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Diagonal Ramsey numbers in multipartite graphs related to stars

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Abstract

Let the star in n vertices, namely $K_{1,n-1}$ be denoted by S_n. If every two coloring of the edges of a complete balanced multipartite graph $K_{j\times s}$ there is a copy of S_n in the first color or a copy of S_m in the second color, then we will say $K_{j\times s} \to (S_n, S_m)$. The size Ramsey multipartite number $m_j(S_n, S_m)$ is the smallest natural number s such that $K_{j\times s} \to (S_n, S_m)$. In this paper, we obtain the **exact** values of the size Ramsey numbers $m_j(S_n, S_m)$ for $n, m \geq 3$ and $j \geq 3$.

Keywords: Ramsey numbers in multipartite graph, diagonal Ramsey number, star
Mathematics Subject Classification : 05C55, 05D10
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1. Introduction

In this paper we concentrate on simple graphs without loops and multiple edges. Let the complete multipartite graph having j uniform sets of size s be denoted by $K_{j\times s}$ and the complete bipartite graph on $n+m$ vertices be denoted by $K_{n,m}$. Given, three graphs K_N, G and H, we say that $K_N \to (G, H)$ if K_N is colored by two colors red and blue and it contains a copy of G(in the first color) or a copy of H(in the second color). Using this notation we define the classical Ramsey number $r(n, m)$ as the smallest integer N such that $K_N \to (K_n, K_m)$. However, even in the case of diagonal classical Ramsey numbers $r(n, n)$ almost nothing significant is known beyond the case $n = 5$ (see [7] for a survey).

In the decades that followed there are several interesting variations that have originated from these classical Ramsey number. One obvious variation is the case of size Ramsey numbers build

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up mainly by Erdős et al. [2]. Another variation introduced recently by Buger et al. [1] and Syafrizal et al. [8], is the concept of balanced multipartite Ramsey numbers. This concept is based on exploring the two colorings of multipartite graphs $K_{j\times s}$ instead of the complete graph. Formally define size Ramsey multipartite number $m_j(G,H)$ as the smallest natural number s such that $K_{j\times s}\rightarrow(G,H)$. However, currently there are very few known multipartite Ramsey numbers between pairs of graphs and pairs of classes of graphs other than the ones introduced initially by Syafrizal et al. [8, 9], Lusiani et al. [6] and Jayawardene et al. [4, 5].

Notation

Given a graph $G=(V,E)$ with the *order* of the graph is denoted by $|V(G)|$ and the *size* of the graph is denoted by $|E(G)|$. For a vertex v of a graph G, the *neighborhood* of v is denoted by $N(v)$ and is defined as the set of vertices adjacent to v. Further the cardinality of this set, denoted $d(v)$, is defined as the *degree* of v. We say that a graph G is a k-regular graph if $d(v)=k$ for all $v\in V(G)$. Given a red-blue coloring of $K_{j\times s}=H_R\oplus H_B$. The red degree and blue degree of any vertex v belonging to $V(K_{j\times s})=\{v_{k,i}|\ 0\leq i\leq s-1, 0\leq k\leq j-1\}$ denoted by $d_R(v)$ and $d_B(v)$ respectively, are defined as the degree of vertex v in H_R and H_B respectively.

Given $w\geq 2$, $0\leq i\leq w-1$ and $0<c\leq w-1$, define $\sigma_{c,w}(i)=\{a_1\}\cup\{a_2\}$, $\sigma^{+}_{c,w}(i)=\{a_1\}$, $\sigma^{-}_{c,w}(i)=\{a_2\}$ and $B_{0,w}(i)=\phi$ and if $k>0$, $B_{k,w}(i)=\cup_{c=1}^{k}\sigma_{c,w}(i)$ where $a_1=(i+c) \mod k$ and $a_2=(i-c) \mod k$.

2. Some Lemmas

In all the following lemmas assume $d>0$ as the results are trivially true when $d=0$.

Lemma 2.1. *There exists a regular induced subgraph of degree d of $K_{j\times s}$ on the vertex set $V(K_{j\times s})$ provided that d is even, j is odd and s is odd.*

Proof. Let $d=2k_1(j-1)+2k_2$ for some non negative integers k_1 and k_2 such that $2k_1\leq s-1$ and $0<2k_2\leq j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold.

a) If $r\in B_{k_1,s}(l)$ and $p\neq i$.

b) If $r=l$ and $p\in B_{k_2,j}(i)$.

We know that $K_{j\times s}$ consists of j partite sets of size s. Given $v_{i,l}$ $0\leq i\leq j-1, 0\leq l\leq s-1$, the set $\{v_{p,r}|\ p\neq i \text{ and } r\in B_{k_1,s}(l)\}$ will represent the vertices not belonging to the i^{th} partite set (denoted by V_i) that are at most $2k_1$ distance apart inside a partite set (with respect to the second coordinate), as illustrated in Figure 2.

More precisely, it will consist of the vertices

$v_{0,l-k_1},\ldots,v_{i-1,l-k_1},v_{i+1,l-k_1},\ldots,v_{j-1,l-k_1}$

$v_{0,l-k_1+1},\ldots,v_{i-1,l-k_1+1},v_{i+1,l-k_1+1},\ldots,v_{j-1,l-k_1+1}$

$\ldots$

$v_{0,l-1},\ldots,v_{i-1,l-1},v_{i+1,l-1},\ldots,v_{j-1,l-1}$

$v_{0,l+1},\ldots,v_{i-1,l+1},v_{i+1,l+1},\ldots,v_{j-1,l+1}$

$\ldots$

$v_{0,l+k_1-1},\ldots,v_{i-1,l+k_1-1},v_{i+1,l+k_1-1},\ldots,v_{j-1,l+k_1-1}$

$v_{0,l+k_1}, \dots, v_{i-1,l+k_1}, v_{i+1,l+k_1}, \dots, v_{j-1,l+k_1}$

that is such a set consists of $2k_1(j-1)$ vertices.

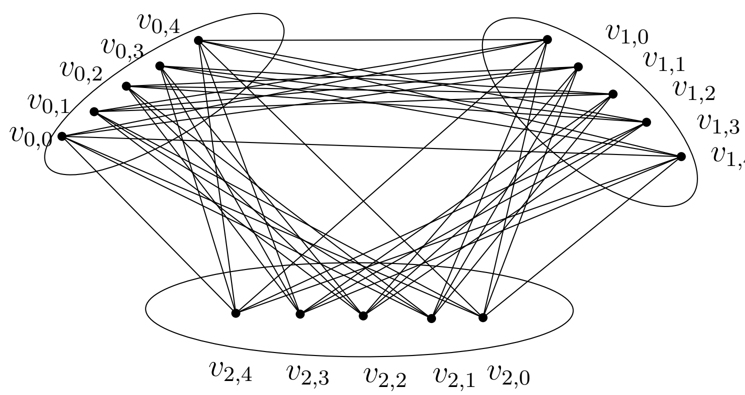

Figure 1. In the case when $d = 6 = 2k_1(j-1) + 2k_2 = (2 \times 1)(3-1) + 2 \times 1$.

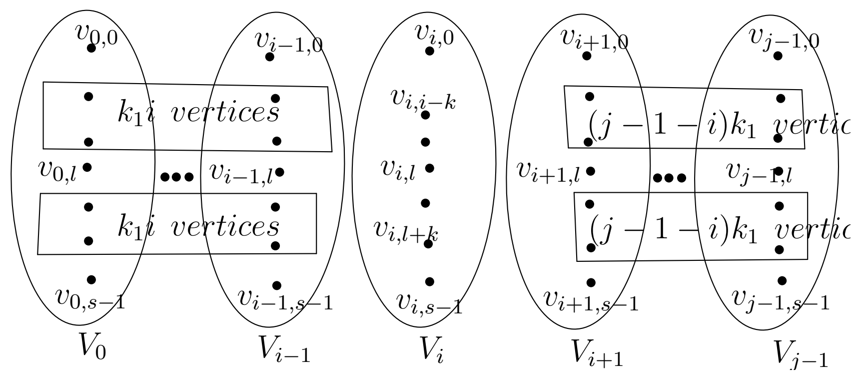

Figure 2. The set consisting of $2k_1(j-1)$ vertices corresponding to part(a), namely $\{v_{p,r} |\ p \neq i$ and $r \in B_{k_1,s}(l)\}$.

Similarly, given $v_{i,l}$ where $0 \leq i \leq j-1$ and $0 \leq l \leq s-1$ the set $\{v_{p,r} |\ r \neq l$ and $p \in B_{k_2,j}(i)\}$ will represent the vertices not belonging to the i^{th} partite set(denoted by V_i) that are at most $2k_2$ distance apart between partite sets (with respect to the first coordinate), as illustrated in the following figure. More precisely, it will consist of the vertices

$v_{i-k_2,l}, \dots, v_{i-1,l}, v_{i+1,l}, \dots, v_{i+k_2,l}$

that is such a set consists of $2k_2$ vertices.

Thus, by the above definition, part (a) will represent $2k_1(j-1)$ vertices adjacent to $v_{i,l}$ belonging to $V_0, V_1, V_2, \dots, V_{i-1}, V_{i+1}, \dots, V_{j-1}$ and part (b) will represent another $2k_2$ vertices adjacent to $v_{i,l}$ belonging to $V_{i-k_2}, V_{i-k_2+1}, \dots, V_{i-1}, V_{i+1}, \dots, V_{i+k_2-1}, V_{i+k_2}$.

Therefore, the degree of $v_{i,l}$ will be equal to $2k_1(j-1) + 2k_2$. Moreover, we get that if any vertex of V_j is adjacent to a vertex of V_k by the rule (a) (or rule(b)) then the exact same rule will

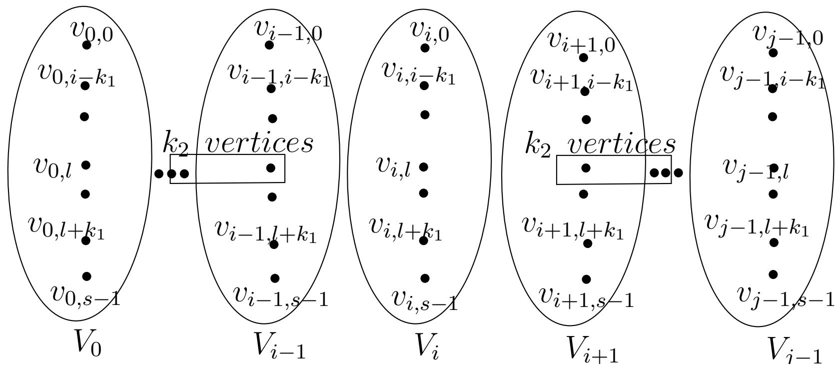

Figure 3. The set consisting of $2k_2$ vertices corresponding to part(b), namely $\{v_{p,r} | \, r = l \text{ and } p \in B_{k_2,j}(i)\}$.

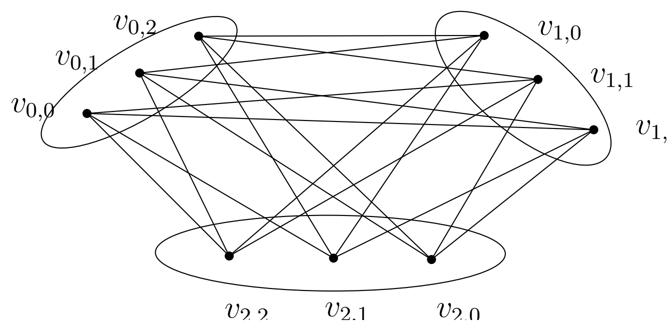

Figure 4. In the case when $d = 4 = (2k_1 + 1)(j - 1) + 2k_2 = (2 \times 0 + 1) \times (3 - 1) + 2 \times 1$.

dictate that particular vertex of V_k also to be adjacent to the exact same vertex of V_j. Thus, the generated graph is well defined.

Next, let $d = (2k_1 + 1)(j - 1) + 2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s - 3$ and $0 < 2k_2 \leq j - 1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold.

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If there exists w such that $4w = (j - 1 + 2k_2)$ and $r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)$.

c) If there exists w such that $(j - 1 + 2k_2) - 4w = 2$ and $r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)$ or else $r = l$ and $p \in B_{1,j}(i)$.

It should be noted that the vertex sets of part (b) and part (c) are disjoint and that $j - 1 + 2k_2$ is even as j is odd. Therefore, given $v_{i,l}$, it will be either adjacent the vertices corresponding to part (a) and part (b) or else adjacent the vertices corresponding to part (a) and part (c) according to whether $4w = (j - 1 + 2k_2)$ or else $(j - 1 + 2k_2) - 4w = 2$, respectively.

$v_{i-w,l-(k_1+1)}, \ldots, v_{i-1,l-(k_1+1)}, v_{i+1,l-(k_1+1)}, \ldots, v_{i+w,l-(k_1+1)}$

$v_{i-w,l+(k_1+1)}, \ldots, v_{i-1,l+(k_1+1)}, v_{i+1,l+(k_1+1)}, \ldots, v_{i+w,l+(k_1+1)}$

Such a set consists of $4w$ vertices. That is, the set consists of $(j - 1) + 2k_2$ vertices.

Similarly, given $v_{i,l}$ the set generated by the later part (c), will represent the two vertices belonging to V_{i-1}, V_{i+1} sets, namely $v_{i-1,l-1}$, $v_{i+1,l+1}$. More precisely the set generated by part (c),

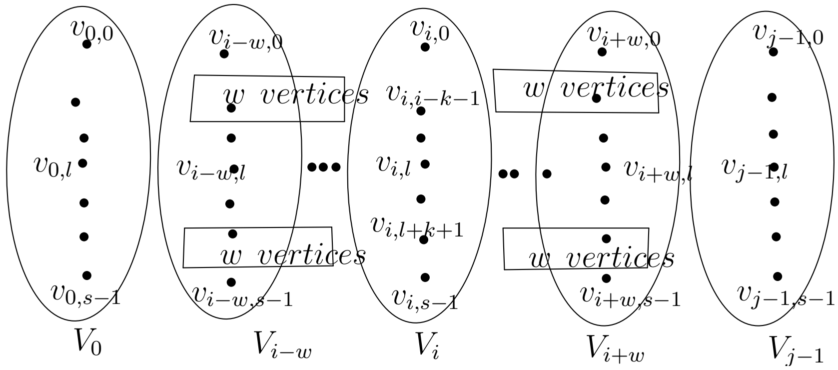

Figure 5. The set consisting of $4w$ vertices corresponding to part (b), namely $\{v_{p,r} |\ r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)\}$.

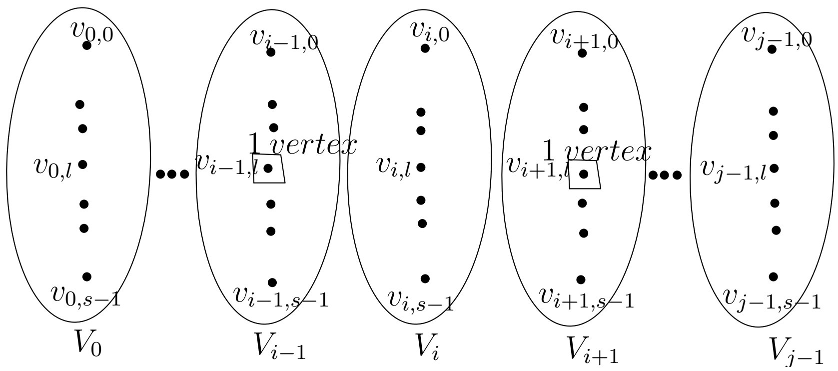

Figure 6. The set consisting of 2 vertices corresponding to later section of part (c), namely $\{v_{p,r} |\ p \in B_{w,j}(i)$ or else $r = l$ and $p \in B_{1,j}(i)\}$.

will consist of the vertices

$v_{i-w,l-(k_1-1)}, \dots, v_{i-1,l-(k_1-1)}, v_{i+1,l-(k_1-1)}, \dots, v_{i+w,l-(k_1-1)}$

$v_{i-1,l-1}, v_{i+1,l+1}$

$v_{i-w,l-(k_1+1)}, \dots, v_{i-1,l-(k_1+1)}, v_{i+1,l-(k_1+1)}, \dots, v_{i+w,l-(k_1+1)}$

Such a set consists of $4w + 2$ vertices. That is, the set consists of $(j - 1) + 2k_2$ vertices.

Therefore, the degree of $v_{i,l}$ will be equal to $2k_1(j - 1) + 2k_2$ when part (a)+(b) situation arises or when part (a)+(c) situation arises. □

Lemma 2.2. *There exist regular graphs of degree d on $V(K_{j\times s})$ if j is even or s is even.*

Proof. We approach this problem by considering the following three cases.

Case 1. If j is even and s is odd.

If $d = 2k_1(j - 1) + 2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s - 1$ and $0 < 2k_2 \leq j - 2$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $r = l$ and $p \in B_{k_2,j}(i)$.

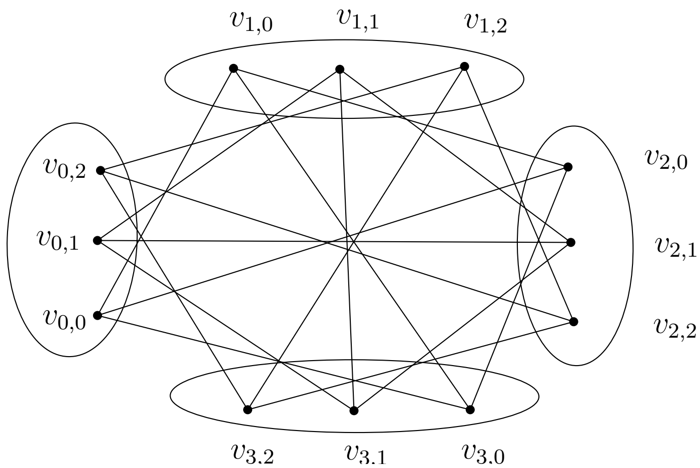

Figure 7. In the Case 1, when $d = 3 = 2k_1(j-1) + 2k_2 + 1 = 2 \times 0 \times (4-1) + 2 \times 1 + 1$.

The vertex $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) or part (b) and they are respectively equal to $2k_1(j-1)$ and $2k_2$. Therefore, we get that the degree of $v_{i,l}$ is equal to $2k_1(j-1) + 2k_2$ as required.

Let $d = 2k_1(j-1) + 2k_2 + 1$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-1$ and $2k_2 \leq j-2$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $r = l$ and $p \in B_{k_2,j}(i)$.

c) If $r = l$ and $p \in \sigma_{\frac{j}{2},j}(i)$.

The vertex $v_{i,l}$ will be either adjacent the vertices corresponding to part (a), part (b) or part (c) and they are respectively equal to $2k_1(j-1)$, $2k_2$ and one. Therefore, we get that the degree of $v_{i,l}$ is equal to $2k_1(j-1) + 2k_2 + 1$ as required.

Let $d = (2k_1+1)(j-1)+m$ for some non negative integers k_1, k_2 and m such that $2k_1 \leq s-3$ and $0 < m \leq j-1$ where $m = 2k_2$ or $m = 2k_2+1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If there exists w such that $w = (j-1+m)$ div 4, $r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)$.

c) If there exists w such that $(j-1+m) - 4w = 1$, $r = l$ and $p \in \sigma_{\frac{j}{2},j}(i)$.

d) If there exists w such that $(j-1+m) - 4w = 2$, $r = l$ and $p \in \sigma_{1,j}(i)$.

e) If there exists w such that $(j-1+m) - 4w = 3$, $r = l$, $p \in \sigma_{1,j}(i)$ and $p \in \sigma_{\frac{j}{2},j}(i)$ (as j is even).

It should be noted that the vertex sets of part (b), (c), (d) and part (e) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) or (a) and part (d) or else part (a) and part (e) according to whether $4w = j-1+2k_2$, $(j-1+2k_2)-4w = 1$, $(j-1+2k_2) - 4w = 2$ or $(j-1+2k_2) - 4w = 3$ respectively. In all these scenarios we get $d = (2k_1+1)(j-1) + m$ as required.

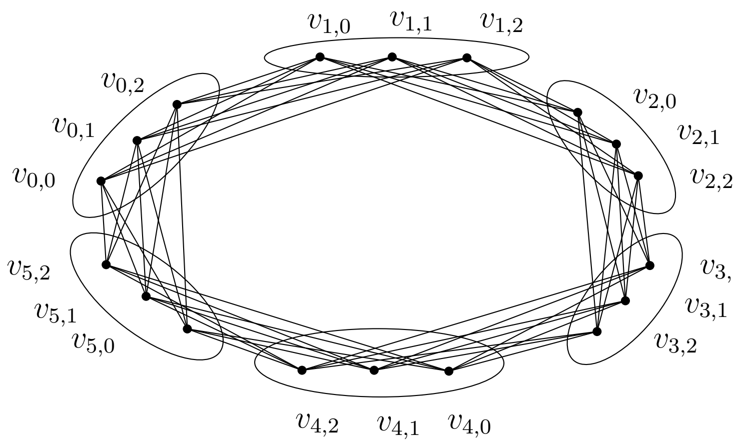

Figure 8. In the Case 1, when $d = 6 = (2k_1 + 1)(j - 1) + 2k_2 + 1$ =$(2 \times 0 + 1)(6 - 1) + 2 \times 0 + 1$.

Case 2. If j is even and s is even.

Let $d = 2k_1(j-1) + 2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-2$ and $0 < k_2 \leq j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 < \frac{j}{2}$, $r = l$ and $p \in B_{k_2,j}(i)$.

c) If $\frac{j}{2} \leq k_2 < j-1$ and (($r = l$ and $p \in B_{\frac{j-2}{2},j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-2)}{2},j}(i)$)).

d) If $k_2 = j-1$ and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p \neq i$)).

It should be noted that the vertex sets of part (b), (c) and part (d) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) or else part (a) and part (d) according to whether $k_2 < \frac{j}{2}$, $\frac{j}{2} \leq k_2 < j-1$ or $k_2 = j-1$ respectively. In all these scenarios we get $d = 2k_1(j-1) + 2k_2$ as required.

Let $d = 2k_1(j-1) + 2k_2 + 1$ for some non negative integers k_1 and k_2 where such that $2k_1 \leq s-2$ and $k_2 < j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 \leq \frac{j-2}{2}$, $r = l$ and ($p \in B_{k_2,j}(i)$ or $p \in \sigma_{\frac{j}{2},j}(i)$).

c) If $k_2 \geq \frac{j}{2}$ and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-2)}{2},j}(i)$)).

It should be noted that the vertex sets of part (b) and part (c) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) according to whether $k_2 \leq \frac{j-2}{2}$ or $k_2 \geq \frac{j}{2}$ respectively. In all these scenarios we get $d = 2k_1(j-1) + 2k_2 + 1$ as required.

Case 3. If j is odd and s is even.

Let $d = 2k_1(j-1) + 2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-2$ and $0 < k_2 \leq j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

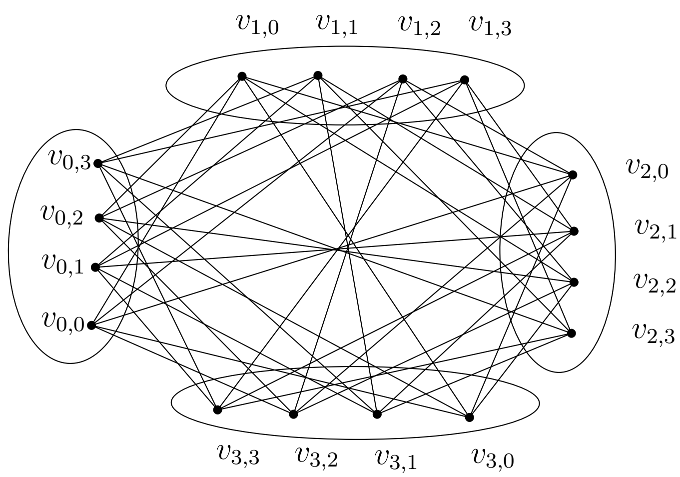

Figure 9. In the Case 2, when $d = 5 = 2k_1(j-1) + 2k_2 + 1 = 2 \times 0 \times (4-1) + 2 \times 2 + 1$.

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 < \frac{j}{2}$, $r = l$ and $p \in B_{k_2,j}(i)$

c) If $k_2 \geq \frac{j}{2}$ and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-1)}{2},j}(i)$)).

It should be noted that the vertex sets of part (b) and part (c) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) according to whether $k_2 < \frac{j}{2}$ or $k_2 \geq \frac{j}{2}$ respectively. In all these scenarios we get $d = 2k_1(j-1) + 2k_2$ as required.

Case 4. If j is odd and s is even.

Let $d = 2k_1(j-1)+2k_2+1$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-2$ and $0 \leq k_2 < j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 < \frac{j-1}{2}$ and (($r = l$ and $p \in B_{k_2,j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r > l$ and $p = \sigma^{+}_{k_2+1,j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r < l$ and $p = \sigma^{-}_{k_2+1,j}(i)$)).

c) If $k_2 = \frac{j-1}{2}$and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r > l$ and $p = \sigma^{+}_{k_2+1,j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r < l$ and $p = \sigma^{-}_{k_2+1,j}(i)$)).

d) If $k_2 > \frac{j-1}{2}$ and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-1)}{2},j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p = \sigma^{+}_{\frac{2k_2-(j-1)}{2}+1,j}(i)$ with $r > l$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p = \sigma^{-}_{\frac{2k_2-(j-1)}{2}+1,j}(i)$ with $r < l$)).

It should be noted that the vertex sets of part (b), (c) and part (d) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) or else (a) and part (d) according to whether $k_2 < \frac{j-1}{2}$, $k_2 = \frac{j-1}{2}$ or $k_2 > \frac{j-1}{2}$ respectively. In all these scenarios we get $d = 2k_1(j-1) + 2k_2 + 1$ as required. □

Lemma 2.3. $m_j(S_n, S_m) \leq \left\lceil \frac{n+m-3}{j-1} \right\rceil$ *where* $j, n, m \geq 3$.

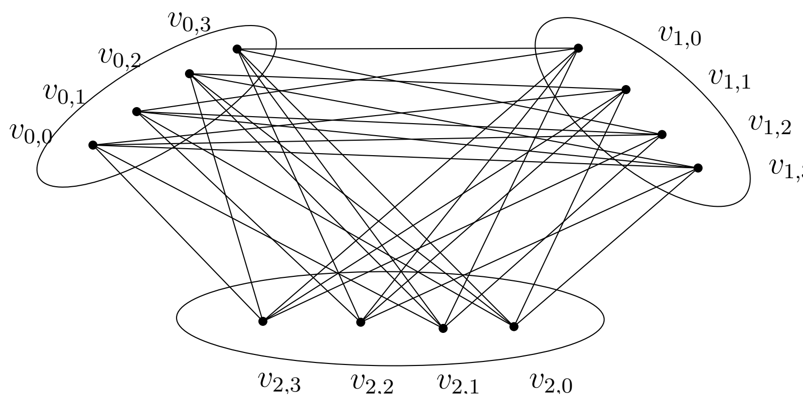

Figure 10. In the Case 3, when $d = 5 = 2k_1(j-1) + 2k_2 + 1 = 2 \times 1 \times (3-1) + 2 \times 0 + 1$.

Proof. Consider any red/blue coloring given by $K_{j \times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{n+m-3}{j-1} \right\rceil$, such that H_R contains no red S_n. Let v be any vertex of $K_{j\times s}$. Then v is incident to at most $n-2$ red edge. Hence,

$$d_B(v) \geq \left\lceil \frac{n+m-3}{j-1} \right\rceil (j-1) - (n-2) \geq m-1$$

Therefore, H_B will contain a blue S_m. Hence the result. □

Lemma 2.4. $m_j(S_n, S_m) \geq \left\lceil \frac{n+m-4}{j-1} \right\rceil$ *where* $j, n, m \geq 3$.

Proof. Consider the red and blue coloring of $K_{j\times s}$ given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{n+m-4}{j-1} \right\rceil - 1$, where all the vertices will have uniform red degree of $n-2$ or $n-3$ (this is possible by Lemma 2.2). Then clearly H_B does not contain a red S_n. Let v be any vertex of $K_{j\times s}$. Then,

$$\begin{aligned} d_B(v) &= \left(\left\lceil \frac{n+m-4}{j-1} \right\rceil - 1 \right)(j-1) - (n-3) \\ &= \left\lceil \frac{n+m-4}{j-1} \right\rceil (j-1) - j + 1 - n + 3 \\ &\geq n + m - 4 - j - n + 4 \geq m - j \end{aligned}$$

Therefore, H_B will not contain a blue S_m. Hence the result. □

Lemma 2.5. $m_j(S_n, S_m) = \left\lceil \frac{n+m-4}{j-1} \right\rceil$ *if* $(n+m-4) \neq 0$ *mod* $(j-1)$ *where* $j, n, m \geq 3$.

Proof. We know that if $(n+m-4) \neq 0$ mod $(j-1)$ then $\left\lceil \frac{n+m-4}{j-1} \right\rceil = \left\lceil \frac{n+m-3}{j-1} \right\rceil$. Hence the result follows by Lemma 2.3 and Lemma 2.4.

□

Lemma 2.6. *Suppose that $j, n, m \geq 3$. Then, $m_j(S_n, S_m) \leq \left\lceil \frac{n+m-4}{j-1} \right\rceil$ provided that $(n+m-4) = 0$ mod $(j-1)$ with j is odd, n is odd and $s = \frac{n+m-4}{j-1}$ is odd.*

Proof. Consider any red/blue coloring given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{n+m-4}{j-1} \right\rceil$, such that H_R contains no red S_n. Since, $j \times s \times (n-2)$ is odd, there will exist at least one vertex $v \in K_{j\times s}$ such it is not incident to $n-2$ red edges, as otherwise by handshake lemma $j \times s \times (n-2) = 2|E(H_R)|$, a contradiction. Hence,

$$d_B(v) \geq \left\lceil \frac{n+m-4}{j-1} \right\rceil (j-1) - (n-3) \geq m-1$$

Therefore, H_B will contain a blue S_m. Hence the result. □

Lemma 2.7. *Suppose that $j, n, m \geq 3$. Then, $m_j(S_n, S_m) \geq \left\lceil \frac{n+m-3}{j-1} \right\rceil$ provided that $(n+m-4) = 0$ mod $(j-1)$ with j is even or $s = \frac{n+m-4}{j-1}$ even or n is even.*

Proof. By Lemma 2.3 and Lemma 2.4, $K_{j\times s}$ where $s = \left\lceil \frac{n+m-3}{j-1} \right\rceil - 1 = \left\lceil \frac{n+m-4}{j-1} \right\rceil$, will have a $n-2$ regular subgraph on $K_{j\times s}$. Using this subgraph generate a red/blue coloring given by $K_{j\times s} = H_R \oplus H_B$, where all the edges of this subgraph are colored red and all other edges colored blue. Then clearly H_R is S_n - free. Furthermore, for any vertex $v \in K_{j\times s}$, $d_B(v) = \left(\frac{n+m-4}{j-1}\right)(j-1) - (n-2) = m-2$. Therefore, H_B will not contain a blue S_m. Hence the result. □

Theorem 2.1. *If $j \geq 3$ and $n, m \geq 2$ then,*

$$m_j(S_n, S_m) = \begin{cases} \left\lceil \frac{\max\{n,m\}-1}{j-1} \right\rceil, & \text{if } n = 2 \text{ or } m = 2, \\ \left\lceil \frac{n+m-4}{j-1} \right\rceil, & \text{if } n+m-4 = (j-1)s; j, s, n \text{ are odd and } n, m \geq 3, \\ \left\lceil \frac{n+m-3}{j-1} \right\rceil, & \text{otherwise,} \end{cases}$$

Proof. The theorem clearly follows from Lemmas 2.5, 2.6, and 2.7 as $m_j(S_2, S_m) = \left\lceil \frac{m-1}{j-1} \right\rceil$ (see Syafrizal et al. 2005). □

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