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The 4-girth-thickness of the complete multipartite graph

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Abstract

The g-girth-thickness $\theta(g,G)$ of a graph G is the smallest number of planar subgraphs of girth at least g whose union is G. In this paper, we calculate the 4-girth-thickness $\theta(4,G)$ of the complete m-partite graph G when each part has an even number of vertices.

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1. Introduction

The *thickness* $\theta(G)$ of a graph G is the smallest number of planar subgraphs whose union is G. Equivalently, it is the smallest number of parts used in any edge partition of $E(G)$ such that each set of edges in the same part induces a planar subgraph.

This parameter was introduced by Tutte [20] in the 60s. The problem to calculate the thickness of a graph G is an NP-hard problem [16] and a few of exact results can be found in the literature, for example, if G is a complete graph [2, 5, 6], a hypercube [15], or a complete multipartite graph for some particular values [21, 22]. Even for the complete bipartite graph there are only partial results [7, 13].

Some generalizations of the thickness for complete graphs have been studied, for instance, the outerthickness θ_o, defined similarly but with outerplanar instead of planar [12], the S-thickness θ_S,

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considering the thickness on a surface S instead of the plane [4], and the k-degree-thickness θ_k taking a restriction on the planar subgraphs: each planar subgraph has maximum degree at most k [9].

The thickness has applications in the design of circuits [1], in the Ringel's earth-moon problem [14], and to bound the achromatic numbers of planar graphs [3], etc. See the survey [17].

In [19], the author introduced the *g-girth-thickness* $\theta(g, G)$ of a graph G as the minimum number of planar subgraphs of girth at least g whose union is G, a generalization of the thickness owing to the fact that the g-girth-thickness is the usual thickness when $g = 3$ and also the *arboricity number* when $g = \infty$ because the *girth* of a graph is the size of its shortest cycle or ∞ if it is acyclic. See also [11].

In this paper, we obtain the 4-girth-thickness $\theta(4, K_{n_1,n_2,\dots,n_m})$ of the complete m-partite graph $K_{n_1,n_2,\dots,n_m}$ when n_i is even for all $i \in \{1, 2, \dots, m\}$.

2. Calculating $\theta(4, K_{n_1,n_2,\dots,n_m})$

Given a simple graph G, we define a new graph $G \bowtie G$ in the following way: If G has vertex set $V = \{w_1, w_2, \dots, w_n\}$, the graph $G \bowtie G$ has as vertex set two copies of V, namely, $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and two vertices $x_i y_j$ are adjacent if $w_i w_j$ is an edge of G, for the symbols $x, y \in \{u, v\}$. For instance, if $w_1 w_2$ is an edge of a graph G, the graph $G \bowtie G$ has the edges u_1u_2, v_1v_2, u_1v_2 and v_1u_2. See Figure 1.

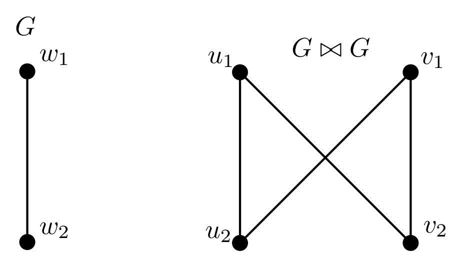

Figure 1. An edge of G produces four edges in $G \bowtie G$.

On the other hand, an acyclic graph of n vertices has at most $n-1$ edges and a planar graph of n vertices and girth $g < \infty$ has at most $\frac{g}{g-2}(n-2)$ edges, see [8]. Therefore, a planar graph of n vertices and girth at least 4 has at most $2(n-2)$ edges for $n \geq 4$ and at most $n-1$, otherwise. In consequence, the 4-girth-thickness $\theta(4, G)$ of a graph G is at least $\left\lceil \frac{|E(G)|}{2(n-2)} \right\rceil$ for $n \geq 4$ and at least $\left\lceil \frac{|E(G)|}{n-1} \right\rceil$, otherwise.

Lemma 2.1. *If G is a tree of order n then $G \bowtie G$ is a bipartite planar graph of size $2(2n-2)$.*

Proof. By induction over n. The basis is given in Figure 1 for $n = 2$. Now, take a tree G with $n+1$ vertices. Since it has at least a leaf, we say, the vertex w_1 incident to w_2 then we delete w_1 from G and by induction hypothesis, $H \bowtie H$ is a bipartite planar of size $2(2n-2)$ edges for $H = G \setminus \{w_1\}$. Since H is connected, the vertex labeled w_2 has at least a neighbour, we say, the vertex labeled w_3, then $u_2v_3v_2$ is a path in $H \bowtie H$ and the edge $u_2v_2 \notin E(H \bowtie H)$. Add the paths $u_2v_1v_2$ and $u_2u_1v_2$ to $H \bowtie H$ such that both of them are "parallel" to $u_2v_3v_2$ and identify the vertices u_2 as a single vertex as well as the vertices v_2. This proves that $G \bowtie G$ is planar. To verify that is bipartite, given a proper coloring of $H \bowtie H$ with two colors, we extend the coloring putting the same color of v_3 to v_1 and u_1. Then the resulting coloring is proper. Due to the fact that we add four edges, $H \bowtie H$ has $2(2n-2)+4 = 2(2(n+1)-2)$ edges and the lemma follows. □

Now, we recall that the arboricity number or ∞-girth-thickness $\theta(\infty, G)$ of a graph G equals (see [18])

$$\max\left\{\left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil : H \text{ is an induced subgraph of } G\right\}.$$

We have the following theorem.

Theorem 2.1. *If G is a simple graph of $n \geq 2$ vertices and e edges, then*

$$\left\lceil \frac{e}{n-1} \right\rceil \leq \theta(4, G \bowtie G) \leq \theta(\infty, G).$$

Proof. Since $G \bowtie G$ has $2n \geq 4$ vertices, $4e$ edges and

$$\frac{|E(G \bowtie G)|}{2(|V(G \bowtie G)|-2)} = \frac{4e}{2(2n-2)} = \frac{e}{n-1},$$

it follows the lower bound

$$\left\lceil \frac{e}{n-1} \right\rceil \leq \theta(4, G \bowtie G).$$

To verify the upper bound, take an acyclic edge partition $\{F_1, F_2, \ldots, F_{\theta(\infty,G)}\}$ of $E(G)$. Therefore, $\{F_1 \bowtie F_1, F_2 \bowtie F_2, \ldots, F_{\theta(\infty,G)} \bowtie F_{\theta(\infty,G)}\}$ is an edge partition of $E(G \bowtie G)$ (where $F_i \bowtie F_i := E(\langle F_i \rangle \bowtie \langle F_i \rangle)$ and $\langle F_i \rangle$ is the induced subgraph of the edge set F_i for all $i \in \{1, 2, \ldots, \theta(\infty, G)\}$). Indeed, an edge $x_jy_{j'} \in E(G \bowtie G)$ is in $F_i \bowtie F_i$ if and only if $w_jw'_j \in E(G)$ is in F_i. By Lemma 2.1, the result follows. □

Corollary 2.1. *If G is a simple graph of $n \geq 2$ vertices and e edges with $\theta(\infty, G) = \lceil \frac{e}{n-1} \rceil$, then*

$$\theta(4, G \bowtie G) = \left\lceil \frac{e}{n-1} \right\rceil.$$

Next, we estimate the arboricity number of the complete m-partite graph.

Lemma 2.2. *If $K_{n_1,n_2,\ldots,n_m}$ is the complete m-partite graph then $\theta(\infty, G) = \lceil \frac{e}{n-1} \rceil$ where $n = n_1 + n_2 + \ldots + n_m$ and $e = n_1n_2 + n_1n_3 + \ldots + n_{m-1}n_m$.*

Proof. By induction over n. The basis is trivial for $K_{1,1}$. Let $G = K_{n_1,n_2,\ldots,n_m}$ with $n > 2$ and $H = G \setminus \{u\}$ a proper induced subgraph of G for any vertex u. By the induction hypothesis, $\theta(\infty, H) = \max\left\{\left\lceil \frac{|E(F)|}{|V(F)|-1} \right\rceil : F \leq H\right\} = \left\lceil \frac{|E(H)|}{(n-1)-1} \right\rceil$, where $F \leq H$ indicates that F is an induced subgraph of H. Since u is an arbitrary vertex and by the hereditary property of the induced subgraphs, we only need to show that

$$\frac{|E(H)|}{n-2} \leq \frac{e}{n-1}$$

because

$$\max\left\{\left\lceil \frac{|E(F)|}{|V(F)|-1}\right\rceil : F \leq G\right\} = \max\left\{\left\lceil \frac{e}{n-1}\right\rceil, \left\lceil \frac{|E(H)|}{n-2}\right\rceil : H = G\setminus\{u\}, u\in V(G)\right\}.$$

We prove it in the following way. Without loss of generality, u is a vertex in a part of size n_m.

Since

$$\begin{array}{cccc} n_1+ & n_1n_2+ & \dots & +n_1n_m+ \\ & n_2+ & \dots & +n_2n_m+ \\ & & \vdots & \\ & & & n_{m-1}n_m \end{array} \leq \begin{array}{cccc} n_1^2+ & n_1n_2+ & \dots & +n_1n_m+ \\ n_2n_1+ & n_2^2+ & \dots & +n_2n_m+ \\ & & \vdots & \\ n_{m-1}n_1+ & n_{m-1}n_2+ & \dots & +n_{m-1}n_m \end{array}$$

then $e+n_1+n_2+\ldots+n_{m-1} \leq n(n_1+n_2+\ldots+n_{m-1})$ and

$$en-e-n(n_1+n_2+\ldots+n_{m-1})+(n_1+n_2+\ldots+n_{m-1}) \leq en-2e$$

$$(n-1)(e-(n_1+n_2+\ldots+n_{m-1})) \leq e(n-2)$$

$$\frac{|E(H)|}{n-2} \leq \frac{e}{n-1}$$

and the result follows. □

Now, we can prove our main theorem.

Theorem 2.2. *If $G = K_{2n_1,2n_2,\ldots,2n_m}$ is the complete m-partite graph then $\theta(4,G) = \left\lceil\frac{e}{n-1}\right\rceil$ where $n = n_1+n_2+\ldots+n_m$ and $e = n_1n_2+n_1n_3+\ldots+n_{m-1}n_m$.*

Proof. We need to show that $G = K_{n_1,n_2,\ldots,n_m} \bowtie K_{n_1,n_2,\ldots,n_m}$. Let $(W_1, W_2, \ldots, W_m)$ be an m-partition of $K_{n_1,n_2,\ldots,n_m}$. The graph $K_{n_1,n_2,\ldots,n_m} \bowtie K_{n_1,n_2,\ldots,n_m}$ has the partition $(U_1\cup V_1, U_2\cup V_2, \ldots, U_m\cup V_m)$ where U_i and V_i are copies of W_i for $i\in\{1,2,\ldots,m\}$. Take two vertices x_i and y_j in different parts, without loss of generality, $U_1\cup V_1$ and $U_2\cup V_2$. If the vertex x_i is in U_1 and y_j is in U_2 then they are adjacent because $w_iw_j$is an edge of $K_{n_1,n_2,\ldots,n_m}$ is m-complete. Similarly for $x_i\in V_1$ and $y_j\in V_2$. If x_i is in U_1 and y_j is in V_2, then also they are adjacent because w_iw_j is an edge of $K_{n_1,n_2,\ldots,n_m}$. By Corollary 2.1 and Lemma 2.2, the theorem follows. □

Due to the fact that $\theta(4,G) = \theta(3,G) = \theta(G)$ for any triangle-free graph G, we obtain an alternative proof for the thickness of the complete bipartite graph $K_{2n_1,2n_2}$ that is given in [7].

Corollary 2.2. *If $G = K_{2n_1,2n_2}$ is the complete bipartite graph then $\theta(G) = \left\lceil\frac{e}{n-1}\right\rceil$ where $n = n_1+n_2$ and $e = n_1n_2$.*

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