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The connected size Ramsey number for matchings versus small disconnected graphs

Hilda Assiyatun, Budi Rahadjeng[1]*, Edy Tri Baskoro

Combinatorial Mathematics Research Group,
Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung,
Jalan Ganesa 10 Bandung, Indonesia

hilda@math.itb.ac.id, budi.rahadjeng@students.itb.ac.id, ebaskoro@math.itb.ac.id

Abstract

Let $F, G,$ and H be simple graphs. The notation $F \to (G, H)$ means that if all the edges of F are arbitrarily colored by red or blue, then there always exists either a red subgraph G or a blue subgraph H. The size Ramsey number of graph G and H, denoted by $\hat{r}(G, H)$ is the smallest integer k such that there is a graph F with k edges satisfying $F \to (G, H)$. In this research, we will study a modified size Ramsey number, namely the connected size Ramsey number. In this case, we only consider connected graphs F satisfying the above properties. This connected size Ramsey number of G and H is denoted by $\hat{r}_c(G, H)$. We will derive an upper bound of $\hat{r}_c(nK_2, H), n \geq 2$ where H is $2P_m$ or $2K_{1,t}$, and find the exact values of $\hat{r}_c(nK_2, H)$, for some fixed n.

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1. Introduction

All graphs in this paper are finite, undirected, and simple. Let $F, G,$ and H be graphs. The number of vertices and edges of graph F will be denoted by $|V(F)|$ and $|E(F)|$, respectively. The notation $F \to (G, H)$ means that in any red-blue coloring of the edges of F there exists a red

[1] Permanent Affiliation: Universitas Negeri Surabaya, Indonesia.
*Corresponding author.
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copy of G or a blue copy of H in F. We denote $F \nrightarrow (G, H)$ to mean that there is some red-blue coloring of the edges of F such that F contains neither a red G nor a blue H. This coloring is called a (G, H)-*coloring* of F.

The *size Ramsey number* for a pair of graphs G and H, denoted by $\hat{r}(G, H)$, is the smallest integer k such that there is a graph F with k edges satisfying $F \longrightarrow (G, H)$. The concept of size Ramsey number of a graph was introduced by Erdős et al. in [2]. A survey of results about the size Ramsey number for a pair of graphs can be seen in [4]. There are only a few results concerning the size Ramsey number for a pair of graphs, namely the size Ramsey numbers involving a complete graph, a star, a cycle or a path. Further results have also been obtained, for instance the size Ramsey number for some regular graphs [5] and the size Ramsey of a directed path [6].

A *matching,* denoted by $nK_2, n \geq 2$, is the graph consisting of $2n$ vertices and n independent edges. In 1978, Burr *et.al* [1] determined the size Ramsey number for a pair of graphs involving matching, $\hat{r}(nK_{1,s}, mK_{1,t}) = (n+m-1)(s+t-1)$, for positive integers $s, t, m,$ and n. The smallest graphs F satisfying this size Ramsey number are $(m+n-1)K_{1,(s+t-1)}$ and $lK_3 \cup (m+n-l-1)K_{1,3}$ for $s = t = 2, 1 \leq l \leq m + n - 1$, namely $(m + n - 1)K_{1,(s+t-1)} \rightarrow (nK_{1,s}, mK_{1,t})$ or $lK_3 \cup (m+n-l-1)K_{1,3} \rightarrow (nK_{1,2}, mK_{1,2})$. These two graphs are disconnected. The other result on the size Ramsey number involving matching was obtained by Erdős and Faudree [3]. They showed that $\hat{r}(2K_2, P_m) = m + 1$, where the smallest graph satisfying the size Ramsey number is a C_{m+1}, namely $C_{m+1} \rightarrow (2K_2, P_m)$. Note that in this case, we have a connected smallest graph F satisfying $F \rightarrow (G, H)$.

Therefore, in general we have either connected or disconnected graph F with smallest size and satisfying $F \rightarrow (G, H)$, for given G and H. In this paper, we are interested in finding a connected graph F with minimum size and satisfying $F \rightarrow (G, H)$. The smallest size of a connected graph F so that $F \rightarrow (G, H)$ is called the connected size Ramsey number and denoted by $\hat{r}_c(G, H)$.

Some results on the connected size Ramsey number for a pairs of graphs were established. Rahadjeng et al. [8] determined the connected size Ramsey number for the pairs $(2K_2, K_{1,m})$ and $(3K_2, K_{1,m})$. Then, in [7], they showed that $\hat{r}_c(nK_2, K_{1,3}) = 4n - 1$, for $n \geq 2$.

In this paper, we will determine an upper bound of $\hat{r}_c(nK_2, H), n \geq 2$ where H is isomorphic to $2P_m$ or $2K_{1,t}$. We also determine the exact values of $\hat{r}_c(nK_2, H)$ for some fixed n.

2. Main Results

In this section, we present the following results.

Theorem 2.1. *For* $m \geq 2$, $\hat{r}_c(2K_2, 2P_m) = 2m + 1$.

Proof. First, we will show that $\hat{r}_c(2K_2, 2P_m) \leq 2m + 1$. To do this, we will define the connected graph F having $2m + 1$ edges satisfying $F \rightarrow (2K_2, 2P_m)$. Consider the graph $F = C_{2m+1}$. Let μ be any red-blue coloring of F such that there is no red $2K_2$. Then, there is no red edge in F or a red subgraph in F is isomorphic to either P_2 or P_3. Let us consider a subgraph $F' = F - E(P_i)$ with $i = 2$ or 3. Certainly, F' is isomorphic to either a path P_{2m+1} or P_{2m}. Since the necessary condition of the path containing $2P_m$ is having at least $2m$ vertices, then obviously F' contains $2P_m$. Hence, $F \rightarrow (2K_2, 2P_m)$.

Now, we will show that $\hat{r}_c(2K_2, 2P_m) \geq 2m+1$. Let G be a connected graph with $|E(G)| \leq 2m$. We will show that $G \nrightarrow (2K_2, 2P_m)$. We are going to prove it by using the number of vertices of G.

First, we assume that $|V(G)| = 2m+1$. In this case, G is a tree. Let $P = v_1, v_2, ..., v_k$ be the longest path in G, with $k \leq 2m+1$. Choose one vertex of $V(P)$, say v_i, so that $G - v_i$ contains no $2P_m$. Color all edges incident with v_i by red and all edges in $G - v_i$ by blue. By this coloring, there is a $(2K_2, 2P_m)$-coloring on F. Thus, $G \nrightarrow (2K_2, 2P_m)$.

Next, suppose that $|V(G)| \leq 2m$. Let us consider a complete graph K_{2m}. For every $v \in V(K_{2m})$, $K_{2m} - v \nsupseteq 2P_m$. Since all graphs of order $2m$ and size $2m$ are proper subgraphs of K_{2m}, then we can color all edges of G with red-blue so that there exists a $(2K_2, 2P_m)$-coloring in G. Thus, $G \nrightarrow (2K_2, 2P_m)$. $\square$

Theorem 2.2. $\hat{r}_c(nK_2, 2P_3) \leq \begin{cases} 3n+1, & \textit{for } n = 3,4,5,6,7, \\ 5(\frac{n}{2})+4, & \textit{for even } n,\ n \geq 8, \\ 5\left(\frac{n+1}{2}\right)+2, & \textit{for odd } n,\ n \geq 9. \end{cases}$

Proof. We will find a connected graph F such that $F \rightarrow (nK_2, 2P_3)$. First, we will prove for the case of $n \in [3,7]$. Let us consider the graph $F = C_{3n+1}$.

Let μ be any red-blue coloring of F that maximizes the number of red edges and contains no red nK_2. The red subgraph of F contains at most $2(n-1)$ edges. The remaining edges, which are blue, are at least $3n+1-2(n-1) = n+3$. This blue subgraph consists of at most $n-1$ disjoint paths. By the pigeon-hole principle, there are at least two disjoint paths of length 2. Thus F contains blue $2P_3$. Hence $F \rightarrow (nK_2, 2P_3)$.

For the case of even n and $n \geq 8$, we consider the graph in Figure 1. The graph G contains $(\frac{n}{2}+1)$ disjoint cycles of length 4 and $\frac{n}{2}$ disjoint edges. Thus, the number of edges of G is $4(\frac{n}{2}+1) + \frac{n}{2} = 5(\frac{n}{2}) + 4$. Let μ be any red-blue coloring of G such that there is no red nK_2.

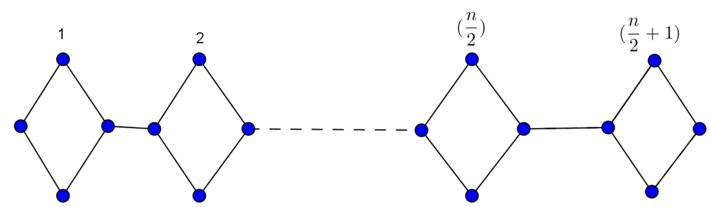


Figure 1. The graph $G \rightarrow (nK_2, 2P_3)$, for even n.

Observe that for each 4-cycle in G, we find at most two red K_2. Since G contains no red nK_2, we have at most $(\frac{n}{2}-1)$ 4-cycles containing two red K_2 and one 4-cycle containing at most one red K_2. As a consequence, we have at least one 4-cycle whose all edges are blue and one 4-cycle which at least 2 consecutive edges are blue. Since those two 4-cycles are separated by at least an edge, G contains a blue $2P_3$. Thus, $G \rightarrow (nK_2, 2P_3)$.

For the case of odd n, $n \geq 9$, let consider the graph in Figure 2. The graph F contains $\left(\frac{n+1}{2}\right)$ disjoint cycles of length 4, $(\left(\frac{n+1}{2}\right)-1)$ disjoint edges and one star $K_{1,3}$. Thus, the number of edges of F is $4\left(\frac{n+1}{2}\right) + (\left(\frac{n+1}{2}\right)-1) + 3 = 5\left(\frac{n+1}{2}\right) + 2$.

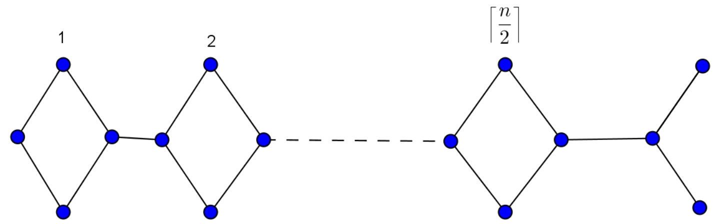


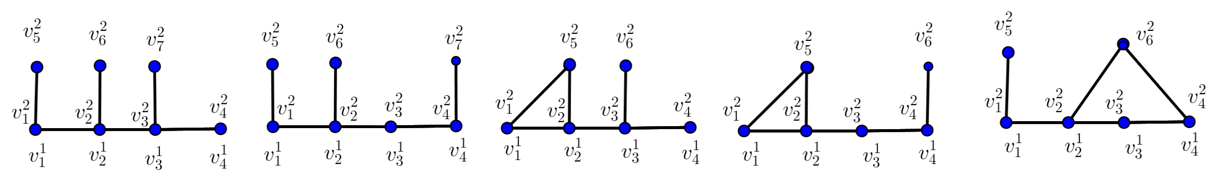
Figure 2. The graph $F \rightarrow (nK_2, 2P_3)$, for odd n.

Let μ be any red-blue coloring of F such that there is no red nK_2. By a similar argument as in the case for even n, there are at most $(\frac{n+1}{2} - 1)$ 4-cycles containing red $2K_2$. As a consequence, we have at least one 4-cycle which all edges are blue and a blue star $K_{1,3}$. Thus, $G \rightarrow (nK_2, 2P_3)$. □

Theorem 2.3. $\hat{r}_c(3K_2, 2P_3) = 10$.

Proof. According Theorem 2.2, $\hat{r}_c(3K_2, 2P_3) \leq 10$. Now, we will prove that $\hat{r}_c(3K_2, 2P_3) \geq 10$. Suppose that F is a connected graph with $|E(F)| \leq 9$. We will show that $F \nrightarrow (3K_2, 2P_3)$.
Decompose F into two connected subgraph F_1 and F_2 with $|E(F_1)| \leq 3$ and $|E(F_2)| \leq 6$. Consider that the subgraph F_1 is isomorphic to a star $K_{1,3}$ or a cycle C_3 or a path P_4. If F_1 is a star $K_{1,3}$ or a cycle C_3, then color all edges in F_1 with red. According Theorem 2.1 $\hat{r}_c(2K_2, 2P_3) = 7$, then there is a $(2K_2, 2P_3)-$ coloring in F_2. Therefore, F contains at most two red K_2 and no blue $2P_3$. So, $F \nrightarrow (3K_2, 2P_3)$.

Now, suppose that F_1 is a path P_4. We claim there are at most 2 common vertices of F_1 and F_2. Suppose there are 3 common vertices of F_2 and F_1. Consider the following graph. Let v_i^1 and v_j^2 be vertices of F_1 and F_2, respectively. Since F_2 is connected, there is a vertex v_k^2 of F_2 adjacent



to v_j^2, $j = 5$ or 6 or 7. Therefore, if we remove the vertex $v = v_2^1$, the graph $F - v$ is connected. Hence, this is the same as the previous case, namely when F_1 is a star $K_{1,3}$. So, there are at most two common vertices of F_1 and F_2, as claimed.

By Theorem 2.1, there is a $(2K_2, 2P_3)-$ coloring in F_2. Observe that, if there are at least two blue paths in F_2, the longest one is P_4. Therefore, we color two consecutive edges in F_1 with red and the other edge with blue so that the blue edge of F_1 is adjacent to the longest blue path in F_2 (if any). Otherwise, the blue edge of F_1 is adjacent to the red edges of F_2. In this coloring, F contains at most two red K_2 and no blue $2P_3$. So, $F \nrightarrow (3K_2, 2P_3)$. Thus, $\hat{r}_c(3K_2, 2P_3) \geq 10$. Combining the two inequalities, we have $\hat{r}_c(3K_2, 2P_3) = 10$.

□

Theorem 2.4. $\hat{r}_c(nK_2, 2P_3) = 3n+1$, *for* $n = 3, 4, 5, 6, 7$.

Proof. By Theorem 2.2, we obtain $\hat{r}_c(nK_2, 2P_3) \leq 3n+1$. Now, we will prove $\hat{r}_c(nK_2, 2P_3) \geq 3n+1$. Suppose that F is a connected graph with $|E(F)| \leq 3n$. We will show that $F \nrightarrow (nK_2, 2P_3)$. We proceed by induction on n. The assertion is true for $n = 3$. Furthermore, we may assume that $\hat{r}_c(kK_2, 2P_3) \geq 3k+1$, for all $n \leq k \leq 6$.

Let F' be a connected graph with $|E(F')| \leq 3(k+1)$. Decompose F' into two connected subgraphs F_1 and F_2 with $|E(F_1)| \leq 3$ and $|E(F_2)| \leq 3k$. Consider that the subgraph F_1 isomorphic to a star $K_{1,3}$ or a cycle C_3 or a path P_4. If F_1 is a star $K_{1,3}$ or a cycle C_3, then color all edges in F_1 with red. Next, by the induction hypothesis, there is a $(kK_2, 2P_3)-$ coloring in F_2. By combining the coloring in F_1 and F_2, there exists at most k red K_2 and no blue $2P_3$ in F'. So, $F \nrightarrow ((k+1)K_2, 2P_3)$.

Now, assume that F_1 is a path P_4. There are at most two common vertices of F_1 and F_2, as in the previous theorem, namely x and y. Consider $(kK_2, 2P_3)-$ coloring in F_2, that maximizes the number of red edges and minimizes the length of blue paths. If at most one of x and y is adjacent with a blue edge in F_2, then we color two consecutive edges in F_1 with red and the other edge with blue so that the blue edge in F_2 is adjacent with red edges in F_1. If both x and y are adjacent with blue edges in F_2, we claim that the longest blue path in F_2 is P_4. Suppose the longest blue path in F_2 is P_5. Let $F_2' = F_2 - P_5$. Observe that $|F_2'| \leq 3k-4$. We can view the coloring in F_2' as a chain of alternating blue and red subgraphs, starting with a blue subgraph and ending with a red subgraph. As the number of red edges is maximized, there are at least $2(k-1)$ red edges in F_2'. Thus, the number of edges in F_2' is at least $(k-1)+2(k-1) = 3k-3$, a contradiction. So, the longest blue path in F_2 is P_4, as claimed. Color two consecutive edges in F_1 with red and the other edge with blue so that the blue edge in F_1 is adjacent with the longest blue path of F_2 (if any). In this coloring, F' contains at most k red K_2 and no blue $2P_3$. So, $F' \nrightarrow ((k+1)K_2, 2P_3)$. Thus, $\hat{r}_c((k+1)K_2, 2P_3) \geq 3(k+1)+1$.

Combining the two inequalities, we conclude that $\hat{r}_c(nK_2, 2P_3) = 3n+1$, for $3 \leq n \leq 7$. □

Theorem 2.5. $\hat{r}_c(8K_2, 2P_3) = 24$.

Proof. By Theorem 2.2, we obtain $\hat{r}_c(8K_2, 2P_3) \leq 24$. Now, we will prove $\hat{r}_c(8K_2, 2P_3) \geq 24$. Suppose that F is a connected graph with $|E(F)| \leq 23$. We will show that $F \nrightarrow (8K_2, 2P_3)$. Decompose F into two connected subgraphs F_1 and F_2 with $|E(F_1)| \leq 2$ and $|E(F_2)| \leq 21$. Color all edges in F_1 with red. According to Theorem 2.4, $\hat{r}_c(7K_2, 2P_3) = 22$. Thus there is a $(7K_2, 2P_3)-$ coloring in F_2. By combining the coloring in F_1 and F_2, there are at most 7 red K_2 and no blue $2P_3$ in F. So, $F \nrightarrow (8K_2, 2P_3)$. Hence, $\hat{r}_c(8K_2, 2P_3) \geq 24$.
Combining the two inequalities, we may conclude that $\hat{r}_c(8K_2, 2P_3) = 24$. □

Theorem 2.6. *For* $m \geq 3, n \geq 3$, $\hat{r}_c(nK_2, 2K_{1,m}) = mn+m+n$.

Proof. First, we will show that $\hat{r}_c(nK_2, 2K_{1,m}) \leq mn+m+n$. Let G be a graph obtained from one cycle C_{2n+1} and $(n+1)$ stars $K_{1,m-1}$ by identifying the vertex of degree $m-1$ of $K_{1,m-1}$ to the vertices of C_{2n+1}, where two vertices of C_{2n+1} are adjacent and the other $n-1$ vertices have distance two from the other, as depicted in Figure 3. The graph G has $2n+1+(m-1)(n+1) = mn+m+n$ edges.

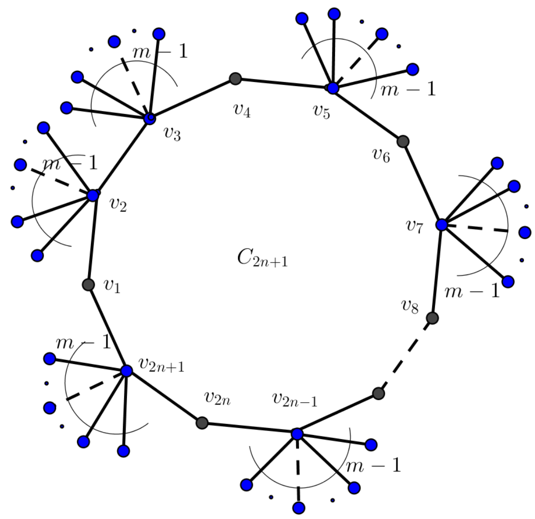


Figure 3. The graph G satisfy $G \to (nK_2, 2K_{1,m})$.

Let μ be any red-blue coloring of G such that there is no red nK_2. Then, all edges of G are colored by blue or the red subgraph G^* of G forms a path of length at most $2(n-1)$ or a subgraph containing at most $(n-1)$ stars $K_{1,i}, i \le m+1$. Let G' be a subgraph of G without edges of the red subgraph G^*. This subgraph G' forms a path of length at least 3 having at least two vertices of degree $\ge m$ or a disconnected graph containing 2 disjoint $K_{1,m}$. Hence, G contains a blue $2K_{1,m}$. So, $G \to (nK_2, 2K_{1,m})$. Thus, $\hat{r}_c(nK_2, 2K_{1,m}) \le mn+m+n$.

Now, we will show that $\hat{r}_c(nK_2, 2K_{1,m}) \ge mn+m+n$. Let G be a connected graph with $|E(G)| \le mn+m+n-1$. We will show that $G \nrightarrow (nK_2, 2K_{1,m})$. Consider the following cases.

Case 1. $\Delta(G) < m$.
Color all edges in G with blue. By this coloring, there is a $(nK_2, 2K_{1,m})-$ coloring in G.

Case 2. $\Delta(G) \ge m$.
Let A be the set of vertices of degree at least m in G. If $|A| \le n-1$, then color all edges incident with all vertices in A by red and the other edges by blue. By this coloring, there is a $(nK_2, 2K_{1,m})-$ coloring in G.
Next, we assume that $|A| \ge n$. Since $|E(G)| \le mn+m+n-1$, there are at most n disjoint $K_{1,m}$ in G, otherwise G has at least $mn+m+n$ edges, a contradiction.

Suppose G contains at most n disjoint stars $K_{1,m}$.
Let C be the set of centers of n disjoint $K_{1,m}$. Observe that, the remaining edges of G are at least m. We consider these remaining edges. If these edges induce no $K_{1,m}$, then we choose $n-1$ vertices of C and then color all edges incident with these vertices by red. Next, we color the remaining edges of G with blue. By this coloring, we obtain a $(nK_2, 2K_{1,m})$-coloring in G.

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Now, suppose these edges induce a $K_{1,m}$ with center u. Since G is connected, then at least one vertex of the $K_{1,m}$ is adjacent to a vertex of C, say v_{i_0}. Therefore, u and v_{i_0} have distance at most 2. If u is adjacent to v_{i_0}, we color all edges incident with u by red. Next, choose at most $(n-2)$ vertices of C that are different with v_{i_0} (if any) and color all edges incident with these vertices by red. By coloring all the remaining edges of G by blue, we obtain a $(nK_2, 2K_{1,m})$-coloring in G. Suppose u is not adjacent to v_{i_0}. In this case, we choose a path P_3 connecting u and v_{i_0} and color the P_3 with red. Furthermore, similar as in the previous case, choose at most $(n-2)$ vertices of C that are different with v_{i_0} (if any) and color all edges incident with these vertices by red. By giving the blue color to the remaining edges of G, we obtain a $(nK_2, 2K_{1,m})-$ coloring in G. Hence, in all cases, we have that $G \nrightarrow (nK_2, 2K_{1,m})$. □

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