in E(G)$. Similarly, if $nm_{ij} \leq 0$ this implies either $i = j$ or $(i,j) \notin E(G)$.

Therefore, we can now define $a_{ij} = \begin{cases} 1, & \text{if } nm_{ij} > 0, \\ 0, & \text{otherwise.} \end{cases}$ □

Example 3.2. From the $\mathcal{NM}(G)$ in Figure 1(b), constructing the adjacency matrix as defined in the above proposition, we get the matrix A as shown in Figure 2. It is immediate that A is the required adjacency matrix.

An alternative interpretation or a way of defining the $\mathcal{NM}(G)$ is to consider the breadth first traversal starting at a vertex i. By inspection of the first two levels in this level decomposition, we can obtain the respective i^{th} row of the $\mathcal{NM}(G)$. We prove this equivalence in the following proposition.

$$A = [a_{ij}] = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 2. Adjacency matrix of G constructed from $\mathcal{NM}(G)$

Proposition 3.3. *Given a graph G, the entries of any row of an $\mathcal{NM}(G)$ corresponds to the subgraph induced by first two levels of level decomposition of the graph rooted at the vertex corresponding to that row.*

Proof. Consider any i^{th} row of the $\mathcal{NM}(G)$. By the definition of $\mathcal{NM}(G)$, vertex i is adjacent to a vertex j if and only if $nm_{ij} > 0$. This gives us the neighbours of i, namely $N_G(i)$, or the first level of the level decomposition. From the following observations, we obtain the vertices that lie in the next level.

1. The diagonal entries are always negative and in particular, if $nm_{ii} = -c$, then the degree of the vertex is c and that there will be exactly c positive entries in that row.
2. For some positive integer c, if $nm_{ij} = c$ then $j \in N_G(i)$ and that there exists $c - 1$ vertices at distance 2 from i through j.
3. If $nm_{ij} = -c$, then the vertex j belongs to the second level of the decomposition and moreover, there exists c paths of length two from vertex i to j. In other words, there exist c common neighbours between vertex i and j.
4. If an entry, $nm_{ij} = 0$ then the distance between vertex i and j is at least 3 or the vertex j is isolated

Combining these observations, one can easily obtain the subgraph induced by two levels of decomposition of G rooted at the vertex i.
On the other hand, from the Breadth first traversal tree rooted at a vertex i and the definition of $\mathcal{NM}(G)$ we can immediately write the corresponding i^{th} row entry by examining the vertices and their position in the first two levels. □

Analogous to $\mathcal{NM}(G)$ we can also define the product matrix $\mathcal{MN}(G)$ as follows.

Definition 4. *The product of Laplacian matrix and adjacency matrix denoted by $\mathcal{MN} = [mn_{ij}]$ is defined as*

$$mn_{ij} = \begin{cases} -|N_G(i)|, & \text{if } i = j, \\ |N_G(i) \setminus N_G(j)|, & \text{if } (i,j) \in E(G), \\ -|N_G(i) \cap N_G(j)|, & \text{if } (i,j) \notin E(G). \end{cases}$$

Remark 3.1. Note that $\mathcal{MN}(G)$ can be obtained by $C(G) \times A = D(G)A - A^2$.

Remark 3.2. For an undirected simple graph G, we have

$$\begin{aligned} (\mathcal{NM})^T &= (A \times C(G))^T \\ &= C(G)^T \times A^T \\ &= C(G) \times A \\ &= \mathcal{MN}. \end{aligned}$$

Proposition 3.4. *The $\mathcal{NM}(G)$ matrix for any graph G is a singular matrix.*

Proof. Let A be the adjacency matrix and C be the Laplacian matrix of a graph G. It is enough to prove $det(\mathcal{NM}) = 0$. Since

$$\begin{aligned} det(\mathcal{NM}(G)) &= det(A \times C) \\ &= det(A) \times det(C) \\ &= 0. \end{aligned}$$

Since it is well know that $det(C) = 0$ we get the last equality and hence the claim. □

Proposition 3.5. *The row sum of $\mathcal{NM}(G)$ for any graph G is zero.*

Proof. Consider any i^{th} row in $\mathcal{NM}(G)$

$$\sum_{j=1}^{n} nm_{ij} = -|N_G(i)| + \sum_{j \in N_G(i)} |N_G(j) \setminus N_G(i)| - \sum_{j \notin N_G[i]} |N_G(i) \cap N_G(j)| \quad (1)$$

$$\sum_{j=1}^{n} nm_{ij} = \sum_{j \in N_G(i)} [|N_G(j) \setminus N_G(i)| - 1] - \sum_{j \notin N_G[i]} |N_G(i) \cap N_G(j)| \quad (2)$$

Consider the level decomposition of the graph G from the vertex i.

Observe that, $\sum\limits_{j \in N_G(i)} [|N_G(j) \setminus N_G(i)| - 1]$ is the number of edges connecting the vertices from level 1 to level 2. Similarly, $\sum\limits_{j \notin N_G[i]} |N_G(i) \cap N_G(j)|$ denote the number of edges connecting the vertices from level 2 to level 1. So, we have

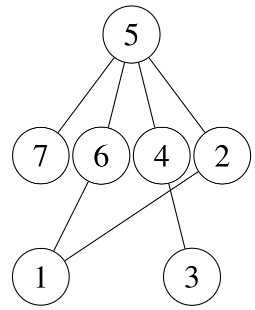
$$\sum_{j \in N_G(i)} [|N_G(j) \setminus N_G(i)| - 1] = \sum_{j \notin N_G[i]} |N_G(i) \cap N_G(j)|. \quad (3)$$

Substitute equation (3) in equation (2) we get the row sum of $\mathcal{NM}(G)$ is zero. □

Remark 3.3. Suppose, given any row of the matrix $\mathcal{NM}(G)$, the minimum value in the row represents the degree of the respective vertex. Hence, considering the position of minimum value as the diagonal position of the row (since $nm_{ii} = -|N_G(i)|$) enables us to identify the vertex that it represent.

$$[nm_{ij}] = \begin{pmatrix} -2 & 2 & -1 & 2 & -4 & 2 & 1 \end{pmatrix}$$

(a) i^{th} row of $\mathcal{NM}$ matrix



(b) A subgraph corresponding to the row matrix

Figure 3. A random row vector of $\mathcal{NM}(G)$ from Example 3.1 and the corresponding rooted subgraph representation

Example 3.3. Consider the row given by Figure 3(a) from the Example 3.1, we see that the minimum value is -4 occurring at 5^{th} position of the row, implying that the row represents vertex 5 in the example. Further, we also get the row sum of $\mathcal{NM}(G)$ of this row vector is zero.

In addition, using the row entries and the two level decomposition, we can construct the induced subgraph rooted at vertex i. Here, observe that nm_{51} is -2 implies that there are two paths between vertices 5 and 1 of length 2. Similar observation leads to the fact that among the vertices $\{2, 4, 6\}$, vertex 1 is adjacent to two of them while vertex 3 is adjacent to the remaining one from the same set. To determine the adjacency of the vertex 1 and vertex 3, we trace the corresponding rows in the $\mathcal{NM}$ matrix, namely $nm_{12}, nm_{14}, nm_{16}$. Figure 3(b) shows the constructed subgraph rooted at vertex 5 by using the corresponding row entries.

Proposition 3.6. *For any $1 \leq i \leq n$, the i^{th} column sum of $\mathcal{NM}(G)$ is equal to*

$$\sum_{j \in N_G(i)} \Bigg(|N_G(i)| - |N_G(j)| \Bigg).$$

Proof. By Remark 3.2, we have $(\mathcal{MN})^T = \mathcal{NM}$. This implies the column sum of $\mathcal{NM}$ matrix is equal to the row sum of $\mathcal{MN}$ matrix. Therefore, we get

$$\begin{aligned}
\sum_{j=1}^{n} nm_{ji} &= \sum_{j=1}^{n} mn_{ij} \\
&= -|N_G(i)| + \sum_{j \in N_G(i)} |N_G(i) \setminus N_G(j)| - \sum_{j \notin N_G[i]} |N_G(i) \cap N_G(j)| \\
&= \sum_{j \in N_G(i)} |N_G(i) \setminus N_G(j)| - \sum_{j \in N_G(i)} |N_G(j) \setminus N_G(i)| (\text{ By Proposition } 3.5) \\
&= \sum_{j \in N_G(i)} \Bigg(|N_G(i)| - |N_G(j)| \Bigg).
\end{aligned}$$

Hence the proof. $\square$

4. Graph characterization using neighbourhood matrix $\mathcal{NM}(G)$

Note that the matrix $\mathcal{NM}(G)$ is not always symmetric. The next result characterizes the graphs for which $\mathcal{NM}(G)$ will be symmetric.

Proposition 4.1. *The matrix $\mathcal{NM}(G)$ is symmetric if and only if the graph G is either regular or contains regular components.*

Proof. Let G be a graph with $w(G)$ components, say $G_1, G_2, \dots, G_w$ such that each G_z is regular with degree r_z, $1 \leq z \leq w(G)$. By the definition of $\mathcal{NM}(G)$ when i is not adjacent to j then $nm_{ij} = nm_{ji}$ and when i is adjacent to j, then $(i, j) \in E(G_z)$, for some z, $1 \leq z \leq w(G)$, and

$$nm_{ij} = |N_G(j)| - |N_G(i) \cap N_G(j)| = r_z - |N_G(i) \cap N_G(j)| \tag{4}$$

$$nm_{ji} = |N_G(i)| - |N_G(i) \cap N_G(j)| = r_z - |N_G(i) \cap N_G(j)| \tag{5}$$

From (4) and (5) we have $nm_{ij} = nm_{ji}$. Therefore the $\mathcal{NM}(G)$ is symmetric when the graph G has regular components.

Conversely, let $\mathcal{NM}(G)$ be symmetric. We know that, $\mathcal{NM}(G)$ can be written as $AD - A^2$. Since sum of symmetric matrices is symmetric and $AD = \mathcal{NM} + A^2$, we must have AD to be symmetric. But from Remark 2.2, it is known that AD is symmetric whenever G is the union of regular components. □

Recall that a graph G is said to be a strongly regular graph with parameters (n, k, μ_1, μ_2), if G is a k-regular graph on n vertices in which every pair of adjacent vertices has μ_1 common neighbours and every pair of non-adjacent vertices has μ_2 common neighbours.

Proposition 4.2. *If a graph G is strongly regular then the entries of $\mathcal{NM}(G)$ contains either two or three distinct values.*

Proof. By the definition of $\mathcal{NM}(G)$ it immediate follows that for a strongly regular graph G,

$$nm_{ij}(G) = \begin{cases} -k, & \text{if } i = j, \\ k - \mu_1, & \text{if } (i, j) \in E(G), \\ -\mu_2, & \text{if } (i, j) \notin E(G). \end{cases}$$

where $\mu_1 = |N_G(i) \cap N_G(j)|$, for $(i, j) \in E(G)$ and $\mu_2 = |N_G(i) \cap N_G(j)|$, for $(i, j) \notin E(G)$. This implies the entries of $\mathcal{NM}(G)$ of a strongly regular graph takes values from $\{-k, k - \mu_1, -\mu_2\}$ or $\{-k, k - \mu_1\}$, when $k = \mu_2$. □

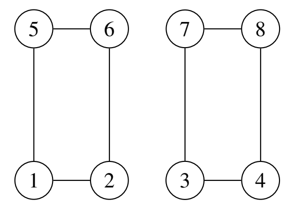
Remark 4.1. Note that the converse of the above proposition need not be true.

Example 4.1. Figure 4(a) is the $\mathcal{NM}(G)$ containing only three distinct values as entries, namely, $\{-2, 0, 2\}$. Figure 4(b) is the corresponding graph of Figure 4(a). Note that the graph is not a strongly regular graph.

Proposition 4.3. *If at least one row of $\mathcal{NM}(G)$ has no zero entries then the graph G has diameter at most 4.*

$$\mathcal{NM}(G) = \begin{pmatrix} -2 & 2 & 0 & 0 & 2 & -2 & 0 & 0 \\ 2 & -2 & 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & -2 & 0 & 0 & -2 & 2 \\ 2 & -2 & 0 & 0 & -2 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & 2 & 0 & 0 & 2 & -2 \end{pmatrix}$$

(a) $\mathcal{NM}$ matrix of a graph G

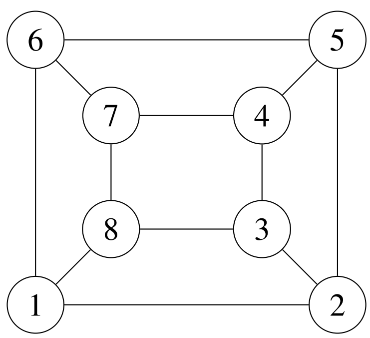


(b) Corresponding Graph G

Figure 4. A graph G and its $\mathcal{NM}$ matrix.

Proof. Let i^{th} row of $\mathcal{NM}(G)$ have no zero entries then by using the two level decomposition definition we have $d_G(i,j) \leq 2$, for all $j \in V(G) - \{i\}$, for otherwise, if $d_G(i,j) = 3$ for some j this implies $nm_{ij} = 0$. Therefore for any $j, k \in V(G) - \{i\}$ we have $d_G(i,j) \leq 2$ and $d_G(i,k) \leq 2$. So, $d_G(j,k) \leq d_G(j,i) + d_G(i,k) \leq 2 + 2 = 4$. □

Remark 4.2. Note that the converse of the above proposition need not be true. It is well known that the cubic graph on 8 verrtices (Q_3) has diameter 3 but every row of $\mathcal{NM}(Q_3)$ contains exactly one zero. Figure 5(a) and 5(b) represents the cubic graph on 8 vertices and $\mathcal{NM}(Q_3)$ respectively.



(a) Cube graph Q_3

$$\mathcal{NM}(Q_3) = \begin{pmatrix} -3 & 3 & -2 & 0 & -2 & 3 & -2 & 3 \\ 3 & -3 & 3 & -2 & 3 & -2 & 0 & -2 \\ -2 & 3 & -3 & 3 & -2 & 0 & -2 & 3 \\ 0 & -2 & 3 & -3 & 3 & -2 & 3 & -2 \\ -2 & 3 & -2 & 3 & -3 & 3 & -2 & 0 \\ 3 & -2 & 0 & -2 & 3 & -3 & 3 & -2 \\ -2 & 0 & -2 & 3 & -2 & 3 & -3 & 3 \\ 3 & -2 & 3 & -2 & 0 & -2 & 3 & -3 \end{pmatrix}$$

(b) $\mathcal{NM}$ matrix corresponding to Q_3

Figure 5. The graph Q_3 and its $\mathcal{NM}$ matrix

Proposition 4.4. *The $\mathcal{NM}(G)$ has no zero entries if and only if the graph G has diameter at most 2.*

Proof. Suppose $\mathcal{NM}(G)$ has no zero entries, that is, every row of $\mathcal{NM}(G)$ has no zero entries. From the proof of Proposition 4.3 it is immediate that for every pair of distinct vertices i and j, $d_G(i,j) \leq 2$. This implies that $diameter(G) \leq 2$. Converse follows directly by the observation that if diameter of the graph is at most 2, then every entry of the matrix becomes non-zero. □

Proposition 4.5. *The graph G is triangle-free if and only if $nm_{ij} = |nm_{jj}|$ $\forall (i,j) \in E(G)$.*

Proof. Graph G is triangle-free if and only if $N_G(i) \cap N_G(j) = \emptyset$ for $(i,j) \in E(G)$. By the definition of $\mathcal{NM}(G)$ if i is adjacent to j then $nm_{ij} = |N_G(j) \setminus N_G(i)| = |N_G(j)| - |N_G(i) \cap N_G(j)| = |N_G(j)|$. Now in the i^{th} row $nm_{ij} = |N_G(j)| = |nm_{jj}|$. □

Proposition 4.6. *Given a graph G, the number of triangles in G is given by*

$$\frac{1}{6}\sum_{i}\sum_{j\in N_G(i)}\Bigg(|nm_{jj}| - nm_{ij}\Bigg).$$

Proof. Given a vertex i, when i is adjacent to j and there exists at least one common neighbour x, for i and j, we get a triangle. Therefore, the number of triangles containing the vertex i is given by $NT(i) = \frac{1}{2}\sum_{j\in N_G(i)} |N_G(i)\cap N_G(j)|$, since a triangle $< i, j, x, i >$ will be counted twice, one for each $j, x \in N_G(i)$. Hence,

$$\begin{aligned}
\text{Total number of triangles in the graph} &= \frac{1}{3}\sum_{i} NT(i) \\
&= \frac{1}{6}\sum_{i}\sum_{j\in N_G(i)} |N_G(i)\cap N_G(j)| \\
&= \frac{1}{6}\sum_{i}\sum_{j\in N_G(i)}\Bigg(|N_G(j)| - |N_G(j) - N_G(i)|\Bigg) \\
&= \frac{1}{6}\sum_{i}\sum_{j\in N_G(i)}\Bigg(|nm_{jj}| - nm_{ij}\Bigg).
\end{aligned}$$

Hence the claim. □

Remark 4.3. It is well known that number of triangle in a graph is equal to $\frac{1}{6}trace(A^3)$ or $\frac{1}{6}\sum_{i=1}^{n}\lambda_i^3$, where A is the adjacency matrix of the graph and $\lambda_i, 1 \le i \le n$ is the eigenvalue of A. Note that if we want to count a triangle using the $\mathcal{NM}(G)$ the computational time involved is very less when compared to computing $\frac{1}{6}trace(A^3)$ or $\frac{1}{6}\sum_{i=1}^{n}\lambda_i^3$.

Proposition 4.7. *Given a graph G, the number of 4 cycles(including induced and non-induced) is equal to* $\frac{1}{4}\sum_{i=1}^{n}\left(\sum_{j\in N_G(i)}\binom{|nm_{jj}| - nm_{ij}}{2} + \sum_{j\notin N_G(i)}\binom{|nm_{ij}|}{2}\right)$.

Proof. Given a graph G on n vertices, the number of 4-cycles containing the vertex i is given by $\sum_{j=1, j\neq i}^{n}\binom{|N_G(i)\cap N_G(j)|}{2}$. Hence the total number of 4-cycles (both induced and not induced) can be given by,

$$\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\binom{|N_G(i)\cap N_G(j)|}{2}$$
$$= \frac{1}{4}\sum_{i=1}^{n}\left(\sum_{j\in N_G(i)}\binom{|N_G(i)\cap N_G(j)|}{2}+\sum_{j\notin N_G(i)}\binom{|N_G(i)\cap N_G(j)|}{2}\right)$$
$$= \frac{1}{4}\sum_{i=1}^{n}\left(\sum_{j\in N_G(i)}\binom{|nm_{jj}|-nm_{ij}}{2}+\sum_{j\notin N_G(i)}\binom{|nm_{ij}|}{2}\right).$$

□

Remark 4.4. Note that in the above proof, $\frac{1}{4}\sum_{i=1}^{n}\sum_{j\notin N_G(i)}\binom{|nm_{ij}|}{2}$ gives a count of the total number of induced C_4 plus half the number of $K_4-\{e\}$. Similarly, $\frac{1}{4}\sum_{i=1}^{n}\sum_{j\in N_G(i)}\binom{|nm_{jj}|-nm_{ij}}{2}$ gives the total number of K_4 along with half the number of $K_4-\{e\}$ in the graph.

Proposition 4.8. *A graph G is C_4-free if and only if $nm_{ij}\geq -1$, for every $(i,j)\notin E(G)$.*

Proof. By the definition of $\mathcal{NM}(G)$, we can conclude that $nm_{ij}\geq -1$ if and only if $|N_G(i)\cap N_G(j)|\leq 1,(i,j)\notin E(G)$. This implies that G has no induced C_4. □

Recall that the girth of a graph is the length of a shortest cycle contained in the graph.

Proposition 4.9. *A graph G has girth at least 5 if and only if $nm_{ij}=|nm_{jj}|$ for every edge $(i,j)\in E(G)$ and $nm_{ij}\geq -1$, for every pair $(i,j)\notin E(G)$.*

Proof. By Proposition 4.5, we have that the graph G is Triangle free if and only if $nm_{ij}=|nm_{jj}|$, for every edge $(i,j)\in E(G)$.
Also, by Proposition 4.8 we have that the graph has no induced C_4 if and only if for every pair $(i,j)\notin E(G)$, $nm_{ij}\geq -1$. Therefore we can conclude G has girth at least 5. □

5. Conclusion and Future directions

In this paper, we have introduced a new graph matrix ($\mathcal{NM}(G)$) that can be associated with a graph to reveal more information when compared to the adjacency matrix. We have also systematically demonstrated the equivalence of the $\mathcal{NM}(G)$ and the product of two other existing graph matrices, namely adjacency and Laplacian matrices. Further, we have endorsed its relationship with the concept of level decomposition of the graph. Further, we have also substantiated the usefulness of the $\mathcal{NM}(G)$ by identifying numerous properties with the aid of this matrix. In this process, we have shown many simple properties, such as counting the number of triangles in a graph, can be done in minimal time.

In our first attempt to analyse a new graph matrix, we have only studied its correctness and a few of its properties in this paper. This graph matrix seems to be quite promising and be applicable in studying problems relating to domination in graphs and graph isomorphism problem. As an extension of this current work, our subsequent research article comprises of the study of the $\mathcal{NM}$ spectrum.

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