Electronic Journal of Graph Theory and Applications **7** (2) (2019), 373–381

Electronic Journal of
Graph Theory and Applications

Orthogonal embeddings of graphs in Euclidean space

Wai Chee Shiu[a], Richard M. Low[b]

[a]*Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong, P.R. China*
[b]*Department of Mathematics and Statistics, San Jose State University, San Jose, CA 95192, USA*

shiu1958@gmail.com, richard.low@sjsu.edu

Abstract

Let $G = (V, E)$ be a simple connected graph. An injective function $f : V \to \mathbb{R}^n$ is called an n-dimensional (or n-D) orthogonal labeling of G if $uv, uw \in E$ implies that $(f(v) - f(u)) \cdot (f(w) - f(u)) = 0$, where $\cdot$ is the usual dot product in Euclidean space. If such an orthogonal labeling f of G exists, then G is said to be embedded in $\mathbb{R}^n$ orthogonally. Let the orthogonal rank $or(G)$ of G be the minimum value of n, where G admits an n-D orthogonal labeling (otherwise, we define $or(G) = \infty$). In this paper, we establish some general results for orthogonal embeddings of graphs. We also determine the orthogonal ranks for cycles, complete bipartite graphs, one-point union of two graphs, Cartesian product of orthogonal graphs, bicyclic graphs without pendant, and tessellation graphs.

Keywords: orthogonal labeling, graph embedding, orthogonal drawing, Euclidean space
2010 Mathematics Subject Classification: 05C78, 05C10
DOI: 10.5614/ejgta.2019.7.2.13

1. Introduction

Graph labelings form an important part of graph theory. First formally introduced in the 1960s by Alex Rosa, this area of research has been the subject matter for many papers in the mathematical literature. Certain types of graph labelings have applications to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader is directed to J.A. Gallian's comprehensive dynamic survey on graph labelings [2].

Received: 30 March 2018, Revised: 21 May 2019, Accepted: 1 June 2019.

For concepts and notation not explicitly defined in this paper, the reader is directed to [1]. In this paper, $G = (V, E)$ is a simple connected graph. Let $\Delta(G)$ denote the maximum degree of G. In [3], the following type of graph labeling was introduced:

- An injective function $f : V \to \mathbb{R}^{\Delta(G)}$ is an *orthogonal labeling* of G if $uv, uw \in E$ implies that $(f(v) - f(u)) \cdot (f(w) - f(u)) = 0$, where $\cdot$ is the usual dot product in Euclidean space. If G has such a labeling, then G is called an *orthogonal* graph.

In their paper [3], Immanuel and Sugeng showed that the hypercubes Q_n $(n \geq 1)$, C_{2k} $(k \geq 2)$ and trees are orthogonal. They also showed that odd cycles, as well as any G containing K_3 as a subgraph, are not orthogonal.

An orthogonal labeling of G can be explicitly constructed from a rectilinear drawing of G in $\mathbb{R}^{\Delta(G)}$, where incident edges of G are perpendicular to each other. This particular type of labeling is quite interesting as it imposes a geometric structure to a graph. In fact, othogonal drawings of G (where $\Delta(G) = 2$) have been studied extensively in computational geometry and computer science. As of this writing, a key phrase search for "orthogonal drawing" yields 111 entries in the MathSciNet database.

In this paper, we give further analysis of orthogonal graphs in a more general context. We start with a few definitions. For a graph $G = (V, E)$, an injective function $f : V \to \mathbb{R}^n$ is an n-*dimensional* (or n-D) *orthogonal labeling* of G if $uv, uw \in E$ implies that $(f(v) - f(u)) \cdot (f(w) - f(u)) = 0$, where $\cdot$ is the usual dot product in Euclidean space. Let the *orthogonal rank* of G (denoted by $or(G)$) be the minimum value of n such that G admits an n-D orthogonal labeling (if any), otherwise define $or(G) = \infty$. If $or(G) = \Delta(G)$, then G is called an *orthogonal* graph.

Observations.

1. If G has an n-D orthogonal labeling, then $n \geq \Delta(G)$.
2. Suppose H is a subgraph of a graph G. Then, $or(H) \leq or(G)$.

2. Some results

For convenience, we will use v to denote the vertex label $f(v)$ and uv to denote the vector $f(v) - f(u)$, for $u, v \in V(G)$, when there is no danger for confusion.

Lemma 2.1. $or(K_{2,3}) = \infty$.

Proof. Let (X, Y) be the vertex set bipartition of $K_{2,3}$, where $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. Suppose there is an n-D orthogonal labeling f of $K_{2,3}$, where $3 \leq n < \infty$. Without loss of generality, assume that $x_1 = (0, \ldots, 0)$, $y_1 = (a_1, 0, \ldots, 0)$, $y_2 = (0, a_2, 0, \ldots, 0)$ and $y_3 = (0, 0, a_3, 0, \ldots, 0)$, for some $a_i \in \mathbb{R} \setminus \{0\}$, $1 \leq i \leq 3$. Now, suppose that $x_2 = (p_1, p_2, p_3, \ldots, p_n)$. Then, $y_1x_2 = (p_1 - a_1, p_2, p_3, \ldots, p_n)$, $y_2x_2 = (p_1, p_2 - a_2, p_3, \ldots, p_n)$ and $y_3x_2 = (p_1, p_2, p_3 - a_3, \ldots, p_n)$. Since y_1x_1 and y_1x_2 are orthogonal, we have $a_1(p_1 - a_1) = 0$. Since $a_1 \neq 0$, this implies that $p_1 = a_1$. Similarly, we have $p_2 = a_2$ and $p_3 = a_3$. Since y_1x_2 and y_2x_2 are orthogonal, we have $\sum_{i=3}^{n} p_i^2 = 0$. This implies $a_3 = p_3 = 0$, which is a contradiction. Hence, the claim is established. □

Corollary 2.2. *Suppose G contains $K_{2,3}$. Then, $or(G) = \infty$.*

Proof. This follows immediately from Lemma 2.1 and Observation 2. □

Corollary 2.3. *$or(K_{m,n}) = \infty$, for $2 \le m \le n$, where m, n are not simultaneously equal to 2. Furthermore, $or(K_{2,2}) = 2$.*

Proof. In [3], it was shown that all even cycles are orthogonal. Thus, $or(K_{2,2}) = 2$. The remainder of the claim follows immediately from Corollary 2.2. □

Let G_i, $1 \le i \le n$, be n disjoint graphs. Suppose $u_i \in V(G_i)$. Let H be the graph obtained from the G_i by identifying all the u_i to a single vertex. We say that H is a *one-point union* of the G_i, where $1 \le i \le n$.

Lemma 2.4. *Let G' be a one-point union of arbitrary graphs G and H, with identified vertex w. Then, $or(G') = \max\{or(G), or(H), \deg(w)\}$.*

Proof. If $or(G) = \infty$ or $or(H) = \infty$, then the result is obvious. So, we assume that $or(G) = g$ and $or(H) = h$. Let w be the vertex in G' after merging $u \in V(G)$ and $v \in V(H)$. Clearly, $or(G') \ge \max\{or(G), or(H), \deg(w)\}$.

We need to show that $or(G') \le \max\{or(G), or(H), \deg(w)\}$. Let $\alpha : V(G) \to \mathbb{R}^g$ and $\beta : V(H) \to \mathbb{R}^h$ be g-D and h-D orthogonal labelings of G and H, respectively. Without loss of generality, let $1 \le h \le g$. Let the neighborhood of u in G be $\{u_1, \dots, u_m\}$ and the neighborhood of v in H be $\{v_1, \dots, v_n\}$. Clearly, $m \le g$ and $n \le h$.

Case 1: Suppose $m + n \le g$. By using rotation and translation, we can assume that $\alpha(u)$ is the origin in $\mathbb{R}^g$ and each edge $\alpha(u)\alpha(u_i)$ lies on the positive i-th axis, $1 \le i \le m$. Moreover, we can also assume that $\alpha(V(G))$ lies in the region $\Omega_+ = \{(x_1, \dots, x_g) \mid x_i \ge 0\}$. Similarly, we can assume that each edge $\beta(v)\beta(v_j)$ lies on the negative $(m + j)$-th axis, $1 \le j \le n$. Moreover, we can also assume that $\beta(V(H))$ lies in the region $\Omega_- = \{(x_1, \dots, x_g) \mid x_i \le 0\}$. Combining these two labelings, we have the required labeling of G' and $or(G') = g = or(G)$.

Case 2: Suppose $m + n > g$. Now, we want to orthogonally embed G' in $\mathbb{R}^{m+n}$. By using rotation and translation, we can assume that $\alpha(u)$ is the origin in $\mathbb{R}^{m+n}$ and each edge $\alpha(u)\alpha(u_i)$ lies on the positive i-th axis, $1 \le i \le m$. Moreover, we can also assume that $\alpha(V(G))$ lies in the region $\Omega_+ = \{(x_1, \dots, x_g, 0, \dots, 0) \in \mathbb{R}^{m+n} \mid x_i \ge 0, 1 \le i \le g\}$. Similarly, we can assume that each edge $\beta(v)\beta(v_j)$ lies on the negative $(m + j)$-th axis, $1 \le j \le n$. Moreover, we can also assume that $\beta(V(H))$ lies in the region $\Omega_- = \{(x_1, \dots, x_g, \dots, x_{m+n}) \mid x_i \le 0, 1 \le i \le m + n\}$. Combining these two labelings, we have the required labeling of G' and $or(G') = m + n = \deg(w)$. □

Figure 1 illustrates Lemma 2.4. In the first graph, G_1 is a one-point union of C_5 and $K_{1,3}$, with the identified vertex being a leaf of $K_{1,3}$. In the second graph, G_2 is another one-point union of C_5 and $K_{1,3}$, where the identified vertex is the vertex of degree three in $K_{1,3}$.

Corollary 2.5. *The one-point union of orthogonal graphs is orthogonal.*

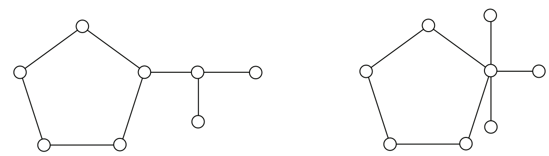

Figure 1. $or(G_1) = 3$ and $or(G_2) = 5$.

Proof. This follows immediately from Lemma 2.4. □

Corollary 2.6. *The one-point union of an orthogonal graph G and a tree T is orthogonal.*

Proof. In [3], it was shown that every tree is orthogonal. Hence by Corollary 2.5, the claim is established. □

We note that Corollary 2.6 was also shown in [3].

Lemma 2.7. *Let $or(G), or(H) < \infty$ for graphs G and H, respectively. Then, $\Delta(G) + \Delta(H) \leq or(G \times H) \leq or(G) + or(H)$.*

Proof. Note that the maximum degree of $G \times H$ is $\Delta(G) + \Delta(H)$. So, the first inequality is obvious. Let $f_G : V(G) \to \mathbb{R}^m$ and $f_H : V(H) \to \mathbb{R}^n$, where $or(G) \leq m$ and $or(H) \leq n$. Define $f : V(G) \times V(H) \to \mathbb{R}^{m+n}$ by $f((u, v)) = (f_G(u), f_H(v))$, for $u \in V(G)$ and $v \in V(H)$. It is straightforward to check that f is an $(m + n)$-D orthogonal labeling of $G \times H$, which establishes the second inequality. □

Corollary 2.8. *If G and H are orthogonal, then $G \times H$ is orthogonal.*

Proof. Let G and H be orthogonal graphs. Then, $or(G) = \Delta(G)$ and $or(H) = \Delta(H)$. Thus, by Lemma 2.7, the claim follows. □

Lemma 2.9. *Let K be a connected graph with cut-edge uv. Suppose $K - uv = G + H$* (where $u \in G$, $v \in H$) *and $or(G), or(H) < \infty$. Then, $or(K) = \max\{or(G), or(H), \deg_K(u), \deg_K(v)\}$.*

Proof. Clearly, $or(K) \geq \max\{or(G), or(H), \deg_K(u), \deg_K(v)\}$. Let $g = or(G)$ and $h = or(H)$. Without loss of generality, assume $1 \leq h \leq g$. Clearly, $\deg_G(u) = m \leq g$ and $\deg_H(v) = n \leq h$. Note that $\deg_K(u) = m + 1 \leq g + 1$ and $\deg_K(v) = n + 1 \leq h + 1 \leq g + 1$.

Let $\alpha : V(G) \to \mathbb{R}^g$ and $\beta : V(H) \to \mathbb{R}^h$ be g-D and h-D orthogonal labelings, respectively. Let the neighborhood of u in G be $\{u_1, \ldots, u_m\}$ and the neighborhood of v in H be $\{v_1, \ldots, v_n\}$.

We first consider $h < g$.

Case 1: Suppose $m < g$. Then, $\max\{or(G), or(H), \deg_K(u), \deg_K(v)\} = g$. It suffices to find a g-D orthogonal labeling for K. By using rotation and translation, we can assume that $\alpha(u)$ is the origin in $\mathbb{R}^g$ and each edge $\alpha(u)\alpha(u_i)$ lies on the positive i-th axis, $1 \leq i \leq m$. Moreover, we can also assume that $\alpha(V(G))$ lies in the region $\Omega_+ = \{(x_1, \ldots, x_g) \mid x_i \geq 0, 1 \leq i \leq g\}$. Similarly, we can embed the graph H in $\mathbb{R}^g$ such that $\beta(v)$ is the origin of $\mathbb{R}^g$ and each edge $\beta(v)\beta(v_j)$ lies on the negative j-th axis, $1 \le j \le n$. Moreover, we can also assume that $\beta(V(H))$ lies in the region $\Omega_- = \{(x_1, \ldots, x_g) \mid x_j \le 0, 1 \le j \le g\}$. Finally, we translate $\beta(V(H))$ one unit downward along the g-th axis. This gives us the required labeling.

Case 2: Suppose $m = g$. Then, $\max\{or(G), or(H), \deg_K(u), \deg_K(v)\} = g + 1$. It suffices to find a $(g + 1)$-D orthogonal labeling for K. The proof is the same as in Case 1, by replacing g with $g + 1$.

Now, suppose $h = g$. Without loss of generality, we can assume that $n \le m$. The proof for this case is the same as above.

Thus, $or(K) \le \max\{or(G), or(H), \deg_K(u), \deg_K(v)\}$. This completes the proof. □

3. Applications

There is a 2-D orthogonal labeling of C_{2k} in [3]. Its restriction is a 2-D orthogonal labeling of $P_{2k} = v_0v_1 \cdots v_{2k-1}$ for $k \ge 2$. For the sake of completeness, we list that labeling ϕ here:
$\phi(v_0) = (0, 0)$; $\phi(v_{2r}) = (r, k-r)$ for $1 \le r \le k-1$; and $\phi(v_{2r-1}) = (r-1, k-r)$ for $1 \le r \le k$. See Figure 2.

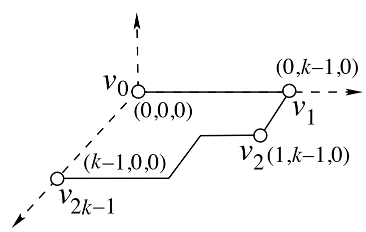

Figure 2. A 2-D orthogonal labeling of P_{2k} (drawn in $\mathbb{R}^3$).

For further use, we now introduce a 3-D orthogonal labeling of $P_{2k+1} = v_0v_1u_0v_2 \cdots v_{2k-1}$ such that the end vertices v_0 and v_{2k-1} are embedded in x-axis, where $k \ge 2$. Starting from the 2-D orthogonal labeling ϕ for $P_{2k} = v_0v_1 \cdots v_{2k-1}$ shown above, we embed P_{2k} in the xy-path. Then, we embed the inserted vertex u_0 at $(1/2, k-1, 1/2)$. Clearly $v_0v_1 \perp v_1u_0$, $v_1u_0 \perp u_0v_2$ and $u_0v_2 \perp v_2v_3$.

Furthermore, we see that, $v_{2k-1}v_0 \perp v_0u_0$. Hence, we have the following theorem.

Theorem 3.1. *For $k \ge 2$, $or(C_{2k}) = 2$ and $or(C_{2k+1}) = 3$.*

For an illustration of the above remarks and Theorem 3.1, see Figure 3.

Lemma 3.2. $or(C_3) = \infty$.

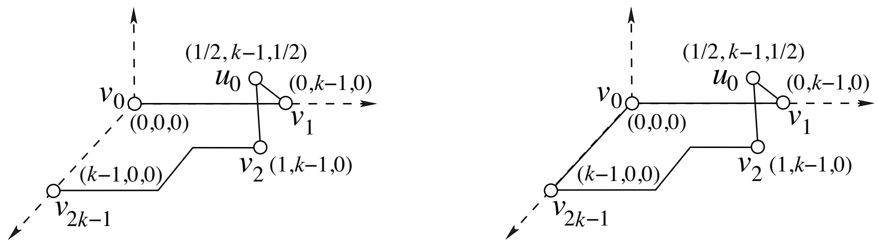

Figure 3. 3-D orthogonal labelings of P_{2k+1} and C_{2k+1}, respectively.

Proof. Assume that $or(C_3) = t$, for some finite t. Without loss of generality, let v_0 be labeled with $(0, 0, 0, \dots, 0)$, v_1 labeled with $(1, 0, 0, \dots, 0)$ and v_2 labeled with $(0, c, 0, \dots, 0)$. Then, $v_2v_1 \perp v_0v_1$ implies $1^2 + 0^2 + 0^2 + \cdots + 0^2 = 0$, which gives the desired contradiction. □

We note that Lemma 3.2 is also mentioned in [3].

Corollary 3.3. *If G is orthogonal, then $G \times C_{2n}$ is orthogonal, for $n \geq 2$.*

Corollary 3.4. *One-point union of n cycles is orthogonal, for $n \geq 2$.*

Proof. Let G be a one-point union of n cycles H_i's. Note that $or(H_i) = 2$ or 3, for each i. Consider $n = 2$ first. By Lemma 2.4, we have $or(G) = \max\{or(H_1), or(H_2), 4\} = 4$. Now, we consider $n \geq 3$. By applying Lemma 2.4 repeatedly, we see that the orthogonal rank of a one-point union of cycles is $2n$, for $n \geq 2$. □

4. Bicyclic graphs without pendant

It is known that a bicyclic graph without pendant is a one-point union of two cycles, a theta graph or a long dumbbell graph [6]. In this section, we show that all bicyclic graphs without pendant are orthogonal.

Let $U(m, n)$ be the one-point union of two cycles $C_m = u_0u_1 \cdots u_{m-1}u_0$ and $C_n = u_0u_mu_{m+1} \cdots u_{n+m-2}u_0$, where $m, n \geq 3$. Corollary 3.4 shows that $U(m, n)$ is orthogonal.

Definition 4.1. A *theta graph* is the union of three internally disjoint paths that have the same two distinct end vertices. Without loss of generality, we may assume that $s \geq t \geq r \geq 1$ such that $Q_s = u_0u_1 \cdots u_s$, $Q_t = u_0u_{s+1}u_{s+2} \cdots u_{s+t-1}u_s$ and $Q_r = u_0u_{s+t}u_{s+t+1} \cdots u_{s+t+r-2}u_s$. Here, Q_i denotes a path of length i. We denote this graph by $\Theta(s, t, r)$. Since we only consider simple graphs, at least two of $s \geq t \geq 2$.

Definition 4.2. A *long dumbbell graph* is a graph obtained from two cycles C_m and C_n, by joining a path Q_l of length l for $m, n \geq 3$ and $l \geq 1$. Without loss of generality, we can assume

$$C_m = u_0u_1 \cdots u_{m-1}u_0, \quad Q_l = u_{m-1}u_m \cdots u_{m+l-1}$$

$$\text{and} \quad C_n = u_{m+l-1}u_{m+l}\cdots u_{m+n+l-2}u_{m+l-1}.$$

We denote this graph by $D(m, n; l)$.

Theorem 4.1. *The long dumbbell graph $D(m, n; l)$ is orthogonal and $or(D(m, n; l)) = 3$, for $m, n \geq 3$ and $l \geq 1$.*

Proof. Let uv be an edge in Q_l. Let $D(m, n; l) - uv = G_1 + G_2$. By Lemma 2.4, we have $2 \leq or(G_1) \leq 3$ and $2 \leq or(G_2) \leq 3$. Now $\deg(u) = 1$ or 3 and $\deg(v) = 1$ or 3. Applying Lemma 2.9, we have $or(D(m, n; l)) = 3$. Hence, $D(m, n; l)$ is orthogonal. □

Since $\Theta(s, 2, 1)$ contains K_3 and $\Theta(2, 2, 2) \cong K_{2,3}$, their orthogonal ranks are infinity. Thus, we now only consider $\Theta(s, t, 1)$ when $s \geq t \geq 3$, and $\Theta(s, t, r)$ when $s \geq t \geq r \geq 2$ but $(s, t, r) \neq (2, 2, 2)$.

Theorem 4.2. *The theta graph $\Theta(s, t, 1)$ is orthogonal and $or(\Theta(s, t, 1)) = 3$, for $s \geq t \geq 3$.*

Proof. We keep the notation defined above. We embed Q_t (as well as $Q_t \cup Q_1 \cong C_{t+1}$) in the first octant such that u_0 is the origin and u_s is the point $(\lceil t/2\rceil - 1, 0, 0)$.

Individually, we may put Q_s in the fourth octant (i.e., the region $\{(x, y, z) \mid x, z \geq 0, y \leq 0\}$) such that u_0 is the origin and $u_s = (\lceil s/2\rceil - 1, 0, 0)$. Scale down the image of Q_s by the rate $\frac{\lceil t/2\rceil-1}{\lceil s/2\rceil-1}$ so that u_s becomes $(\lceil t/2\rceil - 1, 0, 0)$. Hence, this labeling is the required orthogonal labeling. □

Theorem 4.3. *The theta graph $\Theta(s, t, r)$ is orthogonal $[$except $\Theta(2, 2, 2)]$ and $or(\Theta(s, t, r)) = 3$, for $s \geq t \geq r \geq 2$.*

Proof. The labeling is similar to the proof of Theorem 4.2. We embed $Q_t \cup Q_r \cong C_{t+r}$ in the first octant such that u_0 is the origin and u_{s+1} is the point $(\lfloor (t + r)/2\rfloor - 1, 0, 0)$. Suppose the vertex u_s is put at the point $\boldsymbol{w}$.

We view the ray from the origin to w as the positive x-axis on defining ϕ or ψ above. By applying enlargement if necessary, we embed Q_s in the fifth octant (i.e., the region $\{(x, y, z) \mid x, y \geq 0, z \leq 0\}$) such that u_0 is the origin and u_s is $\boldsymbol{w}$. Hence, this labeling is the required orthogonal labeling. □

Hence, all bicyclic graphs without pendant (except $\Theta(s, 2, 1)$ and $\Theta(2, 2, 2)$) are orthogonal.

5. Tessellation graphs

A *tessellation* is a tiling of the plane, using polygons. If a tessellation consists of congruent polygons, it is a *regular* tessellation. Thus, there are only three regular tessellations, utilizing equilateral triangles, squares, or regular hexagons. A *tessellation* graph is a finite subgraph of a regular tessellation, consisting of a grid of congruent polygons where each polygon shares at least one common edge with another.

Definition 5.1. A region Ω in the plane is *n-connected* if the complement of Ω has exactly n components.

Definition 5.2. For $n \geq 2$, an *n-tessellation graph* is a graph which tessellates an n-connected region in the plane.

For example, a 1-tessellation graph tessellates a simply-connected, bounded region in the plane.

Consider a tessellation of the plane, using congruent equilateral triangles. Two triangles are *connected* if they share a common edge. Let T be a connected collection of triangles. Then, T is a connected planar graph, consisting of a grid of C_3's with each C_3 sharing at least one common edge with another. A connected collection of triangles is called a *triomino*. T is called an *n-triomino* if it is an n-tessellation graph.

Theorem 5.1. *Let T be a 1-triomino. Then, $or(T) = \infty$.*

Proof. Every 1-triomino contains K_3. Thus, the result follows immediately from $or(K_3) = \infty$ and Observation 2. □

A *cell* is the boundary of a unit square ($\cong C_4$) in the xy-plane, where the vertices of the square are at lattice points. Two cells are *connected* if they share a common edge. Let S be a connected collection of connected cells. Thus, S can be viewed as a connected planar graph, consisting of a grid of C_4's with each C_4 sharing at least one common edge with another. A connected collection of connected cells is called a *polyomino*.

Theorem 5.2. *Every 1-polyomino is orthogonal.*

Proof. Let P be a 1-polyomino. If $P \cong P_n \times P_2$, then P is orthogonal by Corollary 2.8. Now, suppose that P is not isomorphic to $P_n \times P_2$. Then, P is a subgraph of $P_n \times P_m$, where $m \geq 3$. Also, $\Delta(P) = 4$. Thus, $or(P) \geq 4$. Since $or(P_n \times P_m) = 4$, we have that $or(P) \leq 4$ by Observation 2. Hence, $or(P) = 4$. □

Consider a tessellation of the plane, using congruent hexagons. Two hexagons are *connected* if they share a common edge. Let H be a connected collection of hexagons. Then, H is a connected planar graph, consisting of a grid of C_6's with each C_6 sharing at least one common edge with another. A connected collection of hexagons is called a *honeycomb* graph.

Here, we view such a graph (or *benzenoid system*) as a (finite) connected plane graph, where each inner face is a regular hexagon of side length one. The *wall*, which was defined in [4], is an infinite graph $W = (V, E)$, where $V = \mathbb{Z} \times \mathbb{Z}$ and $\{(y_1, z_1), (y_2, z_2)\} \in E$ if: (1). $z_1 = z_2$ and $|y_1 - y_2| = 1$; or (2). $y_1 = y_2$, $|z_1 - z_2| = 1$ and $y_1 + z_1 + y_2 + z_2 \equiv 1 \pmod 4$. See Figure 4.

Remark. Suppose (y, z) is a vertex of W. The neighborhood $N(y, z)$ of (y, z) is as follows:

- For even y and even z, $N(y, z) = \{(y-1, z),\ (y+1, z),\ (y, z+1)\}$.
- For even y and odd z, $N(y, z) = \{(y-1, z),\ (y+1, z),\ (y, z-1)\}$.
- For odd y and even z, $N(y, z) = \{(y-1, z),\ (y+1, z),\ (y, z-1)\}$.
- For odd y and odd z, $N(y, z) = \{(y-1, z),\ (y+1, z),\ (y, z+1)\}$.

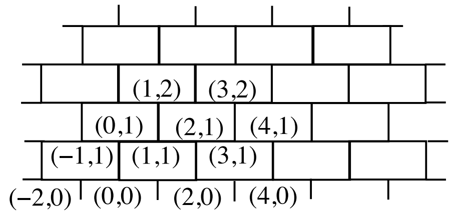

Figure 4. The wall W.

Note that a honeycomb graph is isomorphic to a subgraph of W.

Theorem 5.3. *The wall W is orthogonal and $or(W) = 3$.*

Proof. We embed the wall into the yz-plane first. That is, each vertex (y_i, z_j) of the wall is located at $(0, y_i, z_j)$. Secondly, we move the point $(0, y_i, z_j)$ to $(1, y_i, z_j)$ when y_i is odd.

Now, for the point $(0, y_i, z_j)$ (i.e., y_i is even), its neighbors are $(1, y_i - 1, z_j)$, $(1, y_i + 1, z_j)$, $(0, y_i, z_j + \alpha)$, where α is either 1 or -1. The vector raised from $(0, y_i, z_j)$ to these neighbors are $(1, -1, 0)$, $(1, 1, 0)$ and $(0, 0, \alpha)$. Clearly, they are mutually orthogonal. For the point $(1, y_i, z_j)$ (i.e., y_i is odd), its neighbors are $(0, y_i - 1, z_j)$, $(0, y_i + 1, z_j)$, $(1, y_i, z_j + \alpha)$. The vector raised from $(1, y_i, z_j)$ to these neighbors are $(-1, -1, 0)$, $(-1, 1, 0)$ and $(0, 0, \alpha)$. Clearly, they are also mutually orthogonal and $or(W) = 3$. □

Corollary 5.4. *Every 1-tessellation honeycomb graph is orthogonal.*

6. Acknowledgments

The authors are grateful to the anonymous referees. Their valuable comments and suggestions improved the final manuscript.

References

[1] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications*, MacMillan, New York, (1976).

[2] J.A. Gallian. A dynamic survey of graph labeling, *Electron. J. Combin*. **20** (2017), #DS6.

[3] B. Immanuel and K.A. Sugeng. Orthogonal labeling, *Indonesian Journal of Combinatorics*. **1** (2016), 1-8.

[4] W.C. Shiu and P.C.B. Lam. The Wiener number of the hexagonal net, *Discrete Appl. Math.* **73** (1997), 101-111.

[5] W.C. Shiu and P.C.B. Lam. Wiener numbers of pericondensed benzenoid molecule systems, *Congr. Numer.* **126** (1997), 113-124.

[6] W.C. Shiu and R.M. Low. The integer-magic spectra of bicyclic graphs without pendant, *Congr. Numer*. **214** (2012), 65-73.