Electronic Journal of Graph Theory and Applications **7** (2) (2019), 315–328

Electronic Journal of Graph Theory and Applications

Bounds for graph energy in terms of vertex covering and clique numbers

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Abstract

Let G be a simple graph with n vertices, m edges and having adjacency eigenvalues λ_1, λ_2, $\ldots$, λ_n. The energy $\mathbb{E}(G)$ of the graph G is defined as $\mathbb{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. In this paper, we obtain the upper bounds for the energy $\mathbb{E}(G)$ in terms of the vertex covering number τ, the clique number ω, the number of edges m, maximum vertex degree d_1 and second maximum vertex degree d_2 of the connected graph G. These upper bounds improve some of the recently known upper bounds.

Keywords: graph energy, vertex covering number, clique number, maximum degree
Mathematics Subject Classification : 05C30, 05C50
DOI: 10.5614/ejgta.2019.7.2.9

1. Introduction

Let $G(V, E)$ be a finite and simple graph with n vertices and m edges and having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A = (a_{ij})$ of G is a $(0, 1)$-square matrix of order n whose $(i, j)^{th}$-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. The spectrum of the adjacency matrix is called the adjacency spectrum of the graph G.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of G. Let $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ be the

Received: 24 August 2018, Revised: 27 April 2019, Accepted: 21 May 2019.

absolute value eigenvalues of G. Gutman [17] of G defined the energy of G as

$$\mathbb{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Among the pioneering results of the theory of graph energy are the lower and upper bounds for energy, see [2, 5, 15, 16, 18, 19, 22, 26] and the references therein. For more information about energy of graph see [1, 9, 10, 11, 12, 14, 23] and related results see [1, 24, 25].

A subset S of the vertex set $V(G)$ is said to be a *covering set* of G if every edge of G is incident to at least one vertex in S. A covering set with minimum cardinality among all covering sets is called the *minimum covering set* of G and its cardinality, which is denoted by $\tau = \tau(G)$ is called the *vertex covering number* of the graph G. If H is a subgraph of the graph G, we denote the graph obtained by removing the edges in H from G by $G \setminus H$ (that is, only the edges of H are removed from G).

As usual $P_n,\ C_n,\ K_n$ and $K_{s,t}$, respectively, denote the path on n vertices, the cycle on n vertices, the complete graph on n vertices and the complete bipartite graph on $s+t$ vertices. For other undefined notations and terminology, the readers are referred to [4, 21].

The rest of the paper is organized as follows. In Section 2, we obtain some upper bounds for $\mathbb{E}(G)$ in terms of the vertex covering number τ, the number of edges m, maximum vertex degree d_1 and second maximum vertex degree d_2 of the connected graph G. In Section 3, we obtain the upper bounds for the energy $\mathbb{E}(G)$ in terms of the vertex covering number τ, the number of edges m and the adjacency rank r of the connected graph G. These upper bounds improve some of the recently known upper bounds for the energy $\mathbb{E}(G)$ of a connected graph.

2. Upper bounds in terms of clique number and vertex covering number

Consider a real symmetric matrix M of order n. Let $s_i(M)$, $i = 1, 2, \ldots, n$, be the singular values (the positive square roots of the eigenvalues of the matrix M^*M are called the singular values of the matrix M) and $x_i(M)$ be the eigenvalues of M. Then $s_i(M) = |x_i(M)|$, for all $i = 1, 2, \ldots, n$. In the light of this definition, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of the graph G, the energy $\mathbb{E}(G)$ [20] can also be defined as

$$\mathbb{E}(G) = \sum_{i=1}^{n} s_i(A), \tag{1}$$

where $s_i(A)$ are the singular values of the adjacency matrix A of the graph G.

The following lemma can be found in [6].

Lemma 2.1. *Let X, Y and Z be square matrices of order n such that $Z = X + Y$. Then*

$$\sum_{i=1}^{n} s_i(Z) \leq \sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y).$$

Moreover, equality holds if and only if there exists an orthogonal matrix P such that PX and PY are both positive semi-definite matrices.

Let G be a connected graph of order n having clique number ω. Then K_ω is a subgraph of G. Let $S = \{v_1, v_2, \dots, v_{\omega-1}\}$ be a minimum vertex covering set of K_ω. Since any vertex covering set C of G contains S as its part, so let $C = S \cup \{v_\omega, \dots, v_\tau\}$ be a minimum vertex covering set of G. We define Γ_1 to be the family of all connected graphs of order n having clique number ω except for the graphs G when the vertices in a vertex covering set $S = \{v_1, v_2, \dots, v_{\omega-1}\}$ of the subgraph K_ω have the property that there are pendent vertices incident at some $v_i \in S$ or any two vertices of S form a triangle with a vertex $v \in V(G) \setminus C$, where C is the vertex covering set of G. Let $\Gamma_1^{'}$ be the family of all connected graphs having clique number ω such that the vertices in the vertex covering set $S = \{v_1, v_2, \dots, v_{\omega-1}\}$ of the subgraph K_ω have the property that there are $a \geq 1$ pendent vertices incident at each $v_i \in S$, $i = 1, 2, \dots, t$ where $1 \leq t \leq \omega$. Similarly, let $\Gamma_1^{''}$ be the family of all connected graphs having clique number ω such that the vertices in the vertex covering set $S = \{v_1, v_2, \dots, v_{\omega-1}\}$ of the subgraph K_ω have the property that there are $a_i \geq 1$ pendent vertices incident at each $v_i \in S$, for $i = 1, 2, \dots, t$ where $1 \leq t \leq \omega$.

The adjacency matrix of a graph with some symmetry can be put in the form

$$M = \begin{pmatrix} X & \beta & \dots & \beta & \beta \\ \beta^{\intercal} & B & \dots & C & C \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \beta^{\intercal} & C & \dots & B & C \\ \beta^{\intercal} & C & \dots & C & B \end{pmatrix}, \tag{2}$$

where $X \in R^{t\times t}$, $\beta \in R^{t\times s}$ and $B, C \in R^{s\times s}$, such that $n = t + cs$, where c is the number of copies of B. Let $\sigma^{(k)}(Y)$ indicate the multi-set formed by k copies of the spectrum of Y, denoted by $\sigma(Y)$.

Lemma 2.2. *[7] If M is a matrix as in (2) having $c \geq 1$ copies of the block B, then*

(i) $\sigma^{[c-1]}(B - C) \subseteq \sigma(M)$;

(ii) $\sigma(M) \setminus \sigma^{[c-1]}(B - C) = \sigma(M')$ *is the set of the remaining $t + s$ eigenvalues of M, where* $M' = \begin{pmatrix} X & \sqrt{c}\beta \\ \sqrt{c}\beta^{\intercal} & B + (c-1)C \end{pmatrix}$ *and $\sigma^{[c-1]}(X)$ means that the spectrum of matrix X is repeated $c - 1$ times.*

Let $S_\omega(a_1, a_2, \dots, a_\omega)$, $a_i \geq 0$, $1 \leq i \leq \omega$, be the family of connected graphs of order $n = \sum_{i=1}^{\omega}(a_i + 1)$ with m edges having a_i pendent vertices attached at the i^{th} vertex of the clique K_ω. For the family of graphs $S_\omega(a_1, a_2, \dots, a_\omega)$, we have the following result.

Lemma 2.3. *Let G be a connected graph of order n having m edges which belongs to the family $S_\omega(a_1, a_2, \dots, a_\omega)$ with $a_i = a \geq 1$, for $1 \leq i \leq t$ and $a_i = 0$ for $i \geq t + 1$, $1 \leq t \leq \omega$. Then*

$$\mathbb{E}(G) = \begin{cases} 2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3, & \text{if } 1 \leq t \leq \omega - 1, \\ (\omega - 1)\sqrt{4a+1} + \sqrt{(\omega-1)^2 + 4a}, & \text{if } t = \omega, \end{cases}$$

where b_3 is the smallest root of the polynomial $g(a, x) = x^3 - (\omega-2)x^2 - (a+\omega-1)x + a(\omega-t-1)$.

Proof. Let G be the given graph in the family $S_\omega(a_1, a_2, \ldots, a_\omega)$ with $a_i = a$, for $1 \leq i \leq t$ and $a_i = 0$ for $i \geq t+1$, $1 \leq t \leq \omega$. We first suppose that $1 \leq t \leq \omega - 1$.

Let $N_a = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix}_{a+1}$ and $C_{q\times q} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}_{a+1}$, where $q = a+1$.

By suitably labelling the vertices of G, it can be seen that the adjacency matrix of G can be written as

$$A(G) = \begin{pmatrix} N_a & C_{q\times q} & \ldots & C_{q\times q} & C_{q\times 1} & \ldots & C_{q\times 1} \\ C_{q\times q} & N_a & \ldots & C_{q\times q} & C_{q\times 1} & \ldots & C_{q\times 1} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \\ C_{q\times q} & C_{q\times q} & \ldots & N_a & C_{q\times 1} & \ldots & C_{q\times 1} \\ C_{1\times q} & C_{1\times q} & \ldots & C_{1\times q} & [0] & \ldots & [1] \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \\ C_{1\times q} & C_{1\times q} & \ldots & C_{1\times q} & [1] & \ldots & [1] \\ C_{1\times q} & C_{1\times q} & \ldots & C_{1\times q} & [1] & \ldots & [0] \end{pmatrix}_{\omega}.$$

Taking $X = \begin{pmatrix} N_a & C_{q\times q} & \ldots & C_{q\times q} \\ C_{q\times q} & N_a & \ldots & C_{q\times q} \\ \vdots & \vdots & \ldots & \vdots \\ C_{q\times q} & C_{q\times q} & \ldots & N_a \end{pmatrix}_t$, $\beta = \begin{pmatrix} C_{q\times 1} \\ C_{q\times 1} \\ \vdots \\ C_{q\times 1} \end{pmatrix}$, $B = [0]$ and $C = [1]$ in (2), so from Lemma 2.2, it follows that $\sigma(A(G)) = \sigma^{[\omega-t-1]}([-1]) \cup \sigma(N_a^{'})$, where

$$N_a^{'} = \begin{pmatrix} N_a & \ldots & C_{q\times q} & \sqrt{\omega-t}C_{q\times 1} \\ C_{q\times q} & \ldots & C_{q\times q} & \sqrt{\omega-t}C_{q\times 1} \\ \vdots & \ldots & \vdots & \vdots \\ C_{q\times q} & \ldots & N_a & \sqrt{\omega-t}C_{q\times 1} \\ \sqrt{\omega-t}C_{1\times q} & \ldots & \sqrt{\omega-t}C_{1\times q} & [\omega-t-1] \end{pmatrix}.$$

Interchanging first row with last row and then first column with last column, we obtain a matrix permutation similar to $N_a^{'}$. Since the similar matrices have same spectrum, therefore, from the resulting matrix, taking $X = [\omega-t-1]$, $\beta = [\sqrt{\omega-t}C_{1\times q}]$, $B = N_a$, $C = C_{q\times q}$, in (2), from Lemma 2.2, it follows that $\sigma(N_a^{'}) = \sigma^{[t-1]}([N_a - C_{q\times q}]) \cup \sigma(N_a^{''})$, where

$$N_a^{''} = \begin{pmatrix} \omega-t-1 & \sqrt{t\omega-t^2} & 0\ldots & 0 & \\ \sqrt{t\omega-t^2} & t-1 & 1 & \ldots & 1 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 1 & 0 & \ldots & 0 \end{pmatrix}_{a+2}.$$

Taking

$$X = \begin{pmatrix} \omega - t - 1 & \sqrt{t\omega - t^2} \\ \sqrt{t\omega - t^2} & t-1 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B = [0], C = [0],$$

in (2), from Lemma 2.2, it follows that $\sigma(N_a^{''}) = \sigma^{[a-1]}([0]) \cup \sigma(N_a^{(iv)})$, where

$$N_a^{(iv)} = \begin{pmatrix} \omega - t - 1 & \sqrt{t\omega - t^2} & 0 \\ \sqrt{t\omega - t^2} & t-1 & \sqrt{a} \\ 0 & \sqrt{a} & 0 \end{pmatrix}.$$

For the matrix $N_a - C_{q\times q}$, taking $X = [-1],\ \beta = [1],\ B = [0],\ C = [0]$ in (3), by Lemma 2.2, it can be seen that $\sigma(N_a - C_{q\times q}) = \{0^{[a-1]}, a_1, a_2\}$, where $a_1 = \frac{-1+\sqrt{4a+1}}{2},\ a_2 = \frac{-1-\sqrt{4a+1}}{2}$. Thus the spectrum of the matrix $A(G)$ is

$$\{b_1, b_2, a_1^{[t-1]}, -1^{[\omega-t-1]}, 0^{[t(a-1)]}, a_2^{[t-1]}, b_3\},$$

where $b_1 \geq b_2 \geq b_3$ are the zeros of the polynomial $g(a,x) = x^3 - (\omega-2)x^2 - (a+\omega-1)x + a(\omega-t-1)$ and $a_1 = \frac{-1+\sqrt{4a+1}}{2},\ a_2 = \frac{-1-\sqrt{4a+1}}{2}$.

It is clear that $b_1+b_2+b_3 = \omega-2,\ b_1b_2+b_2b_3+b_3b_1 = -(a+\omega-1),\ b_1b_2b_3 = -a(\omega-t-1)$. $g(a,x) = 0$ is a polynomial equation with real coefficients, it follows by Descarte's rule of signs that $g(a,x)$ has either two positive roots or no positive roots. Since $\omega \geq 2$ implies $b_1+b_2+b_3 = \omega - 2 \geq 0$, it follows that $g(a,x) = 0$ must have exactly two positive root say b_1 and b_2. By using $b_1b_2b_3 = -a(\omega-t-1)$, this implies that the third root b_3 should be negative for $1 \leq t \leq \omega-1$. Thus, for $1 \leq t \leq \omega - 1$, we have

$$\begin{aligned} E(G) &= (\omega-t-1)|-1| \\ &\quad + (t-1)\Big(\Big|\frac{-1+\sqrt{4a+1}}{2}\Big| + \Big|\frac{-1-\sqrt{4a+1}}{2}\Big|\Big) + |b_1| + |b_2| + |b_3| \\ &= 2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3, \quad \text{as} \quad b_1+b_2+b_3 = \omega-2. \end{aligned}$$

If $t = \omega$, then proceeding similarly as above it can be seen that the spectrum of the matrix $A(G)$ is

$$\{b_1, b_2, a_1^{[\omega-1]}, 0^{[\omega(a-1)]}, a_2^{[\omega-1]}\},$$

where $b_1 = \frac{(\omega-1)+\sqrt{(\omega-1)^2+4a}}{2},\ b_2 = \frac{(\omega-1)-\sqrt{(\omega-1)^2+4a}}{2}$ and $a_1 = \frac{-1+\sqrt{4a+1}}{2},\ a_2 = \frac{-1-\sqrt{4a+1}}{2}$.

Therefore, for $t = \omega$, we have $\mathbb{E}(G) = (\omega-1)\sqrt{4a+1} + \sqrt{(\omega-1)^2+4a}$, completing the proof. □

The next observation is for the family of graphs $S_\omega(a_1, a_2, \ldots, a_\omega)$, when the non-zero numbers a_i are not equal.

Lemma 2.4. *Let G be a connected graph of order n having m edges which belongs to the family $S_\omega(a_1, a_2, \ldots, a_\omega)$ with $a_i \geq 1$, for $1 \leq i \leq t$ and $a_i = 0$ for $i \geq t+1$, $1 \leq t \leq \omega$. Then $\mathbb{E}(G) \leq 2\omega - 2 + 2\sum_{i=1}^{t}\sqrt{a_i}$.*

Proof. Let G be the given graph from the family $S_\omega(a_1, a_2, \dots, a_\omega)$ with $a_i \geq 1$, for $1 \leq i \leq t$ and $a_i = 0$ for $i \geq t+1$, $1 \leq t \leq \omega$. Since the clique number of G is $\omega \geq 2$, it follows that K_ω is a subgraph of G. The adjacency spectrum of K_ω is $\{\omega - 1, -1^{[\omega-1]}\}$. Therefore, $\mathbb{E}(K_\omega) = 2\omega - 2$. If we remove the edges of K_ω from G, then the adjacency matrix of G can be decomposed as $A(G) = A(K_\omega \cup (n-\omega)K_1) + A(H)$, where $H = G \setminus K_\omega$ is the graph obtained from G by removing the edges of K_ω. Applying Lemma 2.1 and using the fact $E(K_\omega \cup (n-\omega)K_1) = E(K_\omega)$, we have

$$\mathbb{E}(G) \leq \mathbb{E}(K_\omega) + \mathbb{E}(H) = 2\omega - 2 + \mathbb{E}(H). \tag{3}$$

From the hypothesis of the theorem, it is clear that $H = G \backslash K_\omega$ is a forest having $t \geq 1$ components each of which is a star. For $1 \leq t \leq \omega$, let $T_i = K_{a_i,1}$, $i = 1, 2, \dots, t$ be the i^{th} component of H. It is clear that $H = T_1 \cup T_2 \cup \cdots \cup T_t$. Therefore, by Lemma 2.1 and the fact $\mathbb{E}(T_i) = \mathbb{E}(K_{a_i,1}) = 2\sqrt{a_i}$, we have

$$\mathbb{E}(H) \leq \mathbb{E}(T_1) + \mathbb{E}(T_2) + \cdots + \mathbb{E}(T_t) = 2\sum_{i=1}^{t} \sqrt{a_i}.$$

Using this in (3), the result follows. □

The following result gives an upper bound for the energy $\mathbb{E}(G)$ in terms of the vertex covering number τ, the clique number ω and the number of edges m of the graph G.

Theorem 2.5. *Let G be a connected graph of order $n \geq 2$ with m edges having clique number $\omega \geq 2$ and vertex covering number $\tau \geq 2$. If $G \in \Gamma_1$, then*

$$\mathbb{E}(G) \leq 2\sqrt{(\tau - \omega + 1)(m - \frac{\omega(\omega-1)}{2})} + 2\omega - 2, \tag{4}$$

with equality if $\tau = \omega - 1$ and $G \cong K_\omega$. If $G \in \Gamma_1^{'}$, then

$$\mathbb{E}(G) \leq 2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3 + 2\sqrt{(\tau - \omega + 1)(m - at - \frac{\omega(\omega-1)}{2})}, \tag{5}$$

where b_3 is the smallest root of the polynomial $g(a,x) = x^3 - (\omega-2)x^2 - (a+\omega-1)x + a(\omega-t-1)$. Equality occurs in (5) if $\tau = \omega - 1$ and $G \cong S_\omega(a_1, a_2, \dots, a_\omega)$ with $a_i = a \geq 1$, for $1 \leq i \leq t$ and $a_i = 0$ for $i \geq t+1$, $1 \leq t \leq \omega - 1$. And, if $G \in \Gamma_1^{''}$, then

$$\mathbb{E}(G) \leq 2\omega - 2 + 2\sum_{i=1}^{t} \sqrt{a_i} + 2\sqrt{(\tau - \omega + 1)(m - \sum_{i=1}^{t} \sqrt{a_i} - \frac{\omega(\omega-1)}{2})}.$$

Proof. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. Let τ be the vertex covering number and C be the minimum vertex covering set of G. With out loss of generality let $C = \{v_1, v_2, \dots, v_\tau\}$. Since the clique number of G is ω, so

K_ω is a subgraph of G. Using a well known fact that the vertex covering number of a complete graph on ω vertices is $\omega - 1$, let $v_1, v_2, \dots, v_{\omega-1}$ be the vertices in C, which belong to $V(K_\omega)$. The adjacency spectrum of K_ω is $\{\omega - 1, -1^{[\omega-1]}\}$. Thus, $E(K_\omega) = 2\omega - 2$. We first suppose that $G \in \Gamma_1$. If we remove the edges of K_ω from G, the adjacency matrix of G can be decomposed as $A(G) = A(K_\omega \cup (n-\omega)K_1) + A(G \setminus K_\omega)$, where $G \setminus K_\omega$ is the graph obtained from G by removing the edges of K_ω. So, using Lemma 2.1 and the fact $\mathbb{E}(K_\omega \cup (n-\omega)K_1) = \mathbb{E}(K_\omega)$, we get

$$\mathbb{E}(G) \leq \mathbb{E}(K_\omega) + \mathbb{E}(G \setminus K_\omega). \tag{6}$$

To complete the proof in this case, we need to estimate $\mathbb{E}(G \setminus K_\omega)$, which is done as follows. Let $G_\omega, G_{\omega+1}, \dots, G_\tau$ be the spanning subgraphs of $H = G \setminus K_\omega$ corresponding to the vertices $v_\omega, v_{\omega+1}, \dots, v_\tau$ of C, having vertex set same as H and edge sets defined as

$$\begin{aligned}
&E(G_\omega) = \{v_\omega v_t : v_t \in N(v_\omega)\},\\
&E(G_{\omega+1}) = \{v_{\omega+1} v_t : v_t \in N(v_{\omega+1}) \setminus \{v_\omega\}\}\\
&\vdots\\
&E(G_\tau) = \{v_\tau v_t : v_t \in N(v_\tau) \setminus \{v_\omega, v_{\omega+1}, \dots, v_{\tau-1}\}\}.
\end{aligned}$$

For $i = \omega, \omega+1, \dots, \tau$, let $m_i = |E(G_i)|$. Clearly $E(H) = E(G_\omega) \cup E(G_{\omega+1}) \cup \cdots \cup E(G_\tau)$ and $G_i = K_{1,m_i} \cup (n(H) - m_i - 1)K_1$, for all $i = \omega, \omega+1, \dots, \tau$. Also, it is clear that

$$A(H) = A(G_\omega) + A(G_{\omega+1}) + \cdots + A(G_\tau). \tag{7}$$

The adjacency spectrum of $G_i = K_{1,m_i} \cup (n(H) - m_i - 1)K_1$ is $\{\pm\sqrt{m_i}, 0^{[n(H)-2]}\}$. Therefore,

$$\mathbb{E}(G_i) = \mathbb{E}(K_{1,m_i} \cup (n(H) - m_i - 1)K_1) = 2\sqrt{m_i}, \quad \text{for all } i = 1, 2, \dots, \tau. \tag{8}$$

Using Lemma 2.1 to equation (7) and applying (8) and Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
\mathbb{E}(G \setminus K_\omega) &= \mathbb{E}(H) \leq \mathbb{E}(G_\omega) + \mathbb{E}(G_{\omega+1}) + \cdots + \mathbb{E}(G_\tau)\\
&= 2\sqrt{m_\omega} + 2\sqrt{m_{\omega+1}} + \cdots + 2\sqrt{m_\tau} = 2\sum_{i=\omega}^{\tau}\sqrt{m_i}\\
&\leq 2\sqrt{(\tau - \omega + 1)\sum_{i=\omega}^{\tau} m_i} = 2\sqrt{(\tau-\omega+1)(m - \frac{\omega(\omega-1)}{2})},
\end{aligned}$$

where $\sum\limits_{i=\omega}^{\tau} m_i =$ number of edges of $H = m - \frac{\omega(\omega-1)}{2}$.

This shows that

$$\mathbb{E}(G \setminus K_\omega) \leq 2\sqrt{(\tau - \omega + 1)(m - \frac{\omega(\omega-1)}{2})}.$$

Thus, from (6), we have

$$\mathbb{E}(G) \le \mathbb{E}(K_\omega) + \mathbb{E}(G \setminus K_\omega) \le 2\omega - 2 + 2\sqrt{(\tau - \omega + 1)(m - \frac{\omega(\omega-1)}{2})},$$

proving the first inequality. If equality holds in (4), all the inequalities above occurs as equalities. Clearly equality occurs in (5) if and only if $G \cong K_n$, as G is connected. Since equality occurs in Cauchy-Schwarz's inequality if and only if $m_1 = m_2 = \cdots = m_\tau$, so equality occurs in (4) if and only if $\tau = \omega - 1$ and $G \cong K_\omega$. Conversely, if $\tau = \omega - 1$ and $G \cong K_\omega$, then it is easy to see that equality holds in (4).

Next, suppose that $G \in \Gamma_1^{'}$, then $H = S_\omega(a_1, a_2, \ldots, a_\omega)$ with $a_i = a \ge 1$, for $1 \le i \le t$ and $a_i = 0$ for $i \ge t+1$, $1 \le t \le \omega - 1$ is a subgraph of G. If we remove the edges of H from G, then the adjacency matrix of G can be decomposed as $A(G) = A(H \cup (n - \omega - at)K_1) + A(G \setminus H)$, where $G \setminus H$ is the graph obtained from G by removing the edges of H. Applying Lemma 2.1 and using the fact $\mathbb{E}(H \cup (n - \omega - at)K_1) = \mathbb{E}(H)$, we have

$$\mathbb{E}(G) \le \mathbb{E}(H \cup (n - \omega - at)K_1) + \mathbb{E}(G \setminus H) = \mathbb{E}(H) + \mathbb{E}(G \setminus H). \tag{9}$$

Since $1 \le t \le \omega - 1$, from Lemma 2.3, it follows that

$$\mathbb{E}(H) = 2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3,$$

where b_3 is defined in Lemma 2.3. To estimate $\mathbb{E}(H^{'}) = \mathbb{E}(G \setminus H)$, we can proceed similarly as in the above case to obtain

$$\mathbb{E}(G \setminus H) = \mathbb{E}(H^{'}) \le 2\sqrt{(\tau - \omega + 1)(m - at - \frac{\omega(\omega-1)}{2})}.$$

Therefore from (9), we have

$$\begin{aligned}\mathbb{E}(G) \le& \mathbb{E}(H) + \mathbb{E}(G \setminus H)\\ &\le 2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3 + 2\sqrt{(\tau - \omega + 1)(m - at - \frac{\omega(\omega-1)}{2})},\end{aligned}$$

completing the proof of the second inequality. Equality case for this inequality can be discussed similarly as in the above case.

Lastly, suppose that $G \in \Gamma_1^{''}$, then $H = S_\omega(a_1, a_2, \ldots, a_\omega)$ with $a_i \ge 1$, for $1 \le i \le t$ and $a_i = 0$ for $i \ge t+1$, $1 \le t \le \omega - 1$ is a subgraph of G. If we remove the edges of H from G, then the adjacency matrix of G can be decomposed as $A(G) = A(H \cup (n - \omega - \sum_{i=1}^{t} a_i)K_1) + A(G \setminus H)$, where $G \setminus H$ is the graph obtained from G by removing the edges of H. Applying Lemmas 2.1 and 2.4 and proceeding similarly as in above cases, we obtain

$$\mathbb{E}(G) \le 2\omega - 2 + 2\sum_{i=1}^{t}\sqrt{a_i} + 2\sqrt{(\tau - \omega + 1)(m - \sum_{i=1}^{t} a_i) - \frac{\omega(\omega-1)}{2})},$$

which completes the proof of the theorem. □

Now, we obtain the following upper bound for the energy $\mathbb{E}(G)$.

Theorem 2.6. *Let G be a connected graph of order $n \geq 2$ with m edges having clique number $\omega \geq 2$ and vertex covering number $\tau \geq 2$. Let d_1 and d_2 be the maximum and second maximum degree of the graph G. If $G \in \Gamma_1$, then*

$$\mathbb{E}(G) \leq 2\omega - 2 + 2\sqrt{d_1} + 2(\tau - \omega)\sqrt{d_2},$$

with equality if $\tau = \omega - 1$ and $G \cong K_\omega$. If $G \in \Gamma_1^{'}$, then

$$\mathbb{E}(G) \leq 2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3 + 2\sqrt{d_1} + 2(\tau - \omega)\sqrt{d_2}, \tag{10}$$

where b_3 is the smallest root of the polynomial $g(a,x) = x^3 - (\omega-2)x^2 - (a+\omega-1)x + a(\omega-t-1)$. Equality occurs in (10) if $\tau = \omega - 1$ and $G \cong S_\omega(a_1, a_2, \ldots, a_\omega)$ with $a_i = a \geq 1$, for $1 \leq i \leq t$ and $a_i = 0$ for $i \geq t+1$, $1 \leq t \leq \omega - 1$. And, if $G \in \Gamma_1^{''}$, then

$$\mathbb{E}(G) \leq 2\omega - 2 + 2\sum_{i=1}^{t}\sqrt{a_i} + 2\sqrt{d_1} + 2(\tau - \omega)\sqrt{d_2}.$$

Proof. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let τ be the vertex covering number and C be the minimum vertex covering set of G. With out loss of generality, let $C = \{v_1, v_2, \ldots, v_\tau\}$. Since clique number of G is ω, it follows that K_ω is a subgraph of G. If $G \in \Gamma_1$, then proceeding similarly as in the proof of the Theorem 2.5, we arrive at

$$\begin{aligned}\mathbb{E}(G) &\leq \mathbb{E}(K_\omega) + \mathbb{E}(G \setminus K_\omega)\\ &\leq 2\omega - 2 + \mathbb{E}(G_\omega) + \mathbb{E}(G_{\omega+1}) + \cdots + \mathbb{E}(G_\tau)\\ &= 2\omega - 2 + 2\sqrt{m_\omega} + 2\sqrt{m_{\omega+1}} + \cdots + 2\sqrt{m_\tau}.\end{aligned}$$

Let $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$ be the degree sequence of the graph G, where $d_i = d(v_i)$, for all i. As the cardinality of C is minimum, the vertices in C can be picked as follows.

If v_ω has the maximum degree in graph $H = G \setminus K_\omega$, we pick v_ω as the ω^{th} vertex in C. If all the edges of graph H are incident to v_ω, then $C = \{v_1, v_2, \ldots, v_{\omega-1}, v_\omega\}$ is the minimum vertex covering set, otherwise, if $v_{\omega+1}$ has the maximum degree in graph $H - \{v_\omega\}$, we pick $v_{\omega+1}$ as the $\omega + 1^{th}$ vertex in C. If all the edges of graph $H - \{v_\omega\}$ are incident to $v_{\omega+1}$, then $C = \{v_1, v_2, \ldots, v_{\omega-1}, v_\omega, v_{\omega+1}\}$ is the minimum vertex covering set, otherwise, we proceed similarly, to obtain the other elements $v_{\omega+2}, v_{\omega+3}, \ldots, v_\tau$ of the minimum vertex covering set C.

Let $C = \{v_1, v_2, \ldots, v_{\omega-1}, v_\omega, \ldots, v_\tau\}$ be the minimum vertex covering set obtained in this way. It is clear that degree of v_ω in $G_1 = K_{m_1,1} \cup (n - m_1 - 1)K_1$ is at most d_1, giving $m_1 \leq d_1$. Also, degree of $v_{\omega+1}$ in $G_1 = K_{m_2,1} \cup (n - m_2 - 1)K_1$ is either at most d_2 or at most $d_2 - 1$, depending on whether v_ω and $v_{\omega+1}$ are non-adjacent or adjacent in H, which gives $m_2 \leq d_2$. Similarly, it can be seen that $m_i \leq d_2$, for all $i = \omega + 2, \ldots, \tau$. With this it follows that

$$\begin{aligned}\mathbb{E}(G) &\leq 2\omega - 2 + 2\sqrt{m_\omega} + 2\sqrt{m_{\omega+1}} + \cdots + 2\sqrt{m_\tau}\\ &\leq 2\omega - 2 + 2\sqrt{d_1} + 2(\tau - \omega)\sqrt{d_2}.\end{aligned}$$

This completes the proof of the first inequality. Equality case can be discussed similar to Theorem 2.5. The cases when $G \in \Gamma^{'}$ or when $G \in \Gamma^{''}$ can be discussed similarly. □

Wang [26] proved that

$$\mathbb{E}(G) \leq 2\tau\sqrt{d_1}, \tag{11}$$

with equality if and only if G is the disjoint union of τ copies of K_{1,d_1} together with some isolated vertices.

Remark 2.7. *Let G be a connected graph of order $n \geq 2$ with m edges having clique number $\omega \geq 2$, vertex covering number $\tau \geq 2$, maximum degree d_1 and second maximum degree d_2. If $G \in \Gamma_1$, then the upper bound given by Theorem 2.5 always improves the upper bound (11) as*

$$2\omega - 2 + 2\sqrt{d_1} + 2(\tau - \omega)\sqrt{d_2} \leq 2\omega - 2 + 2(\tau - \omega + 1)\sqrt{d_1} \leq 2\tau\sqrt{d_1},$$

implies that $\sqrt{d_1} \geq 0$, which is always true.

If $G \in \Gamma_1^{'}$, then the upper bound given by Theorem 2.5 is better than the upper bound (11) for $\omega \geq \frac{(t-1)\sqrt{4a+1}+2\sqrt{d_1}-t-3-2b_3}{2\sqrt{d_1}-1}$, $d_1 \neq 1$. As

$$\begin{aligned}
&2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3 + 2\sqrt{d_1} + 2(\tau-\omega)\sqrt{d_2} \\
&\leq 2\omega - t - 3 + (t-1)\sqrt{4a+1} - 2b_3 + 2(\tau-\omega+1)\sqrt{d_1} \\
&\leq 2\tau\sqrt{d_1},
\end{aligned}$$

$$\textit{if} \quad \omega \geq \frac{(t-1)\sqrt{4a+1} + 2\sqrt{d_1} - t - 3 - 2b_3}{2\sqrt{d_1} - 2}. \tag{12}$$

In particular, if $G \cong S_\omega(a_1, a_2, \ldots, a_\omega)$ with $a_i = a \geq 1$, for $1 \leq i \leq t$ and $a_i = 0$ for $i \geq t+1$, $1 \leq t \leq \omega - 1$. Then it can be seen that (12) always holds.

If $G \in \Gamma_1^{''}$, then proceeding similarly as in above cases, we note that the upper bound given by Theorem 2.5 is better than the upper bound (11) for $\omega \geq 1 + \frac{\sum_{i=1}^{t} a_i}{\sqrt{d_1}-1}$, $d_1 \neq 1$.

3. Upper bounds in terms of vertex covering number and adjacency rank

We start with the following lemma [27].

Lemma 3.1. *Let X and Y be Hermitian matrices of order n such that $Z = X + Y$. Then*

$$\begin{aligned}
\lambda_k(Z) &\leq \lambda_j(X) + \lambda_{k-j+1}(Y), \quad n \geq k \geq j \geq 1, \\
\lambda_k(Z) &\geq \lambda_j(X) + \lambda_{k-j+n}(Y), \quad n \geq j \geq k \geq 1,
\end{aligned}$$

where $\lambda_i(M)$ is the i^{th} largest eigenvalue of the matrix M.

The following observation follows from the definition of $\mathbb{E}(G)$ and the fact $tr(A) = 0$.

Lemma 3.2. *Let G be a graph with n vertices and m edges. Let ν_+ and $\nu-$ be respectively, the number of positive and number of negative adjacency eigenvalues of graph G. Then*

$$\mathbb{E}(G) = 2\sum_{i=1}^{\nu_+} \lambda_i = -2\sum_{i=1}^{\nu_-} \lambda_{n-i+1} = 2\max_{1\le k\le n} \left(\sum_{i=1}^{k} \lambda_i\right).$$

The following upper bound for the sum of the k largest Laplacian eigenvalues $S_k(G)$ of graph G can be found in [3, 13]:

$$S_k(G) \le k(\tau+1) + m - \frac{\omega(\omega-1)}{2}, \quad k = 1, 2, \ldots, n, \tag{13}$$

with equality if $k \le \omega - 1$ and $G \cong KS_{n,\omega}$.

Now, we obtain an upper bound for the energy $\mathbb{E}(G)$ of a connected bipartite graph G.

Theorem 3.3. *Let G be a connected bipartite graph of order $n \ge 2$ with m edges having vertex covering number τ and adjacency rank r. Then*

$$\mathbb{E}(G) \le \frac{r(\tau+1)}{2} + m - 1.$$

Proof. We have

$$Q(G) = D(G) + A(G) = D(G) - A(G) + 2A(G) = L(G) + 2A(G).$$

Using Lemma 3.1 with $k = i$ and $j = n$, we obtain

$$q_i \ge 2\lambda_i, \quad for \;\; i = 1, 2, \ldots, n.$$

Taking sum from 1 to ν_+ and using Lemma 3.2, we have

$$\mathbb{E}(G) = 2\sum_{i=1}^{\nu_+} \lambda_i \le \sum_{i=1}^{\nu_+} q_i = S^+_{\nu_+}(G).$$

Since G is a bipartite graph, therefore $S^+_{\nu_+}(G) = S_{\nu_+}(G)$. Now, using inequality (13) with $k = \nu_+$, we have

$$\mathbb{E}(G) \le S_{\nu_+}(G) \le \nu_+(\tau+1) + m - \frac{\omega(\omega-1)}{2},$$

that is,

$$\mathbb{E}(G) \le \nu_+(\tau+1) + m - \frac{\omega(\omega-1)}{2}, \tag{14}$$

Again, using Lemma 3.1 to $Q(G) = L(G) + 2A(G)$, with $k = n$ and $j = i$, we obtain

$$\mu_i \ge q_n - 2\lambda_{n-i+1}, \quad for \;\; i = 1, 2, \ldots, n.$$

Taking sum from 1 to ν_- and using Lemma 3.2, we have

$$\sum_{i=1}^{\nu_-} \mu_i \geq q_n\nu_- - 2\sum_{i=1}^{\nu_-} \lambda_{n-i+1} = q_n\nu_- + E(G),$$

that is

$$\mathbb{E}(G) \leq S_{\nu_-}(G) - q_n\nu_-. \tag{15}$$

Since G is a bipartite graph, therefore $\mu_n = q_n = 0$. Using inequality (3.1) with $k = \nu_-$, from (15), it follows that

$$\mathbb{E}(G) \leq S_{\nu_-}(G) \leq \nu_-(\tau+1) + m - \frac{\omega(\omega-1)}{2}. \tag{16}$$

Adding (14) and (16) and using the fact that $\nu_+ + \nu_- = r$ and $\omega = 2$ for a bipartite graph G, we obtain

$$\mathbb{E}(G) \leq \frac{r(\tau+1)}{2} + m - 1,$$

completing the proof. □

Acknowledgement We are grateful to the anonymous referees for their useful suggestions which improved the presentation of the paper. The research of S. Pirzada is supported by SERB-DST under the research project number MTR/2017/000084.

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