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Degree Associated Edge Reconstruction Number of Graphs with Regular Pruned Graph

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Abstract

An *ecard* of a graph G is a subgraph formed by deleting an edge. A *da-ecard* specifies the degree of the deleted edge along with the ecard. The *degree associated edge reconstruction number* of a graph $G,\ dern(G)$, is the minimum number of da-ecards that uniquely determines G. The *adversary degree associated edge reconstruction number* of a graph $G,\ adern(G)$, is the minimum number k such that every collection of k da-ecards of G uniquely determines G. The maximal subgraph without end vertices of a graph G which is not a tree is the *pruned graph* of G. It is shown that $dern$ of complete multipartite graphs and some connected graphs with regular pruned graph is 1 or 2. We also determine $dern$ and $adern$ of corona product of standard graphs.

Keywords: edge reconstruction number, degree associated edge reconstruction number, vertex-transitive graph
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1. Introduction

All graphs considered in this paper are finite, simple and undirected. Graph theoretic terms which are not defined here are used in sense of [1]. For brevity, a vertex of degree m is called an *m-vertex*; a 1-vertex is called an *end vertex*. The maximal subgraph without end vertices of a graph G which is not a tree is the *pruned graph* of G and is denoted by $\widetilde{G}$. If v is a vertex on $\widetilde{G}$,

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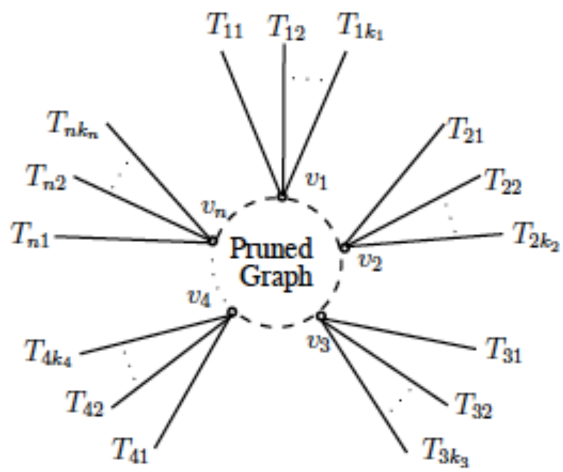

Figure 1. Graph with end vertices

then the maximal subtree T of G such that $V(T) \cap V(\widetilde{G}) = \{v\}$ is called the *rooted tree* at v; the vertex v is called the *root vertex*. A maximal subtree T' of the rooted tree T at v such that v is not a cut vertex of T' is called a *limb* of T. In Figure 1, each T_{ij} is a tree called limb and $\cup_{j=1}^{k_i} T_{ij}$ is called a rooted tree at v_i for $i = 1, 2, \ldots .n$

The graph obtained from $P_{l+1} \cup K_{1,r}$ by identifying an end vertex of P_{l+1} (a path on $l+1$ vertices) and the r-vertex of $K_{1,r}$ is called a *broom graph* and is denoted by $B_{l,r}$; the identified vertex is called the *special vertex* of $B_{l,r}$. The *corona* $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

A *vertex deleted subgraph* or a *card* $G-v$ of a graph G is the unlabeled graph obtained from G by deleting a vertex v and all edges incident with v. The ordered pair $(d(v), G-v)$ is called a *degree associated card* or *dacard* of the graph G, where $d(v)$ is the degree of v in G. The *deck* (*dadeck*) of a graph G is the collection of all its cards (dacards). Following the formulation in [3], a graph G is *reconstructible* if it can be uniquely determined from its dadeck. For a reconstructible graph G, Harary and Plantholt [3] defined the *reconstruction number*, $rn(G)$, to be the minimum number of cards which can only belong to its deck and not in the deck of any other graph $H,\ H \ncong G$, thus uniquely identifying G. An *s-blocking set* of G is a family $\mathscr{F}$ of graphs not isomorphic to G such that every collection of s cards of G will appear in the deck of some graph of $\mathscr{F}$ and every graph in $\mathscr{F}$ will have s cards in common with G. For a reconstructible graph G, Myrvold [7] studied, the *adversary reconstruction number*, which is the minimum number k such that every collection of k cards of G is not contained in the deck of any other graph $H,\ H \cong G$. A graph non-isomorphic to G but having s cards in common with G is called an *s-adversary-blocking graph* of G. For a reconstructible graph G from its dadeck, Ramachandran [8] defined that the *degree associated reconstruction number,* $drn(G)$, is the minimum number of dacards that uniquely determines G. The *adversary degree associated reconstruction number of a graph* $G,\ adrn(G)$, is the minimum number k such that every collection of k dacards of G uniquely determines G. The degree of an edge e, denoted by $d(e)$, is the number of edges adjacent to e. The edge reconstruction number,

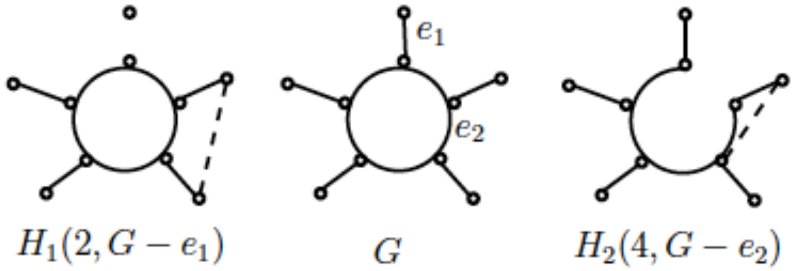

Figure 2. The graph G and its 1-blocking set.

degree associated edge reconstruction number and adversary degree associated edge reconstruction number of a graph are defined analogously with edge deletions instead of vertex deletions. An *extension* of a da-ecard $(d(e), G - e)$ of G is a graph obtained from the da-ecard by adding a new edge which joins two nonadjacent vertices whose degree sum is $d(e)$ and it is denoted by $H(d(e), G - e)$ (or simply by H).

Example. The graph G shown in Figure 2 has $dern$ 2 and $adern$ 3.
For, the 1-blocking set of G is $\{H_1(2, G - e_1), H_2(4, G - e_2)\}$ and the 2-adversary blocking graph is $H_2(4, G - e_2)$.

In [6], it is proved that for a regular graph G, $dern(G) = adern(G) = 1$ and that $dern(K_{m,n})$ is $1,\ 2$ or 3 for $1 \leq m \leq n$. In this paper, we show that $dern$ is 1 or 2 for complete multipartite graphs and some connected graphs with regular pruned graph. Finally, we determine $dern$ and $adern$ for $K_n \circ K_m,\ C_n \circ C_m,\ P_n \circ K_m,\ K_n \circ C_m$ and $K_n \circ P_m$.

2. Complete Multipartite Graph

For a graph G, to prove $dern(G) = k,\ (adern(G) = k)$, we proceed as follows.

(i) First find the da-edeck of G.
(ii) Determine next all possible extensions of every da-ecard of G.
(iii) Finally, show that every extension (other than G) of at least one da-ecard (every da-ecard) has at most $k - 1$ da-ecards in common with those of G, and that at least one extension has precisely $k - 1$ da-ecards in common with those of G.

Recently, it is proved [6] that $dern(K_{m,n})$ is $1,\ 2$ or 3 for $1 \leq m \leq n$. We next determine $dern$ of complete multipartite graph with at least three partite sets. Meijie Ma et al. [4] have also determined $dern$ for complete multipartite graphs.

Theorem 2.1. *Let G be a complete multipartite graph with at least three partite sets. Then, $dern(G) = 1$ or 2.*

Proof. Let G have n-partite sets such that at least two of them have different sizes (as otherwise G is regular and so $dern(G) = 1$). Let the partite sets of G be $A_1,\ A_2,\ \ldots,\ A_n$ such that $|A_1| \geq |A_2| \geq \ldots \geq |A_n|$; let the degree of each vertex in A_i be d_i for $i = 1, \ldots, n$. Then $d_1 \leq d_2 \leq \ldots \leq d_n$. For $i \neq j$, the set of all edges of G joining a vertex of A_i and a vertex of

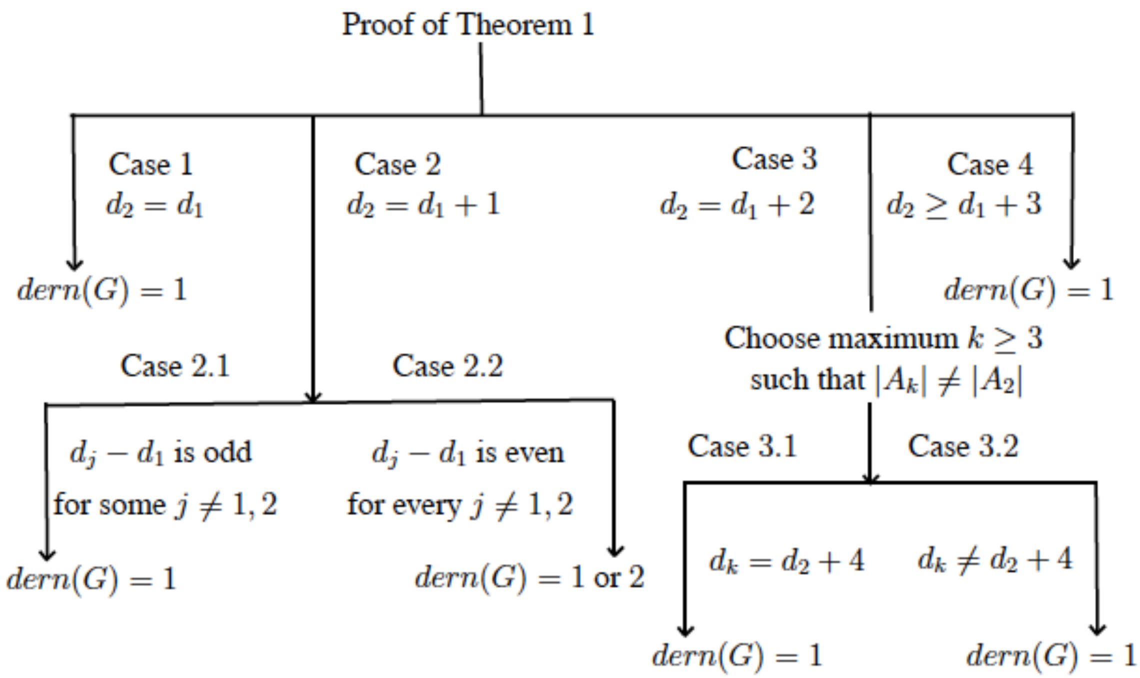

Figure 3. All cases in the proof.

A_j is denoted by $E(A_i, A_j)$. We give a chart (in Figure 3) summarizing the outcome proof. Now consider the following cases.

Case 1: $d_2 = d_1$. In the da-ecard $(2d_1-2, G-e)$ where $e \in E(A_1, A_2)$, exactly two nonadjacent vertices have degree sum equal to $2d_1 - 2$. Hence G can be obtained uniquely from $(2d_1-2, G-e)$ by joining the two $(d_1 - 1)$-vertices. Therefore $dern(G) = 1$.

Case 2: $d_2 = d_1 + 1$. We shall assume that $|A_1| > 2$ (otherwise $|A_1| = 2$ and every extension of the da-ecard $(d_1 + d_2 - 2, G - e)$ is clearly isomorphic to G and hence $dern(G) = 1$).

When $|A_2| = |A_j|$ for $3 \leq j \leq n$, in the da-ecard $(d_1 + d_2 - 2, G - e_1)$ where $e \in E(A_1, A_2)$, if we join the $(d_1 - 1)$-vertex to a $(d_2 - 1)$-vertex, then either $H(d_1 + d_2 - 2, G - e_1)$ is isomorphic to G or it has exactly one da-ecard, namely $(d_1 + d_2 - 2, G - e_1)$, in common with G. The set $\{H_1(d_1 + d_2 - 2, G - e_1), H_2(d_2 + d_j - 2, G - e_2)\}$ is a 1-blocking set of G, where $H_1(d_1 + d_2 - 2, G - e_1)$ and $H_2(d_2 + d_j - 2, G - e_2)$, are obtained, respectively, by joining the $(d_1 - 1)$-vertex to a $(d_2 - 1)$-vertex and two d_1-vertices in the same partite set (Figure 4; the dashed line in them denotes all possible adjacencies between the partite sets). Now assume that $|A_j| \neq |A_2|$ for at least one j, $3 \leq j \leq n$.

Case 2.1: $d_j - d_1$ is odd for at least one j $(3 \leq j \leq n)$. Among these $A_j's$, choose a maximum partite set, say A_k. In the da-ecard $(d_1 + d_k - 2, G - e)$, where $e \in E(A_1, A_k)$, exactly two nonadjacent vertices have degree sum equal to $d_1 + d_k - 2$ and hence $dern(G) = 1$.

Case 2.2: $d_j - d_1$ is even for every j $(3 \leq j \leq n)$. If $d_j - d_1 = 2$ for every j, then consider the da-ecard $(d_1 + d_2 - 2, G - e_1)$ where $e \in E(A_1, A_2)$. In it, if we join the unique $(d_1 - 1)$-vertex to the $(d_2 - 1)$-vertex, then $H(d_1 + d_2 - 2, G - e_1)$ is isomorphic to G or has exactly one da-ecard,

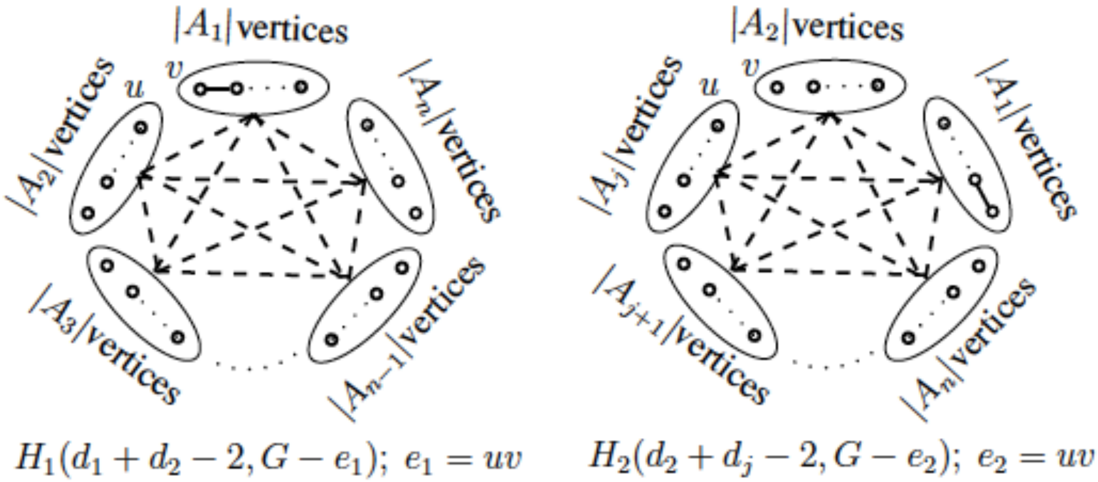

Figure 4. A 1-blocking set of G.

namely $(d_1 + d_2 - 2, G - e_1)$, in common with G. Now the set $\{H_1(2d_1 - 1, G - e_1),\ H_2(2d_1 + 1, G - e_2),\ H_3(2d_1 + 2, G - e_3), H_4\}$ forms a 1-blocking set of G, where $H_1,\ H_2,\ H_3$ are obtained, respectively, by joining the unique $(d_1 - 1)$-vertex to a d_1-vertex, the unique d_1-vertex to a $(d_1 + 1)$-vertex, two $(d_1 + 1)$-vertices (if G has at least two partite sets with $|A_k|$ vertices) in the same partite set of respective da-ecards and H_4 obtained from a da-ecard with associated edge degree $2d_1$ by joining two d_1-vertices in the same partite set. Therefore $dern(G) = 2$.

Now assume that $d_j - d_1 > 2$ for at least one j. Among these A_j's, choose a maximum partite set,say A_k. Then consider the da-ecard $(d_2 + d_k - 2, G - e)$ (obtained by removing an edge $e \in E(A_2, A_k)$). The da-ecard $(d_2 + d_k - 2, G - e)$ has exactly two nonadjacent vertices of degree sum equal to $d_2 + d_k - 2$ and hence $dern(G) = 1$.

Case 3: $d_2 = d_1 + 2$. When $|A_j| = |A_2|$ for $3 \leq j \leq n$, in the da-ecard $(2d_2 - 2 = 2d_1 + 2, G - e)$ where $e \in E(A_2, A_j)$, exactly two nonadjacent vertices have degree sum equal to $2d_1 + 2$ and hence $dern(G) = 1$.

Now assume that $|A_j| \neq |A_2|$ for at least one $j,\ 3 \leq j \leq n$. Choose a maximum partite set, say, A_k $(3 \leq k \leq n)$ such that $|A_k| \neq |A_2|$.

Case 3.1: $d_k = d_2 + 4 = d_1 + 6$. In the da-ecard $(2d_2 + 2, G - e)$ where $e \in E(A_2, A_k)$, there are exactly two nonadjacent vertices whose degree sum equal to $2d_2 + 2$ and hence $dern(G) = 1$.

Case 3.2: $d_k \neq d_2 + 4 = d_1 + 6$. In this case, we use the da-ecard $(d_1 + d_k - 2, G - e)$ and so the proof is just similar to Case 2.1.

Case 4: $d_2 \geq d_1 + 3$. In the da-ecard $(d_1 + d_2 - 2, G - e)$ where $e \in E(A_1, A_2)$, there are exactly two nonadjacent vertices whose degree sum is $d_1 + d_2 - 2$ and hence $dern(G) = 1$, which completes the proof of Theorem 1. □

3. Graphs with Regular Pruned Graph

The maximal subgraph without end vertices of a graph G which is not a tree is the *pruned graph* of G and is denoted by $\widetilde{G}$. In this section, we prove that $dern$ of some connected graphs G

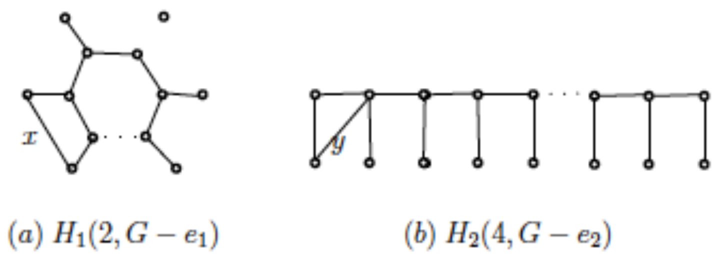

Figure 5. A 1-blocking set of G.

with regular $\widetilde{G}$ is 1 or 2. As a prelude, we determine $dern$ for the following classes of connected graph.

Theorem 3.1. *Let G be a connected graph such that $\widetilde{G}$ is regular. If every rooted tree has exactly $m\ (>0)$ limbs, all are K_2, then*

$$dern(G) = \begin{cases} 2 & \text{if } \widetilde{G} \text{ is a cycle and } m = 1 \\ 1 & \text{otherwise} \end{cases}$$

Proof. Let d be the degree of each root vertex. Then, each da-ecard of G is $(d-1, G-e_1)$ or $(2d-2, G-e_2)$. Now consider the following cases.

Case 1: $m = 1$ and $\widetilde{G}$ is a cycle.
Upper bound: We use the two da-ecards $(2, G-e_1)$ and $(4, G-e_2)$. The da-ecard $(4, G-e_2)$ forces every extension of it to be connected. Hence, in $(2, G-e_1)$, the unique isolated vertex must be joined with the unique 2-vertex and so G is the only extension and so $dern(G) \leq 2$.
Lower bound: The set $\{H_1(2, G-e_1), H_2(4, G-e_2)\}$ forms a 1-blocking set of G, where H_1 is obtained by joining the two 1-vertices and H_2 by joining a 3-vertex to a 1-vertex such that their common neighbour is a 2-vertex (Figures 5(a) and 5(b)). Hence $dern(G) \geq 2$.

Case 2: $m > 1$ or for $m = 1$, $\widetilde{G}$ is not a cycle. In $(d-1, G-e)$, if we join the unique isolated vertex to the unique $(d-1)$-vertex, then $H(d-1, G-e)$ is isomorphic to G. To get an extension non-isomorphic to G, at least one of the two vertices to be joined must be different from these two vertices. But then the degree sum of the two vertices to be joined will be $1,\ 2,\ d,\ d+1,\ 2d-1$ (if $\widetilde{G}$ is not complete and $|\widetilde{G}| > 3$) or $2d$ (if $\widetilde{G}$ is not complete and $|\widetilde{G}| > 3$). Hence $H(d-1, G-e)$ is the unique extension and is isomorphic to G. Therefore $dern(G) = 1$. $\square$

Theorem 3.2. *Let G be a connected graph such that $\widetilde{G}$ is regular. If every rooted tree has exactly $m\ (>0)$ limbs, all are P_k, where k is a constant greater than one, then $dern(G) = 1$ or 2.*

Proof. In view of Theorem 3.1, we can assume that $k > 2$. Let d be the degree of each root vertex (Figure 6). We shall first consider the case $d < 5$.

Upper bound: Consider the two da-ecards $(1, G-e_1)$ and $(d, G-e_2)$. The da-ecard $(1, G-e_1)$ forces every extension H to be connected with $\widetilde{H} \cong \widetilde{G}$ and that the degree of each root vertex to be d. The da-ecard $(d, G-e_2)$ has a root vertex of degree $d-1$ and hence $H(d, G-e_2)$ must be

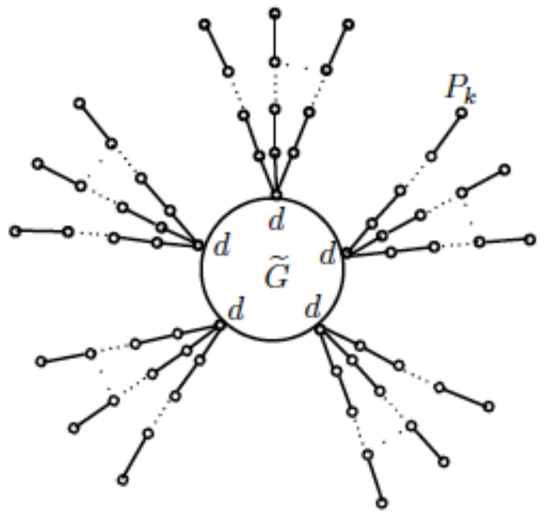

Figure 6. The graph G.

obtained from $(d, G - e_2)$ by adding an edge whose ends are respectively a $(d-1)$-vertex and a 1-vertex of the component P_{k-1}. Hence $dern(G) \leq 2$.

Lower bound: Consider the extension $H_1(1, G - e_1)$ (obtained by joining the end vertex of a limb P_k to the unique isolated vertex) and the extension of a da-ecard with associated edge degree 2 (obtained by adding an edge joining two 1-vertices from two distinct limbs P_k). For $d = 4$, consider the extensions $H_2(4, G - e_2)$ (obtained by adding an edge joining two 2-vertices) and $H_3(6, G - e_3)$ (obtained by adding an edge joining a 4-vertex to a 2-vertex). For $d = 3$, consider the extensions $H_2(3, G - e_2)$ (obtained by adding an edge joining a 1-vertex of a limb P_k to a 2-vertex) and $H_3(4, G - e_3)$ (obtained by adding an edge joining two 2-vertices each adjacent to a 1-vertex). The set of above extensions forms a 1-blocking set of G and hence $dern(G) \geq 2$.

We shall now determine $dern(G)$ for $d \geq 5$. In $(2d-2, G-e)$, if we join two $(d-1)$-vertices, then $H(2d-2, G-e)$ is isomorphic to G; for all other possibilities, the degree sum of the two vertices to be joined will be $2,\ 3,\ 4,\ d,\ d+1,\ d+2,\ 2d-1$ (if $\widetilde{G}$ is not complete and $|\widetilde{G}| > 3$) or $2d$ (if $\widetilde{G}$ is not complete and $|\widetilde{G}| > 4$). Hence $H(2d-2, G-e)$ is the unique extension and is isomorphic to G. Therefore $dern(G) = 1$. □

Theorem 3.3. *Let G be a connected graph such that $\widetilde{G}$ is regular. If every limb is K_2 and every root vertex has degree at least 4, then $dern(G) = 1$.*

Proof. Let d be the minimum degree of the root vertices. In the da-ecard $(d-1, G-e)$, if we join the unique $(d-1)$-vertex to the unique isolated vertex, then $H(d-1, G-e)$ is isomorphic to G; for all other possibilities the degree sum of the two vertices to be joined will be 1, 2 or strictly greater than $d-1$. Therefore $dern(G) = 1$. □

Theorem 3.4. *Let G be a connected graph such that $\widetilde{G}$ is vertex-transitive. If exactly one rooted tree has $m+1$ limbs and the remaining rooted trees have exactly $m\ (>0)$ limbs, all are K_2, then $dern(G) = 1$.*

Proof. Let d be the minimum degree of the root vertices. Consider the da-ecard $(d, G-e)$ obtained by removing an edge e joining a $(d+1)$-vertex and a 1-vertex. In $(d, G-e)$, if we join a d-vertex to the unique isolated vertex, then $H(d, G-e)$ is isomorphic G. If we join at least one vertex different from these two vertices, then the degree sum of the two vertices to be joined will be $1,\ 2,\ d+1$ or $2d$ (if $\widetilde{G}$ is a cycle and $|\widetilde{G}| > 3$). Hence $H(d, G-e)$ is the unique extension and is isomorphic to G. Therefore $dern(G) = 1$. □

When $r = 1$, the graph $B_{l,r}$ is nothing but P_{l+2} and so we assume that r is greater than one in the following theorem.

Theorem 3.5. *Let G be a unicyclic graph in which every rooted tree is $B_{l,r},\ r > 1$ and the special vertex is the root vertex. Then*

$$dern(G) = \begin{cases} 2 & \text{if } l = 1 \text{ and } r \leq 4, \text{ or } l \geq 2 \text{ and } r \leq 5, \text{ or } l \geq 2,\ r = 6 \text{ and } |\widetilde{G}| > 3 \\ 1 & \text{otherwise} \end{cases}$$

Proof. Consider the following cases.

Case 1: $l = 1$. The da-ecards of G are $(r, G - e_1)$ (obtained by removing an edge e_1 whose ends are respectively an $(r + 1)$-vertex and a 1-vertex), $(r + 2, G - e_2)$ (obtained by removing an edge e_2 whose ends are respectively an $(r + 1)$-vertex and a 3-vertex) and $(4, G - e_3)$ (obtained by removing an edge e_3 whose ends are respectively a 3-vertex and a 2-vertex).

Case 1.1: $r \leq 4$. In $(r, G - e_1)$, if we join the unique isolated vertex to an r-vertex or join a 1-vertex to an $(r - 1)$-vertex (when $r = 2, 4$), then either H is isomorphic to G or H has exactly one da-ecard, namely $(r, G - e_1)$, in common with G. Consider the extensions $H_1(r, G - e_1)$ (obtained by joining the unique isolated vertex to a root vertex when $r = 3$ and joining a 1-vertex to an $(r - 1)$-vertex when $r = 2, 4$), $H_2(r + 2, G - e_2)$ (obtained by joining an $(r + 1)$-vertex to a 1-vertex) and $H_3(4, G - e_3)$ (obtained by joining a 3-vertex to a 1-vertex). The set $\{H_1(r, G-e_1), H_2(r+2, G-e_2), H_3(4, G-e_3)\}$ is a 1-blocking set of G and hence $dern(G) = 2$.

Case 1.2: $r \geq 5$. The only possibility to obtain an extension $H(r, G - e_1)$ is to join the unique isolated vertex to the unique r-vertex, since otherwise the degree sum of the two vertices to be joined will be $1,\ 2,\ 3,\ 4,\ 6,\ r+1,\ r+2,\ r+3,\ r+4,\ 2r+1$ or $2r+2$. Hence $H(r, G - e_1)$ is isomorphic to G and $dern(G) = 1$.

Case 2: $l \geq 2$. The da-ecards of G are $(r, G - e_1)$ (obtained by removing an edge e_1 whose ends are respectively an $(r + 1)$-vertex and a 1-vertex), $(r + 1, G - e_2)$ (obtained by removing an edge e_2 whose ends are respectively an $(r + 1)$-vertex and a 2-vertex), $(3, G - e_3)$ (obtained by removing an edge e_3 whose ends are respectively a 3-vertex and a 2-vertex) and $(4, G - e_4)$ (obtained by removing an edge e_4 whose ends are respectively two 3-vertices) and the da-ecards with associated edge degree 2, where the ends of the deleted edge are two 2-vertices.

Case 2.1: $r \leq 5$ or for $r = 6,\ |\widetilde{G}| > 3$. In $(r, G - e_1)$, if we join the unique isolated vertex to an r-vertex or a 1-vertex to an $(r-1)$-vertex (when $r \leq 4$) or a 2-vertex to an $(r-2)$-vertex when ($r = 4, 5$) or two 3-vertices (when $r = 6$), then either $H(r, G - e_1)$ is isomorphic to G or H has exactly one da-ecard, namely $(r, G-e_1)$, in common with G. Consider the extensions $H_1(r, G-e_1)$ (obtained by joining a 1-vertex to an $(r-1)$-vertex (when $r = 2, 3$), a 2-vertex to an $(r-2)$-vertex (when $r = 4, 5$) and two 3-vertices (when $r = 6$ and $|\widetilde{G}| > 3$)) and $H_2(r+1, G-e_2)$ (obtained by joining an r-vertex to 1-vertex of a rooted tree $B_{l,r}$), $H_3(3, G-e_3)$ (obtained by joining a 2-vertex to a 1-vertex of a rooted tree $B_{l,r}$), $H_4(4, G-e_4)$ (obtained by joining a 3-vertex to a 1-vertex) and the extension of a da-card with associated edge degree 2 (obtained by joining two 1-vertices each adjacent to an $(r+1)$-vertex). The set of above extensions forms a 1-blocking set of G and hence $dern(G) = 2$.

Case 2.2: $r = 6,\ |\widetilde{G}| = 3$ or for $r > 6$. The only possibility to obtain an extension H of $(r, G-e_1)$ is to join the unique isolated vertex to the unique r-vertex, since otherwise the degree sum of the two vertices to be joined will be $1,\ 2,\ 3,\ 4,\ 5,\ 6$(if $|\widetilde{G}| > 3$ and $r > 6$), $r+1,\ r+2,\ r+3,\ r+4,\ 2r+1$ or $2r+2$. Therefore $H(r, G-e_1)$ is isomorphic to G and $dern(G) = 1$. □

4. Corona of Graphs

The *corona* $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. In this section, we shall determine $dern$ and $adern$ of corona product of standard graphs.

Theorem 4.1. *Let $G = K_n \circ K_m$. Then*

$$dern(G) = \begin{cases} 2 & \textit{if } n = 3 \textit{ and } m = 1 \\ 1 & \textit{otherwise} \end{cases} \quad and \quad adern(G) = \begin{cases} 3 & \textit{if } n = 3, \textit{ or } n = 2 \textit{ and } m > 1 \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Each da-ecard of G is either $(n+2m-3, G-e_1),\ (2n+2m-4, G-e_2)$ or $(2m-2, G-e_3)$ (when $m > 1$.)
For $n = 1$,$K_n \circ K_m$ is K_{m+1}, which is regular and so $dern(G) = adern(G) = 1$.

We shall determine $dern(G)$ for $n > 1$. For $n = 3$ and $m = 1,\ dern(G) = 2$ by Theorem 3.1. For $n \neq 3$ and $m = 1$, consider the da-ecard $(n-1, G-e_1)$. In $(n-1, G-e_1)$, if we join the unique $(n-1)$-vertex to the unique isolated vertex, then $H(n-1, G-e_1) \cong G$. For all other possibilities, the degree sum of the two vertices to be joined will be 1, 2, n or $n+1$ (when $n > 3$) and 2 (when $n = 2$). Hence $H(n-1, G-e_1)$ is the unique extension and is isomorphic to G. For $m > 1$, consider the da-ecard $(2m-2, G-e_3)$. For $(2m-2, G-e_3)$, the only possibility to obtain an extension H is to join two $(m-1)$-vertices, since otherwise the degree sum of the two vertices to be joined will be strictly greater than $2m-2$. Hence $H(2m-2, G-e_3)$ is the unique extension and is isomorphic to G.

We shall now determine $adern(G)$ for $n > 1$.

Case 1: $n = 2$. For $m = 1$, the da-ecards of G are of the form $(1, G-e_1)$ and $(2, G-e_2)$. Clearly $H(1, G-e_1) \cong G$ and $H(2, G-e_2) \cong G$ and hence $adern(G) = 1$ for $m = 1$. For $m > 1$, the da-ecards of G are of the form $(2m-1, G-e_1)$,$(2m, G-e_2)$ and $(2m-2, G-e_3)$. Clearly $H(2m-2, G-e_3) \cong G$. In $(2m-1, G-e_1)$, if we join the unique $(m-1)$-vertex to an m-vertex, then either $H(2m-1, G-e_1)$ is isomorphic to G or H has two same da-ecards $(2m-1, G-e_1)$ in common with G. The only possibility to obtain an extension H of $(2m, G-e_2)$ is to join the two m-vertices, one from each component, since both components are complete. Clearly $H(2m-1, G-e_1)$ (obtained by joining the unique $(m-1)$-vertex to an m-vertex at a distance 4) is a 2-adversary-blocking graph (Figure 7) and hence $adern(G) = 3$.

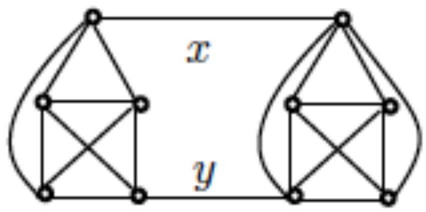

Figure 7. $H(2m-1, G-e_2)$ for $m=4$.

Figure 8. $H(8, G-e_3)$.

Case 2: $n=3$. By Theorem $4,5$ of [5], $adern(G)=3$.

Case 3: $n>3$. For $m=1$, clearly $H(n-1, G-e_1) \cong G$. The only possibility to obtain an extension H of $(2n-2, G-e_2)$ is to join the two $(n-1)$-vertices, since otherwise the degree sum of the two vertices to be joined will be $2,\ n$ or $n+1$. Therefore $adern(G)=1$. For $m>1$, clearly $H(2m-2, G-e_3) \cong G$. In $(n+2m-3, G-e_1)$, if we join the unique $(m-1)$-vertex to the unique $(n+m-2)$-vertex, then $H(n+2m-3, G-e_1) \cong G$. To get an extension non-isomorphic to G at least one of the two vertices to be joined must be different from these two vertices. But then the degree sum of the two vertices to be joined will be $2m-1,\ 2m,\ n+2m-2$ or $n+2m-1$. Hence $H(n+2m-3, G-e_1)$ is the unique extension and is isomorphic to G. For $(2n+2m-4, G-e_2)$, the only possibility to obtain an extension $H(2n-2m-4, G-e_2)$ is to join the two $(n+m-2)$-vertices, since otherwise the degree sum of the two vertices to be joined will be $2m,\ n+2m-2$ or $n+2m-1$. Therefore $adern(G)=1$. $\square$

Theorem 4.2. *Let $G = C_n \circ C_m$. Then, $dern(G)=1$ and*

$$adern(G) = \begin{cases} 3 & \textit{if } m=3 \\ 1 & \textit{otherwise} \end{cases}$$

Proof. The da-ecards of G are of the form $(4, G-e_1), (m+3, G-e_2)$ and $(2m+2, G-e_3)$.

We shall first determine $dern(G)$. In $(4, G-e_1)$, if we join the two 2-vertices, then $H(4, G-e_1) \cong G$. To get an extension non-isomorphic to G, at least one of the two vertices to be joined must be different from these two vertices. But then the degree sum of the two vertices to be joined will be 5, 6, $m+4,\ m+5$ or $2m+4$ (if $n>3$). Therefore $dern(G)=1$.

We shall now determine $adern(G)$. Clearly $H(4, G-e_1) \cong G$. For $m=3$, in $(6, G-e_2)$, if we join the unique 2-vertex to the unique 4-vertex or two 3-vertices, then either $H(6, G-e_2)$ is isomorphic to G or H has exactly one da-ecard, namely $(6, G-e_2)$, in common with G. In

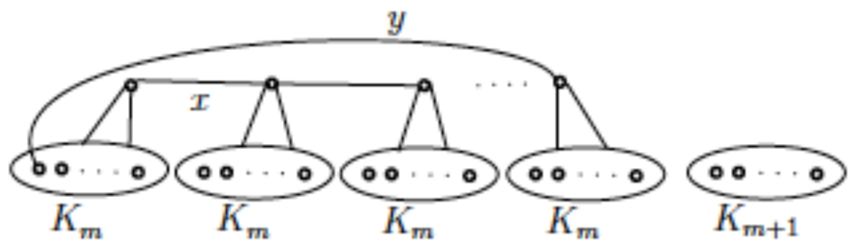

Figure 9. $H(2m+1, G-e_2)$.

$(8, G-e_3)$, if we join the two 4-vertices or a 5-vertex to a 3-vertex, then $H(8, G-e_3)$ is isomorphic to G or has at most two same da-ecards $(8, G-e_3)$ in common with G. The graph $H(8, G-e_3)$ (Figure 8) (obtained by joining a 5-vertex to a 3-vertex whose common neighbour is a 4-vertex) is a 2-adversary-blocking graph. Hence $adern(G) = 3$.

For $m > 3$, the only possibility to obtain an extension $H(m+3, G-e_2)$ is to join the unique $(m+1)$-vertex to the unique 2-vertex, since otherwise the degree sum of the two vertices to be joined will be $5,\ 6,\ m+4,\ m+5,\ 2m+3$ (if $n > 3$) or $2m+4$ (if $n > 3$). In $(2m+2, G-e_3)$, if we join the two $(m+1)$-vertices, then $H(2m+2, G-e_3) \cong G$. For all other possibilities, the degree sum of the two vertices to be joined is $6,\ m+4,\ m+5,\ 2m+3$ (if $n > 3$) or $2m+4$ (if $n > 4$). Therefore $adern(G) = 1$. $\square$

Theorem 4.3. *Let $G = P_n \circ K_m$. Then*

$$dern(G) = \begin{cases} 2 & \textit{if } n \neq 1, 2, 4 \textit{ and } m = 1 \\ 1 & \textit{otherwise} \end{cases} \quad and\ adern(G) = \begin{cases} 1 & \textit{if } n = 1 \textit{ or } n = m+1 = 2 \\ 4 & \textit{if } n > 3 \textit{ and } m = 1 \\ 3 & \textit{otherwise} \end{cases}$$

Proof. If $n = 1$, then G is complete and so $dern(G) = adern(G) = 1$. If $n = 2$, then $dern(G) = 1$ and $adern(G)$ is 3 or 1 according as $m > 1$ or $m = 1$, by Theorem 4.1. If $n > 2$ and $m = 1$, then $dern(G)$ is 1 (if $n = 4$) and 2 (if $n \neq 4$) and $adern(G)$ is 3 (if $n = 3$) and 4 (if $n > 3$), by Theorem 6 of [5]. So it remains to consider the case when $n > 2$ and $m > 1$.

We first determine $dern(G)$. Consider the da-ecard $(2m-2, G-e)$ obtained by removing an edge e joining two m-vertices having an $(m+1)$-vertex as a common neighbour. In $(2m-2, G-e)$, the only possibility to obtain an extension H is to join the two $(m-1)$-vertices, since otherwise the degree sum of the two vertices to be joined will be strictly greater than $2m-2$. Hence $dern(G) = 1$.

We shall now determine $adern(G)$. Consider a da-ecard with associated edge degree $2m-2$. Clearly the resulting extension is unique and is isomorphic to G. In $(2m-1, G-e_1)$, if we join the unique $(m-1)$-vertex to an m-vertex, then either $H(2m-1, G-e_1)$ is isomorphic to G or H has exactly one da-ecard, namely $(2m-1, G-e_1)$, in common with G. Consider a da-ecard with associated edge degree $2m$. If we join the unique $(m-1)$-vertex to an $(m+1)$-vertex or join two m-vetices, then the resulting extension is isomorphic to G or has exactly one da-ecard, with associated edge degree $2m$, in common with G. In $(2m+1, G-e_2)$, if we join an m-vertex to an $(m+1)$-vertex, then $H(2m+1, G-e_2)$ is isomorphic to G or has at most two da-ecards $(2m+1, G-e_2)$ in common with G. Now consider a da-ecard with associated edge degree $2m+2$.

If we join two $(m+1)$-vertices or join an m-vertex to an $(m+2)$-vertex, then either the extension is isomorphic to G or it has at most two da-ecards, with associated edge degree $2m+2$, in common with G. The graph $H(2m+1, G-e_2)$ (obtained by joining an $(m+1)$-vertex to an m-vertex adjacent to an $(m+1)$-vertex at distance $n-1$) (Figure 9) is a 2-adversary-blocking graph. Hence $adern(G) = 3$. □

Theorem 4.4. *Let $G = K_n \circ C_m$. Then $dern(G) = 1$ and*

$$adern(G) = \begin{cases} 3 & \textit{if } n = m-4 = 1 \textit{ or } n = m-1 = 2 \textit{ or } n = m = 3 \\ 2 & \textit{if } n = m-2 = 2 \\ 1 & \textit{otherwise} \end{cases}$$

Proof. It is proved in Theorem 4 of [6] that $dern(G) = 1$ and $adern(G)$ is 3 (if $n = 1$ and $m = 5$) or 1 ($n = 1$ and $m \neq 5$). In view of Theorem 4 of [6] and Theorem 7, we assume that $n > 1$ and $m > 3$. The da-ecards of G are of the form $(4, G-e_1)$ (obtained by removing an edge e_1 whose ends are two 3-vertices), $(n+m, G-e_2)$ (obtained by removing an edge e_2 whose ends are respectively an $(n+m-1)$-vertex and a 3-vertex) and $(2n+2m-4, G-e_3)$ (for $n > 1$ and it is obtained by removing an edge e_3 whose ends are two $(n+m-2)$-vertices).

We shall first determine $dern(G)$. The only possibility to obtain $H(4, G-e_1)$, is to join the two 2-vertices, since otherwise the degree sum of the two vertices to be joined will be $5,\ 6,\ n+m+1$ or $n+m+2$. Hence $dern(G) = 1$.

We shall now determine $adern(G)$. Clearly $H(4, G-e_1) \cong G$. For $n \neq 2$ and $m \neq 4$, $H(n+m, G-e_2) \cong G$. For $n = 2,\ m = 4$, in $(n+m, G-e_2)$, if we join the unique 4-vertex to the unique 2-vertex or two 3-vertices, then either H is isomorphic to G or it has exactly one da-ecard, namely $(n+m, G-e_2)$, in common with G. For $(2n+2m-4, G-e_3)$, the only possibility to obtain an extension is to join the two $(n+m-2)$-vertices, since otherwise the degree sum of the two vetices to be joined will be less than $2n+2m-2$. A 2-adversary-blocking graph is $H(n+m, G-e_2)$, obtained by joining two 3-vertices (when $n = 2,\ m = 4$), which completes the proof. □

Theorem 4.5. *Let $G = K_n \circ P_m$. Then*

$$dern(G) = \begin{cases} 2 & \textit{if } n = 3 \textit{ and } m = 1 \textit{ or } 3 \\ 1 & \textit{otherwise} \end{cases} \qquad \textit{and}$$

$$adern(G) = \begin{cases} 1 & \textit{if } \ n = 1 \textit{ and } m \leq 4, \textit{ or } n = m+1 = 2, \textit{ or } n \geq 4 \textit{ and } m = 1, 2 \\ 2 & \textit{if } \ n \geq 4 \textit{ and } m = 3 \\ 3 & \textit{if } \ n = 1 \textit{ and } m \geq 5, \textit{ or } n = 2 \textit{ and } 2 \leq m \leq 4, \textit{ or } n = 3 \textit{ and} \\ & \quad m = 1, 2, 3 \\ 4 & \textit{if } \ n = m-1 = 3 \\ 5 & \textit{if } \ n = 2 \textit{ and } m \geq 5, \textit{ or } n \geq 4 \textit{ and } m = 4 \\ m+1 & \textit{if } \ n \geq 3 \textit{ and } 5 \leq m < 2n \\ 2n+1 & \textit{if } \ n \geq 3 \textit{ and } m \geq 2n \end{cases}$$

Proof. In view of Theorem 4.1, we assume that $m > 2$. The da-edeck of$G$ consists of the da-ecards $(3, G - e_1)$ (when $m > 2$ and it is obtained by removing an edge e_1 whose ends are respectively a 2-vertex and a 3-vertex such that their common neighbour is an $(n+m-1)$-vertex), $(n + m - 1, G - e_2)$ (when $m > 1$ and it is obtained by removing an edge e_2 whose ends are respectively an $(n + m - 1)$-vertex and a 2-vertex), $(2n + 2m - 4, G - e_3)$ (when $n > 1$ and it is obtained by removing an edge e_3 whose ends are two $(n+m-1)$-vertices), $(n-1, G-e_4)$ (when $m = 1$ and it is obtained by removing an edge e_4 whose ends are respectively a n-vertex and a 1-vertex) and $(2, G - e_5)$ (when $m = 2$ and it is obtained by removing an edge e_5 whose ends are two 2-vertices), the da-ecards with associated edge degree 4 (when $m > 3$ and it is obtained by removing an edge whose ends are two 3-vertices) and the da-ecards with associated edge degree $n + m$ (when $m > 2$ and it is obtained by removing an edge whose ends are respectively an $(n + m - 1)$-vertex adjacent to a 3-vertex).

We shall first determine $dern(G)$. For $n = 1$, the only possibility to obtain an extension $H(3, G-e_1)$ is to join the unique 1-vertex to a 2-vertex, since otherwise the degree sum of the two vertices to be joined will be strictly greater than 3 and hence $dern(G) = 1$. For $n, m \neq 3$, the only possibility to obtain an extension $H(2n+2m-4, G-e_3)$ is to join the two $(n+m-2)$-vertices, since otherwise the degree sum of the two vertices to be joined will be less than $2n + 2m - 4$ and hence $dern(G) = 1$. For $n, m = 3$ in $(3, G - e_1)$, if we join the unique 1-vertex to a 2-vertex, then either the extension is isomorphic to G or it has exactly one da-ecard, namely $(3, G - e_1)$, in common with G. The set $\{H_1(3, G - e_1), H_2(5, G - e_2), H_3(8, G - e_3), H_4(6, G - e_7)\}$ forms a 1-blocking set of G where $H_1(3, G - e_1),\ H_2(5, G - e_2),\ H_3(8, G - e_3)$ and $H_4(6, G - e_7)$ are obtained, respectively, by joining the unique 1-vertex to a 2-vertex at distance 3, a 2-vertex to a 3-vertex, the unique 5-vertex to a 3-vertex at distance 2 and two 3-vertices and hence $dern(G) = 2$.

We shall now determine $adern(G)$.

Case 1: Augmenting the da-ecard $(3, G - e_1)$. For $n = 1$, clearly $H(3, G - e_1) \cong G$. For $n > 1$, if we join the unique 1-vertex to a 2-vertex, then H ($\ncong G$) has exactly one da-ecard, namely $(3, G - e_1)$, in common with G.

Case 2: Augmenting the da-ecard $(n + m - 1, G - e_2)$. We consider the case $n + m - 1 < 7$ (for otherwise $H(n + m - 1, G - e_2) \cong G$).

Case 2.1: $n = 1$. For $m = 3, 4,\ H(n + m - 1, G - e_2) \cong G$. For $m = 5$, if we join a 2-vertex to a 3-vertex, then H ($\ncong G$) has exactly two da-ecards $(n + m - 1, G - e_2)$ in common with G. For $m = 6$, if we join two 3-vertices, then H ($\ncong G$) has exactly one da-ecard, namely $(n + m - 1, G - e_2)$, in common with G.

Case 2.2: $n = 2$. For $m = 4$, if we join a 2-vertex to a 3-vertex, then H ($\ncong G$) has at most two da-ecards $(n+m-1, G-e_2)$ in common with G. For $m = 3$, if we join a 1-vertex to a 3-vertex or two 2-vertices and for $m = 5$, if we join two 3-vertices, then H ($\ncong G$) has exactly one da-ecard, namely $(n + m - 1, G - e_2)$, in common with G.

Case 2.3: $n = 3, 4$. For $n = 3$, if we join a 2-vertex to a 3-vertex (when $m = 3$) and if we join two 3-vertices (when $m = 4$) and for $n = 4$, if we join two 3-vertices (when $m = 3$), then H ($\ncong G$) has exactly one da-ecard, namely $(n + m - 1, G - e_2)$, in common with G.

Case 3: Augmenting the da-ecard $(2n + 2m - 4, G - e_3)$. For $n, m \neq 3,\ H(2n+2m-4, G-e_3) \cong G$. For $n, m = 3$, if we join the unique 5-vertex to a 3-vertex, then H ($\ncong G$) has exactly two da-ecards $H(2n + 2m - 4, G - e_3)$ in common with G.

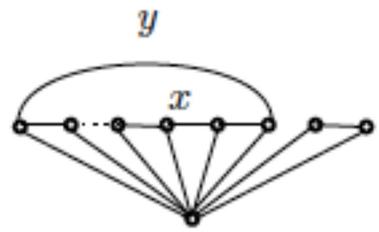

Figure 10. $H(4, G-e)$ for $n = 1$.

Case 4: Augmenting the da-ecard with associated edge degree 4.

Case 4.1: $n < 4$. For $n = 1$, if we join two 2-vertices in da-ecard with associated edge degree 4, then either the extension $H \cong G$ (this happens when $m = 4$) or the extension H $(\ncong G)$ has at most two da-ecards, with associated edge degree 4, in common with G. For $n = 2$, if we join two 2-vertices, then either the extension is isomorphic to G or H $(\ncong G)$ has at most two da-ecards, with associated edge degree 4 (when $m = 4$), and at most four da-ecards, with associated edge degree 4 (when $m > 4$), in common with G. For $n = 3$, if we join two 2-vertices, then either the extension is isomorphic to G or H $(\ncong G)$ has at most three da-ecards, with associated edge degree 4 (when $m = 4$), at most five da-ecards (when $m = 5$) and at most six da-ecards (when $m > 5$), in common with G.

Case 4.2: $n \geq 4$. For $4 \leq m < 2n$, if we join two 2-vertices, then the extension H $(\ncong G)$ has at most m da-ecards, with associated edge degree 4, and for $m \geq 2n$, H $(\ncong G)$ has at most $2n$ da-ecards, with associated edge degree 4, in common with G.

Case 5: Augmenting the da-ecard with associated edge degree $n+m$. For $n \geq 2$, if $n+m \geq 6$, then, in a da-ecard with associated edge degree $n+m$, if we join a 2-vertex to a $(n+m-2)$-vertex, then H $(\ncong G)$ has exactly one da-ecard, with associated edge degree $n+m$, in common with G. For $n = 1$, $H \cong G$ (when $m \neq 5$) and if we join two 3-vertices, then the extension H $(\ncong G)$ has exactly one da-ecard, with associated edge degree $n+m$, in common with G (when $m = 5$). For $n = 2$, if we join a 2-vertex to a 3-vertex, then H $(\ncong G)$ has at most two da-ecards, with associated edge degree $n+m$, in common with G (when $m = 3$).

Consider the da-ecard $(4, G-e)$ obtained by removing an edge e whose ends are adjacent to two 3-vertices of which one 3-vertex has a neighbour of degree 2 and the other has a neighbour of degree 3. Consider the extension $H(4, G-e)$ (obtained by joining two 2-vertices lying in a cycle of length $m-1$ for $n = 1$ and length $m+1$ for $n > 1$). For $n = 1$, $m > 4$, the above extension $H(4, G-e)$, is the 2-adversary blocking graph.

For $n = 2$, the 2-adversary blocking graph for $m = 3$ is $H(5, G-e)$ (obtained by joining a 2-vertex adjacent to a 2-vertex and a 3-vertex to the 3-vertex at distance 3) and the 2-adversary blocking graph for $m = 4$ and the 4-adversary blocking graph for $m \geq 5$ is $H(4, G-e)$. For $n = 3$, the 2-adversary blocking graph is $H(2n+2m-4, G-e_3)$ (obtained by joining the unique 5-vertex to an 3-vertex) for $m = 3$, the 3-adversary blocking graph for $m = 4$, the 5-adversary blocking graph for $m = 5$ and the 6-adversary blocking graph for $m \geq 6$ are $H(4, G-e)$. For $n \geq 4$, the 1-adversary blocking graph for $m = 3$ is $H(3, G-e_2)$ (obtained by joining the unique 1-vertex to

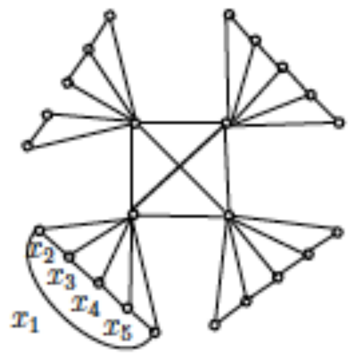

Figure 11. $H(4, G-e)$ for $n = 4$ and $m = 5$.

a 2-vertex at distance 3), the m-adversary blocking graph for $4 \leq m < 2n$ and the $2n$-adversary blocking graph for $m \geq 2n$ are $H(4, G-e)$ (Figure 11), which completes the proof. □

Conclusion

Degree associated edge reconstruction number might be a strong tool for providing evidence to support or reject the Edge Reconstruction Conjecture that remains open. Also the concept of degree associated edge reconstruction number opens up interesting and difficult questions in situations which did not seem to present much difficulty as far as the Reconstruction Problems are concerned.

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