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A numeral system for the middle-levels graphs

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Abstract

A sequence $\mathcal{S}$ of restricted-growth strings unifies the presentation of middle-levels graphs M_k as follows, for $0 < k \in \mathbb{Z}$. Recall M_k is the subgraph in the Hasse diagram of the Boolean lattice $2^{[2k+1]}$ induced by the k- and $(k+1)$-levels. The dihedral group D_{4k+2} acts on M_k via translations mod $2k+1$ and complemented reversals. The first $\frac{(2k)!}{k!(k+1)!}$ terms of $\mathcal{S}$ stand for the orbits of $V(M_k)$ under such D_{4k+2}-action, via the lexical matching colors $0, 1, \ldots, k$ on the $k+1$ edges at each vertex. So, $\mathcal{S}$ is proposed here as a convenient numeral system for the graphs M_k. Color 0 allows to reorder $\mathcal{S}$ via an integer sequence that behaves as an idempotent permutation on its first $\frac{(2k)!}{k!(k+1)!}$ terms, for each $0 < k \in \mathbb{Z}$. Related properties hold for the remaining colors $1, \ldots, k$.

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1. Introduction

This paper complements previous work [5] on reinterpreting the middle-levels theorem [6, 8] via a numeral system that enumerates all ordered trees. Let $0 < k \in \mathbb{Z}$ and let $n = 2k + 1$. The *middle-levels graph* M_k [2, 7] is the subgraph of the Hasse diagram [12] of the Boolean lattice [3], denoted $2^{[n]}$ and induced by its k- and $(k+1)$-th levels (i.e. formed by the k- and $(k+1)$-subsets of $[n] = \{0, \ldots, 2k\}$). The dihedral group D_{2n} acts on M_k via translations mod n (see Section 4) and complemented reversals (see Section 5).

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Let $C_k = \frac{(2k)!}{k!(k+1)!}$ be the k-th Catalan number [13] A000108. Let $\mathcal{S}$ be the sequence [13] A239903 of *restricted-growth strings* or ***RGS's*** ([1] page 325). We will show that the first C_k terms of $\mathcal{S}$ stand for the orbits of $V(M_k)$ under the natural D_{2n}-action on $(V(M_k), E(M_k))$ in two ways: as Stanley's k-***RGS's*** (see below) and as k-*germs*, proposed in this work.

In Section 6, the mentioned D_{2n}-action will allow to project M_k onto a quotient pseudograph R_k whose vertices stand for the first C_k terms of $\mathcal{S}$ via the Kierstead-Trotter lexical-matching [7] color (or *lexical color*) set $[k+1] = \{0, 1, \ldots, k\}$ on the $k+1$ edges incident to each vertex (Sections 7, 8 and 11).

In preparation, RGS's are first tailored in Section 2 into numerical $(k-1)$-strings α that are our k-*germs*. These yield n-strings $F(\alpha)$ (Section 3), each composed by the $k+1$ lexical colors, as well as by k asterisks $*$. The $F(\alpha)$'s represent the k-edge ordered trees (Proposition 3.1) and are obtained via a nested substring-swapping, here called *castling* (Theorem 3.2), that sorts them linearly via pruning and regrafting. These trees (encoded as $F(\alpha)$) represent the vertices of R_k via a corresponding *uncastling* procedure (Section 8).

The mentioned linear sorting arises from an ordered tree $\mathcal{T}_k$ (Theorem 3.1) with $|V(\mathcal{T}_k)| = |V(R_k)| = C_k$. This $\mathcal{T}_k$ controls $V(R_k)$ and allows to lexically visualize $V(M_k)$. On the other hand, an all-RGS's binary tree is given in Section 9, representing the vertices (i.e. the ordered trees) of all R_k's. This is a unifying pattern for the presentation of all the $V(M_k)$'s.

It is known that the k-edge ordered trees (that is, the vertices of R_k) denoted by R. Stanley in [14] page 221 item (e) as "plane trees with $k+1$ vertices", are equivalent to k-strings with initial entry 0, that we shall call k-***RGS***'s, tailored from RGS's in a different way ([14] page 224 item (u)) from that of our k-*germs*. An equivalence of k-germs and k-RGS's is presented in Section 10 via their distinct relation to the k-edge ordered trees.

Our approach yields a stepwise-reversing presentation (i.e., via complemented-reversal adjacency) of the Hamilton cycles of M_k [8, 9, 10, 11] in P. Gregor, T. Mütze and J. Nummenpalo [6], that allows an explicit view of all Kierstead-Trotter lexical colors in ordered trees $F(\alpha)$. The 2-factor W_{01}^k of R_k determined by the colors 0 and 1 is reanalyzed from this viewpoint in [5], Section 9, and W_{01}^k is seen in [5], Section 10, to morph into such Hamilton cycles.

Moreover, an integer sequence $\mathcal{S}_0$ is shown to exist such that, for each $k > 0$, the neighbors of the vertices of R_k via color-k edges have their RGS's ordered as in $\mathcal{S}$ corresponding to an idempotent permutation on the first C_k terms of $\mathcal{S}_0$. This and related properties hold for lexical colors $0, 1, \ldots, k$ (Theorem 11.1 and Remark 11.2) reflecting properties of plane trees (i.e., classes of ordered trees under root rotation).

Incidentally, a sufficient condition [4] (to be compared with [12]), that a path in R_k lifts to a dihedrally invariant Hamilton cycle in M_k, narrows the conjecture on the existence of Hamilton cycles in M_k, solved in [8], to an unsolved unrestricted version; see Remark 11.3.

2. From restricted-growth strings to *k*-germs

Let $0 < k \in \mathbb{Z}$. We can express the mentioned sequence $\mathcal{S}$ as: $\mathcal{S} = (\beta(0), \ldots, \beta(17), \ldots) =$

$$(0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, 1001, 1010, 1011, \ldots) \qquad (1)$$

and note that $\mathcal{S}$ has the lengths of its contiguous pairs $(\beta(i-1),\beta(i))$ constant unless $i = C_k$ for $0 < k \in \mathbb{Z}$, in which case $\beta(i-1) = \beta(C_k - 1) = 12\cdots k$ and $\beta(i) = \beta(C_k) = 10^k = 10\cdots 0$.

To view the continuation of $\mathcal{S}$, each RGS $\beta = \beta(m)$ is transformed, for every $k \in \mathbb{Z}$ with $k \geq \text{length}(\beta)$, into a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ by prefixing $k-$ length(β) zeros to β. As hinted in Section 1, we say that such an α is a *k-germ*. In fact, a *k-germ* α $(1 < k \in \mathbb{Z})$ is a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ such that:

(1) the leftmost position (called position $k-1$) of α contains entry $a_{k-1} \in \{0,1\}$;
(2) given $1 < i < k$, the entry a_{i-1} (at position $i-1$) satisfies $0 \leq a_{i-1} \leq a_i + 1$.

Every k-germ $a_{k-1}a_{k-2}\cdots a_2a_1$ yields the $(k+1)$-germ $0a_{k-1}a_{k-2}\cdots\ a_2a_1$. A *non-null* RGS is obtained by stripping a k-germ $\alpha = a_{k-1}a_{k-2}\cdots a_1 \neq 00\cdots 0$ off all the null entries to the left of its leftmost position containing a 1. We denote such an RGS again by α, convene that the *null* RGS $\alpha = 0$ is stripped from all null k-germs α $(0 < k \in \mathbb{Z})$, and use notation $\alpha = \alpha(m)$ (or $\beta = \beta(m)$, as in (1)) both for a k-germ and for its corresponding RGS.

The k-germs are ordered as follows. Given two k-germs, say $\alpha = a_{k-1}\cdots a_2a_1$ and $\beta = b_{k-1}\cdots b_2b_1$, where $\alpha \neq \beta$, we say that α precedes β, written $\alpha < \beta$, whenever either:

(i) $a_{k-1} < b_{k-1}$ or
(ii) $a_j = b_j$, for $k-1 \leq j \leq i+1$, and $a_i < b_i$, for some $k-1 > i \geq 1$.

The resulting order on k-germs $\alpha(m)$, $(m \leq C_k)$, corresponding biunivocally (via the assignment $m \to \alpha(m)$) with the natural order on m, yields a listing that we call the *natural* (k-germ) *listing*. Note that there are exactly C_k k-germs $\alpha = \alpha(m) < 10^k$, $\forall k > 0$. Subsection 2.1, deals with the determination of these RGS's and k-germs.

2.1. *Catalan's triangle*

Given $0 \leq \in \mathbb{Z}$, to determine $\beta(m)$ or $\alpha(m)$, we use *Catalan's triangle* $\triangle$, i.e. a triangular arrangement of integers starting with the following successive rows $\triangle_j$, for $j = 0,\ldots,8$:

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
1	8	35	110	275	572	1001	1430	1430
..	..	..	..	..	..	..	..	..

where reading is linear, as in [13] A009766. The numbers τ_i^j in $\triangle_j$ $(0 \leq j \in \mathbb{Z})$, given by $\tau_i^j = (j+i)!(j-i+1)/(i!(j+1)!)$, are characterized by the following properties:

1. $\tau_0^j = 1$, for every $j \geq 0$;
2. $\tau_1^j = j$ and $\tau_j^j = \tau_{j-1}^j$, for every $j \geq 1$;
3. $\tau_i^j = \tau_i^{j-1} + \tau_{i-1}^j$, for every $j \geq 2$ and $i = 1,\ldots,j-2$;
4. $\sum_{i=0}^j \tau_i^j = \tau_j^{j+1} = \tau_{j+1}^{j+1} = C_j$, for every $j \geq 1$.

The determination of k-germ $\beta(m)$ proceeds as follows. Let $x_0 = m$ and let $y_0 = \tau_k^{k+1}$ be the largest member of the second diagonal of $\triangle$ with $y_0 \leq x_0$. Let $x_1 = x_0 - y_0$. If $x_1 > 0$, then let $Y_1 = \{\tau_{k-1}^j\}_{j=k}^{k+b_1}$ be the largest set of successive terms in the $(k-1)$-column of $\triangle$ with

$y_1 = \sum Y_1 \leq x_1$. Either $Y_1 = \emptyset$, in which case we take $b_1 = -1$, or not, in which case we take $b_1 = |Y_1| - 1$. Let $x_2 = x_1 - y_1$. If $x_2 > 0$, then let $Y_2 = \{\tau_{k-2}^j\}_{j=k}^{k+b_2}$ be the largest set of successive terms in the $(k-2)$-column of $\triangle$ with $y_2 = \sum Y_2 \leq x_2$. Either $Y_2 = \emptyset$, in which case we take $b_2 = -1$, or not, in which case we take $b_2 = |Y_3| - 1$. Iteratively, we arrive at a null x_k. Then $\alpha(x_0) = a_{k-1}a_{k-2}\cdots a_1$, where $a_{k-1} = 1$, $a_{k-2} = 1 + b_1, \ldots$, and $a_1 = 1 + b_k$.

We note that $\beta(m)$ is recovered from $\alpha(m) = \alpha(x_0)$ by removing the zeros to the left of the leftmost 1 in $\alpha(x_0)$. Given an RGS β or associated k-germ α, the considerations above can easily be played backwards to recover the corresponding integer x_0.

For example, if $x_0 = 38$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 38 - 14 = 24$, $y_1 = \tau_2^3 + \tau_2^4 = 5 + 9 = 14$, $x_2 = x_1 - y_1 = 24 - 14 = 10$, $y_2 = \tau_1^2 + \tau_1^3 + \tau_1^4 = 2 + 3 + 4 = 9$, $x_3 = x_2 - y_2 = 10 - 9 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - y_3 = 1 - 1 = 0$, so that $b_1 = 1$, $b_2 = 2$, and $b_3 = 0$, taking to $a_4 = 1$, $a_3 = 1 + b_1 = 2$, $a_2 = 1 + b_2 = 3$ and $a_1 = 1 + b_3 = 1$, determining the 5-germ $\alpha(38) = a_4a_3a_2a_1 = 1231$. If $x_0 = 20$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 20 - 14 = 6$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 1$, $y_2 = 0$ is an empty sum (since its possible summand $\tau_1^2 > 1 = x_2$), $x_3 = x_2 - y_2 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - x_3 = 1 - 1 = 0$, determining the 5-germ $\alpha(20) = a_4a_3a_2a_1 = 1101$. Moreover, if $x_0 = 19$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 19 - 14 = 5$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 5 - 5 = 0$, determining the 5-germ $\beta(19) = a_4a_3a_2a_1 = 1100$.

3. Nested substring-swaps in n-strings

An *ordered* (rooted) *tree* [6] is a tree T with: **(a)** a node v_0 as its root; **(b)** an embedding of T into the plane with v_0 on top; **(c)** the edges between the nodes at distances j and $j+1$ from v_0 ($0 \leq j <$ height(T)) having parent nodes at the j-level above their children at the $(j+1)$-level; **(d)** the children in (c) ordered from left to right.

Proposition 3.1. *Each k-edge ordered tree T is represented biunivocally by an n-string $F(T)$.*

Proof. We perform a depth first search (→DFS) on T with its vertices from v_0 downward denoted v_i ($i = 0, 1, \ldots, k$) in a right-to-left breadth-first search (←BFS) way. Such DFS yields the claimed $F(\alpha)$ by writing successively from left to right:

(i) the subindex i of each v_i in the →DFS downward appearance and

(ii) an asterisk for each edge e_i with child v_i in the →DFS upward appearance. □

Theorem 3.1. *Each k-germ $\alpha = a_{k-1}\cdots a_1 \neq 0^{k-1}$ with rightmost nonzero entry a_i ($1 \leq i = i(\alpha) < k$) corresponds to a k-germ $\beta(\alpha) = b_{k-1}\cdots b_1 < \alpha$ having $b_i = a_i - 1$ and $a_j = b_j$ for $j \neq i$. Moreover, k-germs are the vertices of an ordered tree $\mathcal{T}_k$ rooted at 0^{k-1}, each k-germ $\alpha \neq 0^{k-1}$ having $\beta(\alpha)$ as its parent so that the edge $\beta(\alpha)\alpha$ of $\mathcal{T}_k$ between $\beta(\alpha)$ and α admits a label $i = i(\alpha)$. Furthermore, the existence of $\mathcal{T}_k$ allows to sort all k-germs linearly.*

Proof. The statement, illustrated for $k = 2, 3, 4$ in the first three columns of Table I, is straightforward. Table I also serves as illustration for the proof of Theorem 3.2, below. □

By representing $\mathcal{T}_k$ with each node β having its children α enclosed between parentheses following β and separating siblings with commas, we can write:

$$\mathcal{T}_4 = 000(001, 010(011(012)), 100(101, 110(111(121)), 120(121(122(123))))).$$

Theorem 3.2. *To each k-germ $\alpha = a_{k-1}\cdots a_1$ corresponds biunivocally an n-string $F(\alpha) = F(T) = f_0 f_1 \cdots f_{2k}$ whose entries are $0, 1, \ldots, k$ (once each) and k asterisks $*$ such that:*

(A) *T is a k-edge ordered tree;* **(B)** $F(0^{k-1}) = 012\cdots(k-1)k * \cdots *$*;*

(C) *if $\alpha \neq 0^{k-1}$, then $F(\alpha)$ is obtained from $F(\beta) = F(\beta(\alpha)) = h_0 h_1 \cdots h_{2k}$ as in Theorem 3.1 via the following Nested String Swapping (Castling) Procedure, where $i = i(\alpha)$:*

1. *let $W^i = h_0 h_1 \cdots h_{i-1} = f_0 f_1 \cdots f_{i-1}$ and $Z^i = h_{2k-i+1} \cdots h_{2k-1} h_{2k} = f_{2k-i+1} \cdots f_{2k-1} f_{2k}$ be respectively the initial and terminal substrings of length $i = i(\alpha)$ in $F(\beta)$;*

2. *let $\Omega > 0$ be the leftmost entry of the substring $U = F(\beta) \setminus (W^i \cup Z^i)$ and consider the concatenation $U = X|Y$, with Y starting at entry $\Omega + 1$; then, $F(\beta) = W^i|X|Y|Z^i$;*

3. *set $F(\alpha) = W^i|Y|X|Z^i$, (the result of swapping the nested substring $X|Y$, yielding $Y|X$).*

In particular: **(a)** *the leftmost entry, f_0, of each $F(\alpha)$ is 0;* **(b)** *$k*$ is a substring of $F(\alpha)$;*

(c) *each $f_j \in [0, k]$ with $f_{j+1} \in [0, k)$ satisfies $f_j < f_{j+1}$, where $j \in [0, 2k)$;*

(d) *each substring $f_j * \cdots * f_{j'}$ of $F(\alpha)$ ($j'' \in (j, j') \subset [0, 2k) \Rightarrow f_{j''} = *$) has $f_{j'} < f_j$;*

(e) *W^i is an i-substring with no asterisks;* **(f)** *Z^i is formed exactly by i asterisks.*

TABLE I

m	α	β	$F(\beta)$	i	$W^i \vert X \vert Y \vert Z^i$	$W^i \vert Y \vert X \vert Z^i$	$F(\alpha)$	α
0	0	–	–	–	–	–	$012**$	0
1	1	0	$012**$	1	$0 \vert 1 \vert 2* \vert *$	$0 \vert 2* \vert 1 \vert *$	$02*1*$	1
0	00	–	–	–	–	–	$0123***$	00
1	01	00	$0123***$	1	$0\vert 1\vert 23**\vert *$	$0\vert 23**\vert 1\vert *$	$023**1*$	01
2	10	00	$0123***$	2	$01\vert 2\vert 3*\vert **$	$01\vert 3*\vert 2\vert **$	$013*2**$	10
3	11	10	$013*2**$	1	$0\vert 13*\vert 2*\vert *$	$0\vert 2*\vert 13*\vert *$	$02*13**$	11
4	12	11	$02*13**$	1	$0\vert 2*1\vert 3*\vert *$	$0\vert 3*\vert 2*3\vert *$	$03*2*1*$	12
0	000	–	–	–	–	–	$01234****$	000
1	001	000	$01234****$	1	$0\vert 1\vert 234***\vert *$	$0\vert 234***\vert 1\vert *$	$0234***1*$	001
2	010	000	$01234****$	2	$01\vert 2\vert 34**\vert **$	$01\vert 34**\vert 2\vert **$	$0134**2**$	010
3	011	010	$0134**2**$	1	$0\vert 134**\vert 2*\vert *$	$0\vert 2*\vert 134**\vert *$	$02*134***$	011
4	012	011	$02*134***$	1	$0\vert 2*1\vert 34**\vert *$	$0\vert 34**\vert 2*1\vert *$	$034**2*1*$	012
5	100	000	$01234****$	3	$012\vert 3\vert 4*\vert ***$	$012\vert 4*\vert 3\vert ***$	$0124*3***$	100
6	101	100	$0124*3***$	1	$0\vert 1\vert 24*3**\vert *$	$0\vert 24*3**\vert 1\vert *$	$024*3**1*$	101
7	110	100	$0124*3***$	2	$01\vert 24*\vert 3*\vert **$	$01\vert 3*\vert 24*\vert **$	$013*24***$	110
8	111	110	$013*24***$	1	$0\vert 13*\vert 24**\vert *$	$0\vert 24**\vert 13*\vert *$	$024**13**$	111
9	112	111	$024**13**$	1	$0\vert 24**1\vert 3*\vert *$	$0\vert 3*\vert 24**1\vert *$	$03*24**1*$	112
10	120	110	$013*24***$	2	$01\vert 3*2\vert 4*\vert **$	$01\vert 4*\vert 3*2\vert **$	$014*3*2**$	120
11	121	120	$014*3*2**$	1	$0\vert 14*3*\vert 2*\vert *$	$0\vert 2*\vert 14*3*\vert *$	$02*14*3**$	121
12	122	121	$02*14*3**$	1	$0\vert 2*34*\vert 3*\vert *$	$0\vert 3*\vert 2*14*\vert *$	$03*2*14**$	122
13	123	122	$03*2*14**$	1	$0\vert 3*2*1\vert 4*\vert *$	$0\vert 4*\vert 3*2*1\vert *$	$04*3*2*1*$	123

Proof. Let $\alpha = a_{k-1}\cdots a_1 \neq 0^{k-1}$ be a k-germ. In the sequence of (nested substring-swap) applications of steps 1-3 along the path from root 0^{k-1} to α in $\mathcal{T}_k$, unit augmentation of a_i for larger values of i ($0 < i < k$) must occur earlier, and then in strictly descending order of the entries i of the intermediate k-germs. As a result, the length of the inner substring $X|Y$ is kept non-decreasing after each application. This is illustrated in Table I, where the order of presentation of X and Y is reversed in successively decreasing steps. In the process, items (a)-(e) are seen to be fulfilled.

The three successive subtables in Table I have C_k rows each, where $C_2 = 2$, $C_3 = 5$ and $C_4 = 14$; in the subtables, the k-germs α are shown both on the second and last columns via natural enumeration in the first column; the images $F(\alpha)$ of those α are shown on the penultimate column; the remaining columns in the table are filled, from the second row on, as follows: **(i)** $\beta = \beta(\alpha)$, arising in Theorem 3.1; **(ii)** $F(\beta)$, taken from the penultimate column in the previous row; **(iii)** the length i of W^i and Z^i ($1 \leq i \leq k-1$); **(iv)** the decomposition $W^i|Y|X|Z^i$ of $F(\beta)$; **(v)** the nested swapping $W^i|X|Y|Z^i$ of $W^i|Y|X|Z^i$, re-concatenated in the following, penultimate, column as $F(\alpha)$, with $\alpha = F^{-1}(F(\alpha))$ in the last column. □

In the context of the results above, let $T = T_\alpha$, so $F(T_\alpha) = F(\alpha)$. For each k-germ $\alpha \neq 0^{k-1}$, Theorem 3.2 carries a *tree-surgery transformation* from T_β onto T_α by *pruning-and-regrafting* of an adequate subtree of T_β via the vertices v_i and the edges e_i, with parent vertices reattached in a substring swapping way. Proposition 3.1 was used in Sections 9-10 [5] in giving a stepwise-reversing view of Hamilton cycles [6] in the M_k's.

TABLE II

m	α	$\theta(\alpha)$	$\hat{\theta}(\alpha)$	$\hat{\aleph}(\theta(\alpha)) = \aleph(\hat{\theta}(\alpha))$	$\aleph(\theta(\alpha))$
0	0	00011	$0_0 0_1 0_2 1_* 1_*$	$0_* 0_* 1_2 1_1 1_0$	00111
1	1	00101	$0_0 0_2 1_* 0_1 1_*$	$0_* 1_1 0_* 1_2 1_0$	01011
0	00	0000111	$0_0 0_1 0_2 0_3 1_* 1_* 1_*$	$0_* 0_* 0_* 1_3 1_2 1_1 1_0$	0001111
1	01	0001101	$0_0 0_2 0_3 1_* 1_* 0_1 1_*$	$0_* 1_1 0_* 0_* 1_3 1_2 1_0$	0100111
2	10	0001011	$0_0 0_1 0_3 2_* 0_1 1_* 1_*$	$0_* 0_* 1_2 0_* 1_3 1_1 1_0$	0010111
3	11	0010011	$0_0 0_2 1_* 0_1 0_3 1_* 1_*$	$0_* 0_* 1_3 1_1 0_* 1_2 1_0$	0011011
4	12	0010101	$0_0 0_3 1_* 0_2 1_* 0_1 1_*$	$0_* 1_1 0_* 1_2 0_* 1_3 1_0$	0101011

Each $F(\alpha)$ corresponds to a binary n-string $\theta(\alpha)$ of weight k obtained by replacing each number in $[k+1]$ by 0 and each asterisk $*$ by 1. By attaching the entries of $F(\alpha)$ as subscripts to the corresponding entries of $\theta(\alpha)$, a subscripted binary n-string $\hat{\theta}(\alpha)$ is obtained, as shown for $k = 2, 3$ in the fourth column of Table II. Let $\aleph(\theta(\alpha))$ be given by the *complemented reversal* of $\theta(\alpha)$, that is:

$$\text{if } \theta(\alpha) = a_0 a_1 \cdots a_{2k}, \text{ then } \aleph(\theta(\alpha)) = \bar{a}_{2k} \cdots \bar{a}_1 \bar{a}_0, \tag{2}$$

where $\bar{0} = 1$ and $\bar{1} = 0$. A subscripted version $\hat{\aleph}$ of $\aleph$ is obtained for $\hat{\theta}(\alpha)$, as shown in the fifth column of Table II, with the subscripts of $\hat{\aleph}$ reversed with respect to those of $\aleph$. Each image of a k-germ α under $\aleph$ is an n-string of weight $k+1$ and has the 1's indexed with subscripts in $[k+1]$ and the 0's indexed with asterisk subscript. The subscripts in $[k+1]$ reappear from Section 7 on as lexical colors for the graphs M_k.

4. Translations mod n in M_k

The n-*cube graph* H_n is the Hasse diagram of the Boolean lattice $2^{[n]}$ on the set $[n]$. We will express each vertex v of H_n in three equivalent ways, namely, as:

- **(a)** ordered set $A = \{a_0, a_1, \ldots, a_{j-1}\} = a_0 a_1 \cdots a_{j-1} \subseteq [n]$ that v represents, $(0 < j \leq n)$;
- **(b)** characteristic binary n-vector $B_A = (b_0, b_1, \ldots, b_{n-1})$ of ordered set A in (a) above, where $b_i = 1$ if and only if $i \in A$, $(i \in [n])$;
- **(c)** polynomial $\epsilon_A(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}$ associated to B_A in (b) above.

The ordered set A and the vector B_A in (a) and (b) respectively are written for short as $a_0 a_1 \cdots a_{j-1}$ and $b_0 b_1 \cdots b_{n-1}$. A is said to be the *support* of B_A.

For each $j \in [n]$, let $L_j = \{A \subseteq [n]; |A| = j\}$ be the j-*level* of H_n. Then, M_k is the subgraph of H_n induced by $L_k \cup L_{k+1}$, for $1 \leq k \in \mathbb{Z}$. By viewing the elements of $V(M_k) = L_k \cup L_{k+1}$ as polynomials, as in (c) above, a regular (i.e., free and transitive) translation mod n action Υ' of $\mathbb{Z}_n$ on $V(M_k)$ is seen to exist, given by:

$$\Upsilon' : \mathbb{Z}_n \times V(M_k) \to V(M_k), \text{ with } \Upsilon'(i, v) = v(x)x^i \text{ (mod } 1 + x^n), \tag{3}$$

where $v \in V(M_k)$ and $i \in \mathbb{Z}_n$. Now, Υ' yields a quotient graph M_k/π of M_k, where π stands for the equivalence relation on $V(M_k)$ given by:

$$\epsilon_A(x)\pi\epsilon_{A'}(x) \Longleftrightarrow \exists\, i \in \mathbb{Z} \text{ with } \epsilon_{A'}(x) \equiv x^i \epsilon_A(x) \text{ (mod } 1 + x^n),$$

with $A, A' \in V(M_k)$. This is used in the proof of Theorem 6.1. Clearly, M_k/π is the graph whose vertices are the equivalence classes of $V(M_k)$ under π. Notice that π induces a partition of $E(M_k)$ into equivalence classes that are the edges of M_k/π.

5. Complemented reversals in M_K

Let $(b_0 b_1 \cdots b_{n-1})$ denote the class of $b_0 b_1 \cdots b_{n-1} \in L_i$ in L_i/π. Let $\rho_i : L_i \to L_i/\pi$ be the canonical projection given by $\rho(b_0 b_1 \cdots b_{n-1}) = (b_0 b_1 \cdots b_{n-1})$, for $i \in \{k, k+1\}$. The definition of the complemented reversal $\aleph$ in display (2) is easily extended to a bijection, again denoted $\aleph$, from L_k onto L_{k+1}. Let $\aleph_\pi : L_k/\pi \to L_{k+1}/\pi$ be given by $\aleph_\pi((b_0 b_1 \cdots b_{n-1})) = (\bar{b}_{n-1} \cdots \bar{b}_1 \bar{b}_0)$. Note $\aleph_\pi$ is a bijection and the identities $\rho_{k+1}\aleph = \aleph_\pi \rho_k$ and $\rho_k \aleph^{-1} = \aleph_\pi^{-1}\rho_{k+1}$.

The following geometric representations are handy. List vertically the vertex parts L_k and L_{k+1} of M_k (resp. L_k/π and L_{k+1}/π of M_k/π) so as to display a splitting of $V(M_k) = L_k \cup L_{k+1}$ (resp. $V(M_k)/\pi = L_k/\pi \cup L_{k+1}/\pi$) into pairs, each pair contained in a horizontal line, the two composing vertices of such pair equidistant from a vertical line ϕ (resp. ϕ/π, depicted through M_2/π on the left of Figure 1, Section 6 below). In addition, we impose that each resulting horizontal vertex pair in M_k (resp. M_k/π) be of the form $(B_A, \aleph(B_A))$ (resp. $((B_A), (\aleph(B_A)) = \aleph_\pi((B_A))))$, disposed from left to right at both sides of ϕ. In this context, a non-horizontal edge of M_k/π is said to be a *skew edge*.

Theorem 5.1. *Each skew edge $e = (B_A)(B_{A'})$ of M_k/π corresponds to another skew edge $\aleph_\pi((B_A))\aleph_\pi^{-1}((B_{A'}))$ obtained from e by reflection on the line ϕ/π. Moreover:*

(i) *the skew edges of M_k/π appear in pairs, with the endpoints of the edges in each pair forming two horizontal pairs of vertices equidistant from ϕ/π;*

(ii) *each horizontal edge of M_k/π has multiplicity equal either to 1 or to 2.*

Proof. The skew edges $B_A B_{A'}$ and $\aleph^{-1}(B_{A'})\aleph(B_A)$ of M_k are reflection of each other about ϕ. Their endopoints form two horizontal pairs $(B_A, \aleph(B_{A'}))$ and $(\aleph^{-1}(B_A), B_{A'})$ of vertices. Now, ρ_k and ρ_{k+1} extend together to a covering graph map $\rho : M_k \to M_k/\pi$, since the edges accompany the projections correspondingly, exemplified for $k = 2$ as follows:

$$\aleph((B_A))= \aleph((00011))= \aleph(\{00011,10001,11000,01100,00110\})=\{00111,01110,11100,11001,10011\}=(00111),$$
$$\aleph^{-1}((B'_A))=\aleph^{-1}((01011))=\aleph^{-1}(\{01011,10110,10110,11010,10101\})=\{00101,10010,01001,10100,01010\}=(00101).$$

Here, the order of the elements in the image of class (00011) (resp. (01011)) mod π under $\aleph$ (resp. $\aleph^{-1}$) are shown reversed, from right to left (cyclically between braces, continuing on the right once one reaches the leftmost brace). Such reversal holds for every $k > 2$:

$$\aleph((B_A))= \aleph((b_0\cdots b_{2k}))= \aleph(\{b_0\cdots b_{2k}, b_{2k}\ldots b_{2k-1}, \ldots, b_1\cdots b_0\})=\{\bar b_{2k}\cdots\bar b_0, \bar b_{2k-1}\cdots\bar b_{2k}, \ldots, \bar b_1\cdots\bar b_0\}=(\bar b_{2k}\cdots\bar b_0),$$
$$\aleph^{-1}((B'_A))=\aleph^{-1}((\bar b'_{2k}\cdots\bar b'_0))=\aleph^{-1}(\{\bar b'_{2k}\cdots\bar b'_0, \bar b'_{2k-1}\cdots\bar b'_{2k}, \ldots, \bar b'_1\cdots\bar b'_0\})=\{b'_0\cdots b'_{2k}, b'_{2k}\cdots b'_{2k-1}, \ldots, b'_1\cdots b'_0\}=(b'_0\cdots b'_{2k}),$$

where $(b_0 \cdots b_{2k}) \in L_k/\pi$ and $(b'_0 \cdots b'_{2k}) \in L_{k+1}/\pi$. This establishes (i).

Every horizontal edge $v\aleph_\pi(v)$ of M_k/π has $v \in L_k/\pi$ represented by $\bar b_k \cdots \bar b_1 0 b_1 \cdots b_k$ in L_k, (so $v = (\bar b_k \cdots \bar b_1 0 b_1 \cdots b_k)$). There are 2^k such vertices in L_k and at most 2^k corresponding vertices in L_k/π. For example, $(0^{k+1}1^k)$ and $(0(01)^k)$ are endpoints in L_k/π of two horizontal edges of M_k/π, each. To prove that this implies (ii), we have to see that there cannot be more than two representatives $\bar b_k \cdots \bar b_1 b_0 b_1 \cdots b_k$ and $\bar c_k \cdots \bar c_1 c_0 c_1 \cdots c_k$ of a vertex $v \in L_k/\pi$, with $b_0 = 0 = c_0$. Such a v is expressible as $v = (d_0 \cdots b_0 d_{i+1} \cdots d_{j-1} c_0 \cdots d_{2k})$, with $b_0 = d_i$, $c_0 = d_j$ and $0 < j - i \leq k$. Let the substring $\sigma = d_{i+1} \cdots d_{j-1}$ be said $(j-i)$-*feasible*. Let us see that every $(j-i)$-feasible substring σ forces in L_k/π only vertices ω leading to two different (parallel) horizontal edges in M_k/π incident to v. In fact, periodic continuation mod n of $d_0 \cdots d_{2k}$ both to the right of $d_j = c_0$ with minimal cyclic substring $\bar d_{j-1} \cdots \bar d_{i+1} 1 d_{i+1} \cdots d_{j-1} 0 = P_r$ and to the left of $d_i = b_0$ with minimal cyclic substring $0 d_{i+1} \cdots d_{j-1} 1 \bar d_{j-1} \cdots \bar d_{i+1} = P_\phi$ yields a 2-way infinite string that winds up onto a class $(d_0 \cdots d_{2k})$ containing such an ω. For example, some pairs of feasible substrings σ and resulting vertices ω are:

$$(\sigma,\omega) = (\emptyset,(\mathrm{oo}1)),\ (0,(\mathrm{o}0\mathrm{o}11)),\ (1,(\mathrm{o}1\mathrm{o})),\ (0^2,(\mathrm{o}00\mathrm{o}111)),\ (01,(\mathrm{o}01\mathrm{o}011)),\ (1^2,\mathrm{o}11\mathrm{o}0)),$$
$$(0^3,\mathrm{o}000\mathrm{o}1111)),\ (010,(\mathrm{o}010\mathrm{o}101101)),\ (01^2,(\mathrm{o}011\mathrm{o})),\ (101,(\mathrm{o}101\mathrm{o})),\ (1^3,(\mathrm{o}111\mathrm{o}00)),$$

with 'o' replacing $b_0 = 0$ and $c_0 = 0$, and where $k = \lfloor \frac{n}{2} \rfloor$ has successive values $k = 1, 2, 1, 3, 3, 2, 4, 5, 2, 2, 3$. If σ is a feasible substring and $\bar\sigma = \aleph(\sigma)$, then the possible symmetric substrings $P_\phi \sigma P_r$ about $\mathrm{o}\sigma\mathrm{o} = 0\sigma 0$ in a vertex v of L_k/π are in order of ascending length:

$$0\sigma 0,$$
$$\bar\sigma 0\sigma 0\bar\sigma,$$
$$1\bar\sigma 0\sigma 0\bar\sigma 1,$$
$$\sigma 1\bar\sigma 0\sigma 0\bar\sigma 1\sigma,$$
$$0\sigma 1\bar\sigma 0\sigma 0\bar\sigma 1\sigma 0,$$
$$\bar\sigma 0\sigma 1\bar\sigma 0\sigma 0\bar\sigma 1\sigma 0\bar\sigma,$$
$$1\bar\sigma 0\sigma 1\bar\sigma 0\sigma 0\bar\sigma 1\sigma 0\bar\sigma 1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

where we use again '0' instead of 'o' for the entries immediately preceding and following the shown central copy of σ. The lateral periods of P_r and P_ϕ determine each one horizontal edge at v in M_k/π up to returning to b_0 or c_0, so no entry $e_0 = 0$ of $(d_0 \cdots d_{2k})$ other than b_0 or c_0 happens such that $(d_0 \cdots d_{2k})$ has a third representative $\bar{e}_k \cdots \bar{e}_1 0 e_1 \cdots e_k$ (besides $\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 0 c_1 \cdots c_k$). Thus, those two horizontal edges are produced solely from the feasible substrings $d_{i+1} \cdots d_{j-1}$ characterized above. □

To illustrate Theorem 5.1, let $1 < h < n$ in $\mathbb{Z}$ be such that $\gcd(h, n) = 1$ and let $\lambda_h : L_k/\pi \to L_k/\pi$ be given by $\lambda_h((a_0 a_1 \cdots a_n)) \to (a_0 a_h a_{2h} \cdots a_{n-2h} a_{n-h})$. For each such $h \leq k$, there is at least one h-feasible substring σ and a resulting associated vertex $v \in L_k/\pi$ as in the proof of Theorem 5.1. For example, starting at $v = (0^{k+1}1^k) \in L_k/\pi$ and applying λ_h repeatedly produces a number of such vertices $v \in L_k/\pi$. If we assume $h = 2h'$ with $h' \in \mathbb{Z}$, then an h-feasible substring σ has the form $\sigma = \bar{a}_1 \cdots \bar{a}_{h'} a_{h'} \cdots a_1$, so there are at least $2^{h'} = 2^{\frac{h}{2}}$ such h-feasible substrings.

6. Dihedral quotient pseudograph R_k of M_k

An *involution* of a graph G is a graph map $\aleph : G \to G$ such that $\aleph^2$ is the identity. If G has an involution, an $\aleph$-*folding* of G is a graph H, possibly with loops, whose vertices v' and edges or loops e' are respectively of the form $v' = \{v, \aleph(v)\}$ and $e' = \{e, \aleph(e)\}$, where $v \in V(G)$ and $e \in E(G)$; e has endvertices v and $\aleph(v)$ if and only if $\{e, \aleph(e)\}$ is a loop of G.

Note that both maps $\aleph : M_k \to M_k$ and $\aleph_\pi : M_k/\pi \to M_k/\pi$ in Section 5 are involutions. Let $\langle B_A \rangle$ denote each horizontal pair $\{(B_A), \aleph_\pi((B_A))\}$ (as in Theorem 5.1) of M_k/π, where $|A| = k$. An $\aleph$-folding R_k of M_k/π is obtained whose vertices are the pairs $\langle B_A \rangle$ and having:

(1) an edge $\langle B_A \rangle \langle B_{A'} \rangle$ per skew-edge pair $\{(B_A)\aleph_\pi((B_{A'})), (B_{A'})\aleph_\pi((B_A))\}$;

(2) a loop at $\langle B_A \rangle$ per horizontal edge $(B_A)\aleph_\pi((B_A))$; because of Theorem 5.1, there may be up to two loops at each vertex of R_k.

Theorem 6.1. *R_k is a quotient pseudograph of M_k under an action $\Upsilon : D_{2n} \times M_k \to M_k$.*

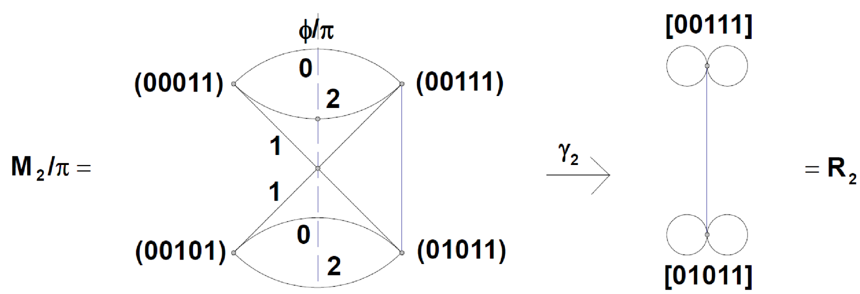


Figure 1. Reflection symmetry of M_2/π about a line ϕ/π and resulting graph map γ_2.

Proof. D_{2n} is the semidirect product $\mathbb{Z}_n \rtimes_\varrho \mathbb{Z}_2$ via the group homomorphism $\varrho : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_n)$, where $\varrho(0)$ is the identity and $\varrho(1)$ is the automorphism $i \to (n - i)$, $\forall i \in \mathbb{Z}_n$. If $* : D_{2n} \times$

$D_{2n} \to D_{2n}$ indicates group multiplication and $i_1, i_2 \in \mathbb{Z}_n$, then $(i_1, 0) * (i_2, j) = (i_1 + i_2, j)$ and $(i_1, 1) * (i_2, j) = (i_1 - i_2, \bar{j})$, for $j \in \mathbb{Z}_2$. Set $\Upsilon((i,j), v) = \Upsilon'(i, \aleph^j(v))$, $\forall i \in \mathbb{Z}_n, \forall j \in \mathbb{Z}_2$, where Υ' is as in display (3). Then, Υ is a well-defined D_{2n}-action on M_k. By writing $(i,j) \cdot v = \Upsilon((i,j), v)$ and $v = a_0 \cdots a_{2k}$, we have $(i,0) \cdot v = a_{n-i+1} \cdots a_{2k} a_0 \cdots a_{n-i} = v'$ and $(0,1) \cdot v' = \bar{a}_{i-1} \cdots \bar{a}_0 \bar{a}_{2k} \cdots \bar{a}_i = (n-i, 1) \cdot v = ((0,1) * (i,0)) \cdot v$, leading to the compatibility condition $((i,j) * (i',j')) \cdot v = (i,j) \cdot ((i',j') \cdot v)$. □

Theorem 6.1 yields a graph projection $\gamma_k : M_k/\pi \to R_k$ for the action Υ, given for $k = 2$ in Figure 1. In fact, γ_2 is associated with reflection of M_2/π about the dashed vertical symmetry axis ϕ/π so that R_2 (containing two vertices and one edge between them, with each vertex incident to two loops) is given as its image. Both the representations of M_2/π and R_2 in the figure have their edges indicated with colors 0,1,2, as arising inSection 7.

7. Lexical procedure

Let P_{k+1} be the subgraph of the unit-distance graph of $\mathbb{R}$ (the real line) induced by the set $[k+1] = \{0, \dots, k\}$. We draw the grid $\Gamma = P_{k+1} \square P_{k+1}$ in the plane $\mathbb{R}^2$ with a diagonal ∂ traced from the lower-left vertex $(0,0)$ to the upper-right vertex (k,k). For each $v \in L_k/\pi$, there are $k+1$ n-tuples of the form $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ that represent v with $b_0 = 0$. For each such n-tuple, we construct a $2k$-path D in Γ from $(0,0)$ to (k,k) in $2k$ steps indexed from $i = 0$ to $i = 2k-1$. This leads to a lexical edge-coloring implicit in [7]; see the following statement and Figure 2 (Section 8), containing examples of such a $2k$-path D in thick trace.

Theorem 7.1. *[7] Each $v \in L_k/\pi$ has its $k+1$ incident edges assigned colors $0, 1, \dots, k$ by means of the following* Lexical Procedure'*, where $0 \le i \in \mathbb{Z}$, $w \in V(\Gamma)$ and D is a path in Γ. Initially, let $i = 0$, $w = (0,0)$ and D contain solely the vertex w. Repeat $2k$ times the following sequence of steps (1)-(3), and then perform once the final steps (4)-(5):*

(1) *If $b_i = 0$, then set $w' := w + (1,0)$; otherwise, set $w' := w + (0,1)$.*
(2) *Reset $V(D) := v(D) \cup \{w'\}$, $E(D) := E(D) \cup \{ww'\}$, $i := i+1$ and $w := w'$.*
(3) *If $w \neq (k,k)$, or equivalently, if $i < 2k$, then go back to step (1).*
(4) *Set $\check{v} \in L_{k+1}/\pi$ to be the vertex of M_k/π adjacent to v and obtained from its representative n-tuple $b_0 b_1 \cdots\ b_{n-1} = 0 b_1 \cdots b_{n-1}$ by replacing the entry b_0 by $\bar{b}_0 = 1$ in $\check{v}$, keeping the entries b_i of v unchanged in $\check{v}$ for $i > 0$.*
(5) *Set the color of the edge $v\check{v}$ to be the number c of horizontal (alternatively, vertical) arcs of D above ∂.*

Proof. If addition and subtraction in $[n]$ are taken modulo n and we write $[y,x) = \{y, y+1, y+2, \dots, x-1\}$, for $x, y \in [n]$, and $S^c = [n] \setminus S$, for $S = \{i \in [n] : b_i = 1\} \subseteq [n]$, then the cardinalities of the sets $\{y \in S^c \setminus x : |[y,x) \cap S| < |[y,x) \cap S^c|\}$ yield all the edge colors, where $x \in S^c$ varies. □

The Lexical Procedure of Theorem 7.1 yields a 1-factorization not only for M_k/π but also for R_k and M_k. This is clarified by the end of Section 8.

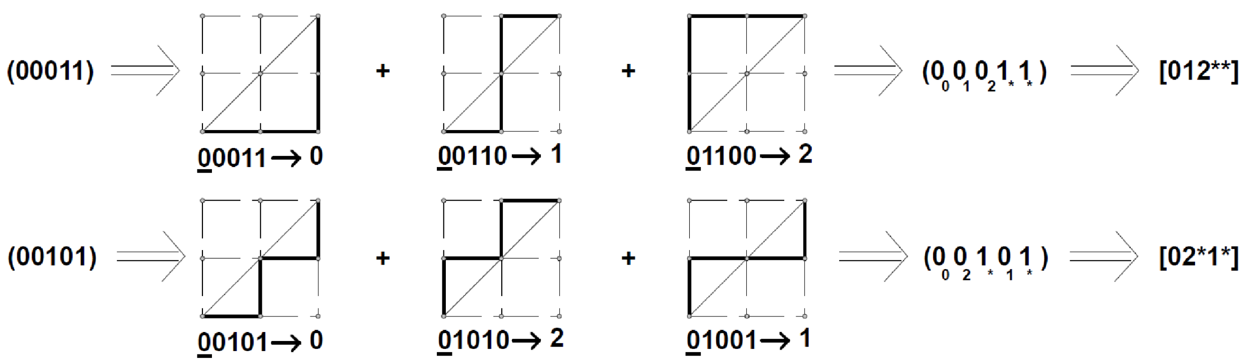


Figure 2. Representing lexical-color assignment for $k = 2$.

8. Lexical 1-factorization

A notation $\delta(v)$ is assigned to each pair $\{v, \aleph_\pi(v)\} \in R_k$, where $v \in L_k/\pi$, so that there is a unique k-germ $\alpha = \alpha(v)$ with $\langle F(\alpha)\rangle = \delta(v)$, where the notation $\langle\cdot\rangle$ appeared for example as in $\langle B_A\rangle$ in Section 6. We exemplify $\delta(v)$ for $k = 2$ in Figure 2, with the Lexical Procedure (indicated by arrows "⇒") departing from $v = (00011)$ (top) and $v = (00101)$ (bottom), passing to sketches of Γ (separated by symbols "+"), one sketch (in which to trace the edges of $D \subset \Gamma$ as in Theorem 7.1) per representative $b_0b_1\cdots b_{n-1} = 0b_1\cdots b_{n-1}$ of v shown under the sketch (where $b_0 = 0$ is underscored) and pointing via an arrow "→" to the corresponding color $c \in [k+1]$. Recall this c is the number of horizontal arcs of D below ∂.

In each of the two cases in Figure 2 (top, bottom), an arrow "⇒" to the right of the sketches points to a modification $\hat{v}$ of $b_0b_1\cdots b_{n-1} = 0b_1\cdots b_{n-1}$ obtained by setting as a subindex of each 0 (resp. 1) its associated color c (resp. an asterisk "$*$"). Further to the right, a third arrow "⇒" points to the n-tuple $\delta(v)$ formed by the string of subindexes of entries of $\hat{v}$ in the order they appear from left to right.

Theorem 8.1. *Let $\alpha(v^0) = a_{k-1}\cdots a_1 = 00\cdots 0$. Each $\delta(v)$ corresponds to a sole k-germ $\alpha = \alpha(v)$ with $\langle F(\alpha)\rangle = \delta(v)$ by means of the following Uncastling Procedure: Given $v \in L_k/\pi$, let $W^i = 01\cdots i$ be the maximal initial numeric (i.e., colored) substring of $\delta(v)$, so that the length of W^i is $i+1$ $(0 \le i \le k)$. If $i = k$, let $\alpha(v) = \alpha(v^0)$; else, set $m = 0$ and:*

1. *set $\delta(v^m) = \langle W^i|X|Y|Z^i\rangle$, where Z^i is the terminal j_m-substring of $\delta(v^m)$, with $j_m = i+1$, and let X, Y (in that order) start at contiguous numbers Ω and $\Omega - 1 \ge i$;*

2. *set $\delta(v^{m+1}) = \langle W^i|Y|X|Z^i\rangle$;*

3. *obtain $\alpha(v^{m+1})$ from $\alpha(v^m)$ by increasing its entry a_{j_m} by 1;*

4. *if $\delta(v^{m+1}) = [01\cdots k * \cdots *]$, then stop; else, increase m by 1 and go to step 1.*

Proof. This is a procedure inverse to that of castling (Section 3), so 1-4 follow. □

Theorem 8.1 allows to produce a finite sequence $\delta(v^0), \delta(v^1), \ldots, \delta(v^m), \ldots, \delta(v^s)$ of n-strings with $j_0 \ge j_1 \ge \cdots \ge j_m \cdots \ge j_{s-1}$ as in steps 1-4, and k-germs $\alpha(v^0), \alpha(v^1), \ldots, \alpha(v^m), \ldots,$ $\alpha(v^s)$, taking from $\alpha(v^0)$ through the k-germs $\alpha(v^m)$, $(m = 1, \ldots, s-1)$, up to $\alpha(v) = \alpha(v^s)$ via unit incrementation of a_{j_m}, for $0 \le m < s$, where each incrementation yields the corresponding

$\alpha(v^{m+1})$. Recall F is a bijection from the set $V(\mathcal{T}_k)$ of k-germs onto $V(R_k)$, both sets being of cardinality C_k. Thus, to deal with $V(R_k)$ it is enough to deal with $V(\mathcal{T}_k)$, a fact useful in interpreting Theorem 8.2 below. For example $\delta(v^0) = \langle 04*3*2*1*\rangle = \langle 0|4*|3*2*1|*\rangle = \langle W^0|X|Y|Z^0\rangle$ with $m=0$ and $\alpha(v^0)=123$, continued in Table III with $\delta(v^1) = \langle W^0|Y|X|Z^0\rangle$, finally arriving to $\alpha(v^s) = \alpha(v^6) = 000$.

TABLE III

$j_0{=}0$	$\delta(v^1)$	$=$	$\langle 0\|3{*}2{*}1\|4{*}\|{*}\rangle$	$=$	$\langle 03{*}2{*}14{*}{*}\rangle$	$=$	$\langle 0\|3{*}\|2{*}14{*}\|{*}\rangle$	$\alpha(v^1){=}122$	$\langle F(122)\rangle{=}\delta(v^1)$
$j_1{=}0$	$\delta(v^2)$	$=$	$\langle 0\|2{*}14{*}\|3{*}\|{*}\rangle$	$=$	$\langle 02{*}14{*}3{*}{*}\rangle$	$=$	$\langle 0\|2{*}\|14{*}3{*}\|{*}\rangle$	$\alpha(v^2){=}121$	$\langle F(121)\rangle{=}\delta(v^2)$
$j_2{=}0$	$\delta(v^3)$	$=$	$\langle 0\|14{*}3{*}\|2{*}\|{*}\rangle$	$=$	$\langle 014{*}3{*}2{*}{*}\rangle$	$=$	$\langle 01\|4{*}\|3{*}2\|{*}{*}\rangle$	$\alpha(v^3){=}120$	$\langle F(120)\rangle{=}\delta(v^3)$
$j_3{=}1$	$\delta(v^4)$	$=$	$\langle 01\|3{*}2\|4{*}\|{*}{*}\rangle$	$=$	$\langle 013{*}24{*}{*}{*}\rangle$	$=$	$\langle 01\|3{*}\|24{*}\|{*}{*}\rangle$	$\alpha(v^4){=}110$	$\langle F(110)\rangle{=}\delta(v^4)$
$j_4{=}1$	$\delta(v^5)$	$=$	$\langle 01\|24{*}\|3{*}\|{*}{*}\rangle$	$=$	$\langle 0124{*}3{*}{*}{*}\rangle$	$=$	$\langle 012\mid 4*\|3\|{*}{*}\rangle$	$\alpha(v^5){=}100$	$\langle F(100)\rangle{=}\delta(v^5)$
$j_5{=}2$	$\delta(v^6)$	$=$	$\langle 012\|3\|4{*}\|{*}{*}{*}\rangle$	$=$	$\langle 01234{*}{*}{*}{*}\rangle$			$\alpha(v^6){=}000$	$\langle F(000)\rangle{=}\delta(v^6)$

A pair of skew edges $(B_A)\aleph_\pi((B_{A'}))$ and $(B_{A'})\aleph((B_A))$ in M_k/π, to be called a *skew reflection edge pair* (*SREP*), provides a color notation for any $v \in L_{k+1}/\pi$ such that in each particular edge class mod π:

(I) all edges receive a common color in $[k+1]$ regardless of the endpoint on which the Lexical Procedure (or its modification immediately below) for $v \in L_{k+1}/\pi$ is applied;

(II) the two edges in each SREP in M_k/π are assigned a common color in $[k+1]$.

The modification in step (I) consists in replacing in Figure 2 each v by $\aleph_\pi(v)$ so that on the left we have instead now (00111) (top) and (01011) (bottom) with respective sketch subtitles

$$0011\underline{1}{\to}0,\quad 1001\underline{1}{\to}1,\quad 1100\underline{1}{\to}2,$$
$$0101\underline{1}{\to}0,\quad 1010\underline{1}{\to}2,\quad 0110\underline{1}{\to}1,$$

resulting in similar sketches when the steps (1)-(5) of the Lexical Procedure are taken with right-to-left reading and processing of the entries on the left side of the subtitles (before the arrows "$\to$"), where the values of each b_i must be taken complemented, (i.e., as $\bar{b_i}$).

Since an SREP in M_k determines a unique edge ϵ of R_k (and vice versa), the color received by the SREP can be attributed to ϵ, too. Clearly, each vertex of either M_k or M_k/π or R_k defines a bijection from its incident edges onto the color set $[k+1]$. The edges obtained via $\aleph$ or $\aleph_\pi$ from these edges have the same corresponding colors.

Theorem 8.2. *A 1-factorization of M_k/π by the colors $0,1,\ldots,k$ is obtained via the Lexical Procedure and can be lifted to a covering 1-factorization of M_k and subsequently collapsed onto a folding 1-factorization of R_k. This validates the notation $\delta(v)$, for each $v \in V(R_k)$, so that there is a unique k-germ $\alpha = \alpha(v)$ with $\langle F(\alpha)\rangle = \delta(v)$.*

Proof. As pointed out in (II) above, each SREP in M_k/π has its edges with a common color in $[k{+}1]$. Thus, the $[k{+}1]$-coloring of M_k/π induces a well-defined $[k{+}1]$-coloring of R_k. This yields the claimed collapsing to a folding 1-factorization of R_k. The lifting to a covering 1-factorization in M_k is immediate. The arguments above determine that the collapsing 1-factorization in R_k induces the claimed k-germs $\alpha(v)$. □

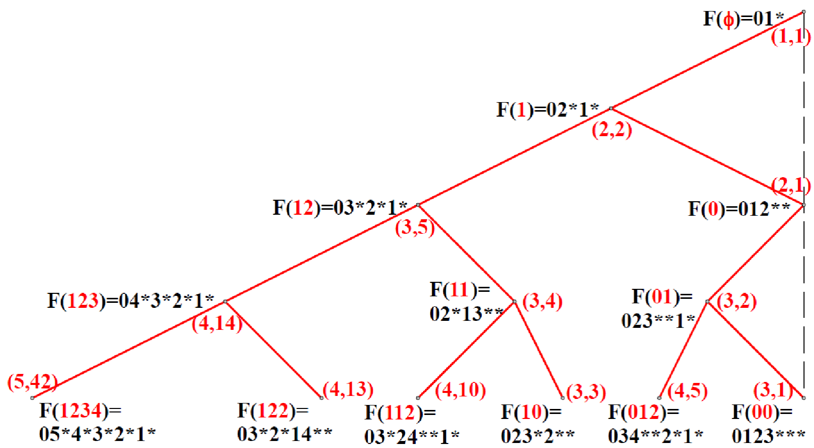


Figure 3. Restriction of T to its first five levels.

9. All-germs binary tree

The graph R_1 has just one vertex 001 with $\delta(001) = 01*$ (δ as in Section 8) and two loops. Note that the correspondence F in Section 3 has $01*$ as the image of the empty set: $F(\emptyset) = 01*$. While Theorem 3.2 allows to sort all k-germs for a fixed k, the following theorem allows to sort all k-germs.

Theorem 9.1. *A binary tree T exists with node set $\cup_{k=1}^{\infty} V(R_k)$ and such that:* **(A)** *its root is $01*$;* **(B)** *the left child of a node $\delta(v) = 0|X$ in T with $||X|| = 2k$ ($||X||$ = length of X) exists and is $0|X+1|1*$, where $X+1 = (x_1+1)\cdots(x_{2k}+1)$ if $X = x_1 \cdots x_{2k}$ with color number addition and $*+1 = *$;* **(C)** *unless $\delta(v) = 01\cdots(k-1)k*\cdots*$, it is $\delta(v) = 0|X|Y|*$, where X and Y are strings starting at some $j > 1$ and $j-1$, respectively, in which case there is a right child of $\delta(v)$, namely $0|Y|X|*$, via uncastling. In terms of k-germs, T has each node $a_{k-1}a_{k-2}\cdots a_2a_1$ as a parent of a left child $b_kb_{k-1}\cdots b_1 = a_{k-1}a_{k-2}\cdots a_2a_1(a_1+1)$, and as a parent of a right child ρ only if $a_1 > 0$, in which case $\rho = c_{k-1}\cdots c_2c_1 = a_{k-1}\cdots a_2(a_1-1)$.*

Proof. Figure 3 shows the first five levels of T with edges in red and nodes, expressed in terms of red k-germs via F, in otherwise black equalities. To stress the claimed unifying pattern mentioned in Section 1, the figure also assigns to each node a red-colored ordered pair of positive integers (i,j), where $j \le C_i$. The root, given by $F(\emptyset) = 01*$, is assigned red $(i,j) = (1,1)$. The left child of a node assigned red (i,j) is assigned red $(k,j') = (i+1,j')$, where j' is the order of appearance of the k-germ α corresponding to (k,j') in its presentation via castling as in Table I; α becomes the k-germ corresponding to j' in the sequence $\mathcal{S}$ (A239903), once the extra zeros to the left of its leftmost nonzero entry are removed. Note $j' = j'(j)$ arises from the series associated to A076050, deducible from items 1-4 in Subsection 2.1. The right child of a red (i,j) is defined only if $j > 1$ (strictly to the left of the vertical dotted line); in that case, it is assigned red $(i, j-1)$. □

10. Comparing k-germs and k-RGS's

We show now that the k-germs of Section 2, that were used in all of the above, are equivalent to the sequences of item (u) page 224 [14]. These sequences, that we call k-RGS's in the present context to distinguish them from our k-germs, are indicated in the form $a_0a_1\cdots a_{k-1}$ satisfying $a_0 = 0$ and $0 \le a_{i+1} \le a_i + 1$. Item (r) page 224 [14] can be used to show that these k-RGS's represent bijectively the k-edge ordered trees, also presented in item (e) page 221 [14]. In fact, let $b_i = a_i - a_{i+1} + 1$ and replace a_i with one "1" followed by b_i "-1"s, for $1 \le i \le k-1$, where we assume $a_k = 0$, to get a sequence as in item (r), i.e. sequences of $k-1$ "1"s and $k-1$ "-1"s such that every partial sum is nonnegative, with "-1" denoted simply as "$-$".

TABLE IV

$\underline{01}\overset{1}{-}\underline{\underline{00}}\overset{2}{-}10\overset{1}{-}11\overset{1}{-}\underline{12}$	$\underline{012}\overset{1}{-}011\overset{1}{-}010\overset{2}{-}\underline{\underline{000}}\overset{3}{-}100\overset{2}{-}110\overset{2}{-}120\overset{1}{-}121\overset{1}{-}122\overset{1}{-}\underline{123}$
	${}_1\vert \quad {}_1\vert \quad {}_1\vert$
	$\underline{001}\quad \underline{101}\quad 111\overset{1}{-}\underline{112}$
$\underline{10}\overset{1}{-}\underline{\underline{12}}\overset{2}{-}11\overset{1}{-}01\overset{1}{-}\underline{00}$	$\underline{100}\overset{1}{-}012\overset{1}{-}121\overset{2}{-}\underline{\underline{123}}\overset{3}{-}122\overset{2}{-}112\overset{2}{-}111\overset{1}{-}011\overset{1}{-}001\overset{1}{-}\underline{000}$
	${}_1\vert \quad {}_1\vert \quad {}_1\vert$
	$\underline{120}\quad \underline{110}\quad 101\overset{1}{-}\underline{010}$

For a bijection of the k-edge ordered trees with the sequence in item (r), a depth-first (preorder) search through each k-edge ordered tree is performed: When going "down" an edge (away from the root) records a "1", and when going "up" an edge records a "-1". Thus, the k-germs are in 1-1 correspondence with the RGS's, as claimed. However, each k-germ and its correspondent k-RGS have different expressions, as can be seen by comparing, in the pair of graph subtables in TABLE IV, the tree $\mathcal{T}_k$ presented with its nodes expressed first as k-germs (top table) and then as k-RGS's (bottom table), for $k = 3, 4$, where the root is doubly underlined and the leaves are simply underlined, and where k-RGS's are written $a_1\cdots a_{k-1}$ instead of $a_0a_1\cdots a_{k-1} = 0a_1\cdots a_{k-1}$:

TABLE V

i	*edge label subseq of* ℓ_i	*first node in* ℓ_i	*2nd node in* ℓ_i	*etc.*	*etc.*	*etc.*	*etc.*	*etc.*
1	k_1	$01...k_3k_2k_1$	$01...k_3k_2^2$	$-$	$-$	$-$	$-$	$-$
2	k_2^2	$01...k_3k_2^2$	$01...k_3^2k_2$	$01...k_4k_3^3$	$-$	$-$	$-$	$-$
3	k_3^3	$01...k_4k_3^3$	$01...k_4^2k_3^2$	$01...k_4^3k_3$	$01...k_4^4$	$-$	$-$	$-$
...	...	...	...	...	...	$-$	$-$	$-$
j	k_j^j	$01...k_{j+1}k_j^j$	$01...k_{j+1}^2k^{j_1}$	$01...k_{j+1}^3k_j^{j_2}$	...	$01...k_j^j$	$-$	$-$
...	...	...	...	...	...	...	$-$	$-$
k_3	3	0123^{k_3}	$012^23^{k_4}$	$012^33^{k_5}$	...	012^{k_2}	$-$	$-$
k_2	2	012^{k_2}	$01^22^{k_3}$	$01^32^{k_4}$	...	$01^{k_2}2$	01^{k_1}	$-$
k_1	1	01^{k_1}	0^21^{k-2}	$0^31^{k_3}$	...	$0^{k_2}1^2$	$0^{k_1}1$	0^k

In these representations of $\mathcal{T}_k$ each edge is given as a short segment with a label $i = i(\alpha)$ as in Theorem 3.1. Thus, each path from the root to a leaf in $\mathcal{T}_k$ can be presented by the associated subsequence of edge labels. From the tables above, we see that the collection of such subsequences for $k = 3$ is $\{211, 1\}$, and for $k = 4$ is $\{322111, 3211, 31, 211\}$.

Let χ be the assignment that to each k-germ α assigns its associated k-RGS. Expressing k-RGS's as $a_0a_1 \cdots a_{k-1} = 0a_1 \cdots a_{k-1}$, for example $k = 3$ yields

$$\chi(\underline{\underline{00}}) = \underline{\underline{012}},\ \chi(\underline{01}) = \underline{010},\ \chi(10) = 011,\ \chi(11) = 001,\ \chi(\underline{12}) = \underline{000}.$$

The lower table above can be taken to represent the trees $\chi(\mathcal{T}_3)$ and $\chi(\mathcal{T}_4)$.

The following properties are seen to hold for $1 < k \in \mathbb{Z}$

1. The root of $\chi(\mathcal{T}_k)$ and its farthest leaf in $\chi(\mathcal{T}_i)$ are $\chi(0^{k-1}) = 012\cdots(k-1)$ and $\chi(12\cdots(k-1)) = 0^k$. Furthermore, the leaves of $\chi(\mathcal{T}_k)$ are those RGS's $a_0a_1\cdots a_{k-1}$ with $a_{k-1} = 0$.
2. Each maximum path ℓ_i of $\chi(\mathcal{T}_k)$ whose edges have a constant label $i \in [1, n]$ has initial and terminal nodes of the form $A_1 = 0a_1a_2\cdots a_n$ and $A_h = 0(a_2 - 1)\cdots(a_n - 1)(i - 1)$.
3. By writing $k_j = k-j$, for $j = 1, \dots, k-1$, the longest path ℓ in $\chi(\mathcal{T}_k)$ departing from its root has associated edge-label sequence $k_1k_2^2k_3^3\cdots k_j^j\cdots 2^{k_2}1^{k_1}$ and is the result of concatenating successively its subpaths ℓ_i as in item 2, described in Table V.
4. Each node A of $\chi(\mathcal{T}_{k+1})$ that is a $(k+1)$-RGS's having a maximal substring of the form $012...j$ of length $j+1$, where j is the sole maximum entry in A, yields a node of $\chi(\mathcal{T}_k)$ by just removing j from A. All such nodes A of $\chi(\mathcal{T}_{k+1})$ yield, by these indicated removals, all of $\chi(\mathcal{T}_k)$. To be used below, let χ''_{k+1} be the set of all the nodes A above in this item and let $\chi'_{k+1} = \chi(\mathcal{T}_{k+1}) \setminus \chi''_{k+1}$.
5. Let $(A_1, A_2, \dots, A_h)$ be a path as in item two in $\chi'_k \setminus \ell$. To obtain A_{i-1} from $A_i = 0a_1\cdots a_{k-1}$, for $i = h, h-1, \dots, 2$, let $A_i = A'_i|A''_i$ be obtained by the concatenation of the strings $A'_i = a_0a_1\cdots a_j$ and $A''_i = a_{j+1}\cdots a_{k-1}$, where $A'_i = a_0 = 0$, if $a_1 = 0$, and A'_i is the maximal initial nondecreasing substring of A_i, otherwise, and where $A''_i = A_i \setminus A'_i$. Then $A_{i-1} = 0|(A''_i \setminus a_{j+1})|(A'_i + 1) = 0a_{j+2}\cdots a_{k-1}(a_0+1)(a_1+1)\cdots(a_j+1)$.

TABLE VI

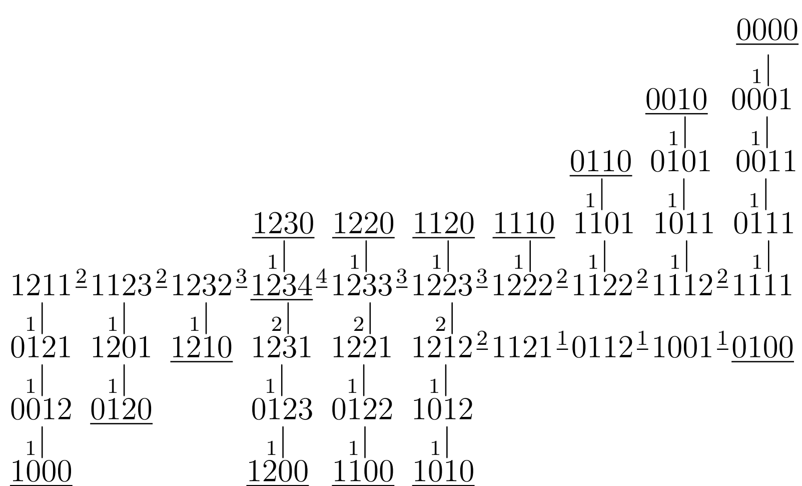
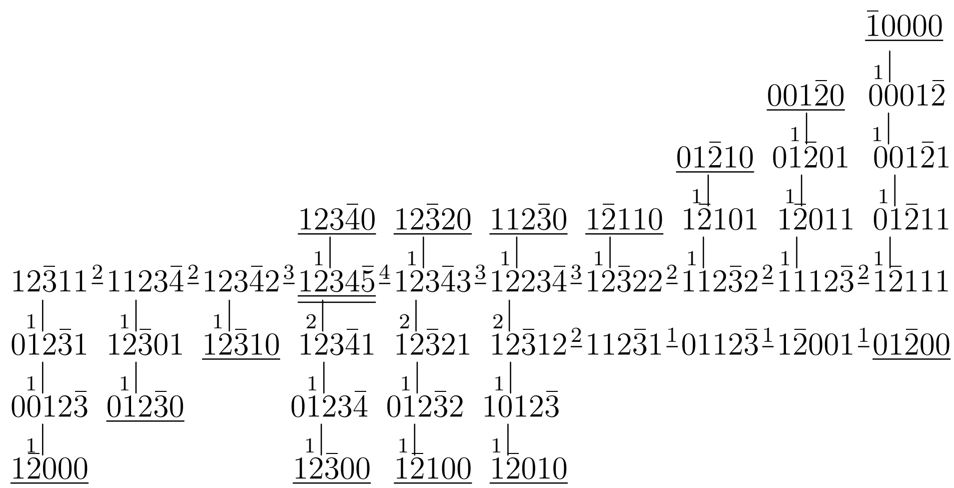


TABLE VII



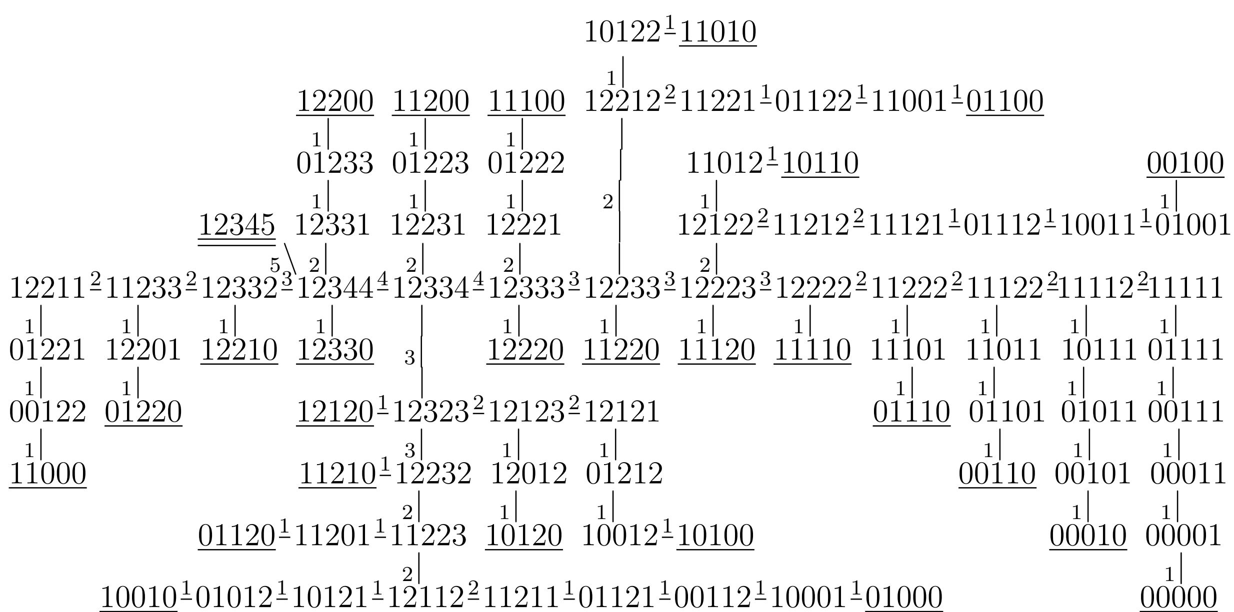
Tables VI and VII contain respective representations of $\chi(\mathcal{T}_5)$ and χ''_6, the latter one here with a bar over the maximal entry of each RGS node , as in item 4, entry whose removal yields a corresponding node of $\chi(\mathcal{T}_4)$.

As an additional example here, Table VIII contains a representation of χ'_6.

By considering the order-number permutations (as in the left column in Table I above) via χ we obtain permutations as follows:

$k=3$	$(0,4)(1,2,3)$
$k=4$	$(0,13)(1,8,6,7,9,2,11,3,4,5,12)$
$k=5$	$(0,41)(1,37,22,18,19,36,2,38,8,29,21,32,7,27,5,39,3,13,14,40)$ $(4,28,35,6,30,26,15,33,23,16,34,17,12)(9,10,31,20,24,25)(11)$

TABLE VIII



11. Colored germ adjacency

TABLE IX

m	α	$F(\alpha)$	$F^3(\alpha)$	$F^2(\alpha)$	$F^1(\alpha)$	$F^0(\alpha)$	α^3	α^2	α^1	α^0
0	0	012 * *	–	012 * *	02 * 1*	1 2 **0	–	0	1	0
1	1	02 * 1*	–	1 * 02*	012 * *	2*1* 0	–	1	0	1
0	00	0123***	0123 ***	013*2**	023* *1*	123*** 0	00	10	01	00
1	01	023**1*	1*023**	1*03*2*	0123 ***	2*13** 0	01	12	00	11
2	10	013*2**	02*20**	0123***	03*2*1*	13*2** 0	11	00	12	10
3	11	02*13**	013*2**	13**02*	02*13**	10**2* 3	10	11	11	01
4	12	03*2*1*	2*1*03*	1*023**	013*2**	3*2*1* 0	12	01	10	12

Given a k-germ α, let (α) represent the dihedral class $\delta(v) = \langle F(\alpha)\rangle$ with $v \in L_k/\pi$. Recall W_{01}^k is the 2-factor given by the union of the 1-factors of colors 0, 1 in M_k (namely those formed by lifting the edges $\alpha\alpha^0$, $\alpha\alpha^1$ of R_k in the notation below in this section, instead of those of colors k, $k-1$, as in [6]).

We present each $c \in V(R_k)$ via the pair $\delta(v) = \{v, \aleph_\pi(v)\} \in R_k$ ($v \in L_k/\pi$) of Section 8 and via the k-germ α for which $\delta(v) = \langle F(\alpha)\rangle$, and view R_k as the graph whose vertices are the k-germs α, with adjacency inherited from that of their δ-notation via F^{-1} (i.e. uncastling). So, $V(R_k)$ is presented as in the natural (k-germ) listing (see Section 2).

To start with, examples of such presentation are shown in Table IX for $k = 2$ and 3, where m, $\alpha = \alpha(m)$ and $F(\alpha)$ are shown in the first three columns, for $0 \le m < C_k$. The neighbors of $F(\alpha)$ are presented in the central columns of the table as $F^k(\alpha)$, $F^{k-1}(\alpha)$, …, $F^0(\alpha)$ respectively for the edge colors $k, k-1, \ldots, 0$, with notation given via the effect of function $\aleph$. The last columns yield the k-germs α^k, α^{k-1}, …, α^0 associated via F^{-1} respectively to the listed neighbors $F^k(\alpha)$, $F^{k-1}(\alpha)$, …, $F^0(\alpha)$ of $F(\alpha)$ in R_k.

TABLE X

m	α	α^4	α^3	α^2	α^1	α^0	m	α	α^4	α^3	α^2	α^1	α^0
–	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –
0	000	000	100	010	001	000	7	110	100	111	110	012	010
1	001	001	101	012	000	011	8	111	111	110	122	011	111
2	010	011	121	000	112	110	9	112	101	122	112	010	112
3	011	010	120	011	111	001	10	120	122	011	100	123	120
4	012	012	123	001	110	122	11	121	121	010	121	122	101
5	100	110	000	120	101	100	12	122	120	112	111	121	012
6	101	112	001	123	100	121	13	123	123	012	101	120	123
–	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –
		3**	***	3**	*2*	**1			3**	***	3**	*2*	**1

For $k = 4$ and 5, Tables X and XI have a similar respective natural enumeration adjacency disposition. We can generalize these tables directly to *Colored Adjacency Tables* denoted *CAT*(k), for $k > 1$. This way, Theorem 11.1(A) below is obtained as indicated in the aggregated last row upending Tables X and XI citing the only non-asterisk entry, for each of $i = k, k-2, \ldots, 0$, as a number $j = (k-1), \ldots, 1$ that leads to entry equality in both columns $\alpha = a_{k-1} \cdots a_j \cdots a_1$ and $\alpha^i = a^i_{k-1} \cdots a^i_j \cdots a^i_1$, that is $a_j = a^i_j$. Other important properties are contained in the remaining items of Theorem 11.1, including (B), that the columns α^0 in all CAT(k), ($k > 1$), yield an (infinte) integer sequence.

Theorem 11.1. *Let:* $k > 1$*,* $j(\alpha^k) = k-1$ *and* $j(\alpha^{i-1}) = i$*,* $(i = k-1, \dots, 1)$*. Then:* **(A)** *each column* α^{i-1} *in* $\mathrm{CAT}(k)$*, for* $i \in [k] \cup \{k+1\}$*, preserves the respective* $j(\alpha^{i-1})$*-th entry of* α*;* **(B)** *the columns* α^k *of all* $\mathrm{CAT}(k)$*'s for* $k > 1$ *coincide into an RGS sequence and thus into an integer sequence* $\mathcal{S}_0$*, the first* C_k *terms of which form an idempotent permutation for each* k*;* **(C)** *the integer sequence* $\mathcal{S}_1$ *given by concatenating the* m*-indexed intervals* $[0,2), [2,5), \dots, [C_{k-1}, C_k)$*, etc. in column* α^{k-1} *of the corresponding tables* $\mathrm{CAT}(2)$*,* $\mathrm{CAT}(3), \dots$*,* $\mathrm{CAT}(k)$*, etc. allows to encode all columns* α^{k-1}*'s;* **(D)** *for each* $k > 1$*, there is an idempotent permutation given in the* m*-indexed interval* $[0, C_k)$ *of the column* α^{k-1} *of* $\mathrm{CAT}(k)$*; such permutation equals the one given in the interval* $[0, C_k)$ *of the column* α^{k-2} *of* $\mathrm{CAT}(k+1)$*.*

Proof. (A) holds as a continuation of the observation made above with respect to the last aggregated row in Tables X and XI. Let α be a k-germ. Then α shares with α^k (e.g. the leftmost column α^i in Tables VIII to X, for $0 \le i \le k$) all the entries to the left of the leftmost entry 1, which yields (B). Note that if $k = 3$ then $m = 2, 3, 4$ yield for α^{k-1} the idempotent permutation $(2,0)(4,1)$, illustrating (C). (D) can be proved similarly. □

TABLE XI

m	α	α^5	α^4	α^3	α^2	α^1	α^0	m	α	α^5	α^4	α^3	α^2	α^1	α^0
0	0000	0000	1000	0100	0010	0001	0000	21	1110	1111	1100	1221	0110	1112	1110
1	0001	0001	1001	0101	0012	0000	0011	22	1111	1110	1111	1220	0122	1111	0111
2	0010	0011	1011	0121	0000	0112	0110	23	1112	1122	1101	1233	0112	1110	1222
3	0011	0010	1010	0120	0011	0111	0001	24	1120	1011	1222	1121	0100	1123	1120
4	0012	0012	1012	0123	0001	0110	0122	25	1121	1010	1221	1120	0121	1122	0101
5	0100	0110	1210	0000	1120	1101	1100	26	1122	1112	1220	1223	0111	1121	1122
6	0101	0112	1212	0001	1123	1100	1121	27	1123	1012	1233	1123	0101	1120	1223
7	0110	0100	1200	0111	1110	0012	0010	28	1200	1220	0110	1000	1230	1201	1200
8	0111	0111	1211	0110	1122	0011	1111	29	1201	1223	0112	1001	1234	1200	1231
9	0112	0101	1201	0122	1112	0010	0112	30	1210	1210	0100	1211	1220	1012	1011
10	0120	0122	1232	0011	1100	1223	1220	31	1211	1222	0111	1210	1233	1011	1221
11	0121	0121	1231	0010	1121	1222	1101	32	1212	1212	0101	1232	1223	1010	1212
12	0122	0120	1230	0112	1111	1221	0012	33	1220	1200	1122	1111	1210	0123	0120
13	0123	0123	1234	0012	1101	1220	1233	34	1221	1221	1121	1110	1232	0122	1211
14	1000	1100	0000	1200	1010	1001	1000	35	1222	1211	1120	1222	1222	0121	1112
15	1001	1101	0001	1201	1012	1000	1011	36	1223	1201	1223	1122	1212	0120	1123
16	1010	1121	0011	1231	1000	1212	1210	37	1230	1233	0122	1011	1200	1234	1230
17	1011	1120	0010	1230	1011	1211	1001	38	1231	1232	0121	1010	1231	1233	1201
18	1012	1123	0012	1234	1001	1210	1232	39	1232	1231	0120	1212	1221	1232	1012
19	1100	1000	1110	1100	0120	0101	0100	40	1233	1230	1123	1112	1211	1231	0123
20	1101	1001	1112	1101	0123	0100	0121	41	1234	1234	0123	1012	1201	1230	1234
		4***	****	4***	*3**	**2*	***1			4***	****	4***	*3**	**2*	***1

The sequences in Theorem 11.1 (B)-(C) start as follows, with intervals ended in ";":

$\{0\}\cup\mathbb{Z}^+=$	0,	1;	2,	3,	4;	5,	6,	7,	8,	9,	10,	11,	12,	13;	14	15,	16,...
$(B)=$	0,	1;	3,	2,	4;	7,	9,	5,	8,	6,	12,	11,	10,	13;	19,	20,	25,...
$(C)=$	1,	0;	0,	3,	1;	0,	1,	8,	7,	12,	3,	2,	9,	4;	0,	1,	3,...

Given a k-germ $\alpha = a_{k-1} \cdots a_1$, we want to express $\alpha^k, \alpha^{k-1}, \dots, \alpha^0$ as functions of α. Given a substring $\alpha' = a_{k-j} \cdots a_{k-i}$ of α $(0 < j \le i < k)$, let: **(a)** the *reverse string* off α' be $\psi(\alpha') = a_{k-i} \cdots a_{k-j}$; **(b)** the *ascent* of α' be **(i)** its maximal initial ascending substring, if $a_{k-j} = 0$, and **(ii)** its maximal initial non-descending substring with at most two equal nonzero terms, if $a_{k-j} > 0$. Then, the following remarks allow to express the k-germs $\alpha^p = \beta = b_{k-1} \cdots b_1$ via the colors $p = k, k-1, \dots, 0$, independently of F^{-1} and F.

Remark 11.1. Assume $p = k$. If $a_{k-1} = 1$, take $0|\alpha$ instead of $\alpha = a_{k-1} \cdots a_1$, with $k-1$ instead of k, removing afterwards from the resulting β the added leftmost 0. Now, let $\alpha_1 = a_{k-1} \cdots a_{k-i_1}$ be the ascent of α. Let $B_1 = i_1 - 1$, where $i_1 = ||\alpha_1||$ is the length of α_1. It can be seen that β has ascent $\beta_1 = b_{k-1} \cdots b_{k-i_1}$ with $\alpha_1 + \psi(\beta_1) = B_1 \cdots B_1$. If $\alpha \neq \alpha_1$, let α_2 be the ascent of $\alpha \setminus \alpha_1$. Then there is a $||\alpha_2||$-germ β_2 with $\alpha_2 + \psi(\beta_2) = B_2 \cdots B_2$ and $B_2 = ||\alpha_1|| + ||\alpha_2|| - 2$. Inductively when feasible for $j > 2$, let α_j be the ascent of $\alpha \setminus (\alpha_1|\alpha_2|\cdots|\alpha_{j-1})$. Then there is a $||\alpha_j||$-germ β_j with $\alpha_j + \psi(\beta_j) = B_j \cdots B_j$ and $B_j = ||\alpha_{j-1}|| + ||\alpha_j|| - 2$. This way, $\beta = \beta_1|\beta_2|\cdots|\beta_j|\cdots$.

Remark 11.2. Assume $k > p > 0$. By Theorem 11.1 (A), if $p < k-1$, then $b_{p+1} = a_{p+1}$; in this case, let $\alpha' = \alpha \setminus \{a_{k-1} \cdots a_q\}$ with $q = p+1$. If $p = k-1$, let $q = k$ and let $\alpha' = \alpha$. In both cases (either $p < k-1$ or $p = k-1$) let $\alpha'_1 = a_{q-1} \cdots a_{k-i_1}$ be the ascent of α'. It can be seen that $\beta' = \beta \setminus \{b_{k-1} \cdots b_q\}$ has ascent $\beta'_1 = b_{k-1} \cdots b_{k-i_1}$ where $\alpha'_1 + \psi(\beta'_1) = B'_1 \cdots B'_1$ with $B'_1 = i_1 + a_q$. If $\alpha' \neq \alpha'_1$ then let α'_2 be the ascent of $\alpha' \setminus \alpha'_1$. Then there is a $||\alpha'_2||$-germ β'_2 where $\alpha'_2 + \psi(\beta'_2) = B'_2 \cdots B'_2$ with $B'_2 = ||\alpha'_1|| + ||\alpha'_2|| - 2$. Inductively when feasible for $j > 2$, let α_j be the ascent of $\alpha' \setminus (\alpha'_1|\alpha'_2|\cdots|\alpha'_{j-1})$. Then there is a $||\alpha'_j||$-germ β'_j where $\alpha'_j + \psi(\beta'_j) = B'_j \cdots B'_j$ with $B'_j = ||\alpha'_{j-1}|| + ||\alpha'_j|| - 2$. This way, $\beta' = \beta'_1|\beta'_2|\cdots|\beta'_j|\cdots$.

We process the left-hand side from position q. If $p > 1$, we set $a_{a_q+2} \cdots a_q + \psi(b_{b_q+2} \cdots b_q)$ to equal a constant string $B \cdots B$, where $a_{a_q+2} \cdots a_q$ is an ascent and $a_{a_q+2} = b_{b_q+2}$. Expressing all those numbers a_i, b_i as a_i^0, b_i^0, respectively, in order to keep an inductive approach, let $a_q^1 = a_{a_q+2}$. While feasible, let $a_{q+1}^1 = a_{a_q+1}$, $a_{q+2}^1 = a_{a_q}$ and so on. In this case, let $b_q^1 = b_{b_q+2}$, $b_{q+1}^1 = b_{b_q+1}$, $b_{q+2}^1 = b_{b_q}$ and so on. Now, $a^1_{a^1_q+2} \cdots a^1_q + \psi(b^1_{b^1_q+2} \cdots b^1_q)$ equals a constant string, where $a^1_{a^1_q+2} \cdots a^1_q$ is an ascent and $a^1_{a^1_q+2} = b^1_{b^1_q+2}$. The continuation of this procedure produces a subsequent string a_q^2 and so on, until what remains to reach the leftmost entry of α is smaller than the needed space for the procedure itself to continue, in which case, a remaining initial ascent is shared by both α and β. This allows to form the left-hand side of $\alpha^p = \beta$ by concatenation.

Remark 11.3. Incidental problem: To find a Hamilton path in each R_k between 2-looped RGSs 0 and $12\ldots(k-1)$, which lifts to a Hamilton cycle in M_k/π. A lifting of such cycle to a Hamilton cycle in M_k would be D_{2n}-invariant under the action Υ of Theorem 6.1.

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