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Clique roots of K_4-free chordal graphs

Hossein Teimoori Faal

Department of Mathematics and Computer Science, Allameh Tabataba'i University, Tehran, Iran.

hossein.teimoori@atu.ac.ir

Abstract

The clique polynomial $C(G, x)$ of a finite, simple and undirected graph $G = (V, E)$ is defined as the ordinary generating function of the number of complete subgraphs of G. A real root of $C(G, x)$ is called a clique root of the graph G. Hajiabolhasan and Mehrabadi showed that every simple graph G has at least a clique root in the interval $[-1, 0)$. Moreover, they showed that the class of triangle-free graphs has only clique roots. In this paper, we extend their result by showing that the class of K_4-free chordal graphs has also only clique roots. In particular, we show that this class always has a clique root -1. We conclude our paper with some interesting open questions and conjectures.

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1. Introduction and Motivation

Throughout this paper graphs are finite, simple and undirected. For the terminology and notations which are not defined here, we refer the readers to the book [1]. For a given graph $G = (V, E)$, a complete subgraph of G on k vertices is called a k-*clique*. For a subset $U \subseteq V(G)$, the subgraph induced on U will be denoted by $G[U]$. We recall that an edge which joins two vertices of a cycle but is not itself an edge of the cycle is called a *chord* of the cycle. A graph is *chordal* if each cycle of length at least four has a chord. We also recall that the *clique polynomial* [2] of the graph G,

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denote it by $C(G,x)$ is defined, as follows

$$C(G,x) := 1 + \sum_{\emptyset \neq U \subseteq V(G) : G[U] \text{ is a clique}} x^{|U|}, \tag{1}$$

or equivalently

$$C(G,x) := 1 + \sum_{k=1}^{\omega(G)} c_k(G), \tag{2}$$

in which $c_k(G)$ is the number of k-cliques in G and $\omega(G)$ is the size of the largest clique of G. From now on, any *real* root of a clique polynomial $C(G,x)$ will be called the *clique root* of G. Hajiabolhasan and Mehrabadi [2] showed that the clique polynomial of every simple graph always has a clique root in the interval $[-1,0)$. They also showed that the class of triangle-free graphs has only clique roots.

Polynomials with only real roots arise often in *graph theory* and other branches of mathematics. Such polynomials with *positive* coefficients have attracted the attention of many researchers because of their implication on *unimodality* and *log-concavity*.

Our main goal here is to contribute in this active line of research by extending the above result for the class of K_4-free chordal graphs. More precisely, we will prove the following.

Theorem 1.1. *The class of K_4-free connected chordal graphs has only clique roots. In particular, this class always has a clique root -1.*

We will also give the following immediate corollary of the above theorem, which is indeed a new algebraic proof of Turan's graph theorem for planar K_4-free graphs.

Corollary 1.1. *If G is a K_4-free connected graph with n vertices and m edges, then we have*

$$m \leq \frac{n^2}{3}.$$

2. Chordal Graphs and Clique Polynomials

In this section, we investigate the important class of *chordal graphs*. This class of graphs is very important in computer science, specially from the computational complexity view point. Many hard problems in general graphs have easy solutions in the class of chordal graphs. As we will see, the clique polynomial of chordal graphs can give us important insights into the structure of these graphs.

Definition 2.1. *A graph is* chordal *if every cycle of length greater than three has a chord. A vertex of a graph is* simplicial *if its neighbors induces a clique in the graph.*

One of the important properties of a chordal graph is that it always has a *clique decomposition*. For the sake completeness, here we quickly review the idea of decomposing a chordal graph into cliques. For detailed information, one can refer to [3].

Definition 2.2. *For given graphs G_1 and G_2, we say that a graph G arises from G_1 and G_2* by pasting along *S if we have $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = S$. In this case, the graphs G_i are called the* simplicial summands *of G.*

Remark 2.1. From the above definition, it is clear that a graph is chordal if it can be constructed recursively by pasting complete graphs along cliques. It is not hard to see that this process is independent of the order in which complete graphs paste to each other. Indeed, this recursive construction gives us a clique decomposition of chordal graphs which is essential to obtain their clique polynomials.

For simplicity of arguments, we use the notation $G_1 \cup_S G_2$ whenever G_1 and G_2 are pasted along S. The following lemma is key to obtain an explicit formula for the clique polynomial of chordal graphs. The proof is straight forward application of the *inclusion-exclusion principle* and left to the reader as a simple exercise.

Proposition 2.1. *Let G_1 and G_2 be two simple graphs and $G = G_1 \cup_Q G_2$ be their pasting along an i - clique Q. Then, we have*

$$C(G,x) = C(G_1,x) + C(G_2,x) - (x+1)^i, \qquad (i \geq 1). \tag{3}$$

By the successive application of the formula (3) and the recursive construction of chordal graphs, we can obtain the following explicit formula for the clique polynomial of chordal graphs.

Theorem 2.1. *Let G be a chordal graph defined as a pasting of the complete graphs $\{G_i\}_{i=1}^r$ of sizes n_i's, respectively. That is, $G = G_1 \cup_{Q_1} G_2 \cup_{Q_2} \cdots \cup_{Q_{r-1}} G_r$, where $\{Q_j\}_{j=1}^{r-1}$ are cliques of sizes l_j's, respectively. Then, we have*

$$C(G,x) = \sum_{i=1}^{r} (x+1)^{n_i} - \sum_{j=1}^{r-1} (x+1)^{l_j}. \tag{4}$$

As an immediate consequence of the above theorem, we have the following interesting result.

Corollary 2.1. *Every chordal graph G without isolated vertices always has a clique root -1. The multiplicity of this root is equal to the size of the smallest clique in the pasting process of the recursive construction of G.*

Remark 2.2. It is worth to note that the notation $G_1 \cup_Q G_2$ is also called the *clique-sum* of two graphs G_1 and G_2. Indeed, based on the formula 3, the clique-sum of graphs preserve the property of having (at least) a clique root -1. Unfortunately, the (ordinary) sum of two graphs which is also called the (disjoint) union of two graphs has not this property. For instance, the graph G obtained by the sum of a 1-clique (an isolated vertex) and a 2-clique (an edge) with the clique polynomial $C(G,x) = 1 + 3x + x^2$ has no clique root -1. Therefore, the clique polynomial of a graph is not *additive* over it's *connected components*. Thus, from now on, we will concentrate on *connected* chordal graphs.

3. Proofs of the Main Results

Here, we first give a proof of the following proposition which is a weaker version of Theorem 1.1. From now on, we will assume that our graphs are connected. Before proceeding the proof, we need to have a quick review of *breadth-first search* (BFS, for short) form the theory of graph algorithms.

Recall that for a given graph $G = (V, E)$ and a root (node) $r \in V$, the breadth-first search produces a *search-tree* T by exploring first the neighbors of r, then the neighbors of it's children. A tree obtained by running a breadth-first search is called a *breadth-first search tree* (*BFS-tree*, for short) rooted at r. It can be easily shown that when G is a connected graph, then the BFS-tree of G is indeed a *spanning tree* of G rooted at r.

A nice property of a breadth-first search tree is that it can give the *distance* from the root r to all other vertices. Therefore we just have to have a value $l(v)$ to every vertex called *level* of v which corresponds to $distT(r, v)$ (the length of the unique path form the root r to the vertex v in the BFS-tree T). Moreover, for any *non-tree* edge $e \in E(G)$ (an edge e which is not in $E(T)$), it's end vertices can only lie on the same level or the consecutive levels.

Proposition 3.1. *The class of K_4-free planar chordal graphs has only clique roots. In particular, -1 is always a clique root.*

Proof. For a given K_4-free planar graph G, by *Euler Formula*, we have

$$n - m + f = 2, \tag{5}$$

where n, m and f are the number of vertices, edges and faces of a *planar embedding* of G, respectively. Moreover, if G is a chordal graph, then we observe that

$$f = t + 1, \tag{6}$$

where t is the number of triangles (triangular faces) of G. Hence, form (5) and (6), we conclude that

$$n - m + t = 1,$$

or equivalently

$$1 - n + m - t = 0. \tag{7}$$

By last identity and considering the fact that the clique polynomial of a K_4-free graph G is $C(G, x) = 1 + nx + mx^2 + tx^3$, we get

$$C(G, -1) = 1 - n + m - t = 0.$$

That is, every K_4-free planar chordal graph G always has -1 as a clique root. Therefore, we obtain the following multiplicative decomposition of $C(G, x)$

$$C(G, x) = (1 + x)(1 + (n - 1)x + (m - n + 1)x^2). \tag{8}$$

The final step of the proof is to show that the quadratic polynomial:

$$Q(G,x) = 1 + (n-1)x + (m-n+1)x^2, \tag{9}$$

always has a real root. To this end, we actually prove that $Q(G,x)$ is a clique polynomial of a *triangle-free* graph $\tilde{G}$ which can be obtained from the original graph G based on the idea of the BFS-tree of G.

For a given K_4-free chordal graph G, pick up an arbitrary vertex $r \in V(G)$. Now, we construct a BFS-tree of G rooted at the vertex r. We will denote it by T_G. Clearly, this tree has n vertices and $n-1$ edges. Now in the graph G, delete all $n-1$ edges of the tree T_G and call the resulting graph $\hat{G}$. This graph has clearly n vertices and $m-(n-1)$ edges. By the construction, it is clear that v is an isolated vertex of $\hat{G}$. Next, we prove that $\hat{G}$ is a triangle - free graph.

Here is the argument. In contrary, let $\Delta = abc$ be a triangle in $\hat{G}$. Then, the end vertices of it's non-tree edges can only lie on the same level or the consecutive levels. Hence, at least two vertices of Δ (let say $e = ab$) lies in the same level. For simplicity of arguments, we will assume that the vertex a appears in BFS-tree before the vertex b. For a given vertex $v \in V$, the unique (shortest) paths form the root r to the vertex v is denoted by $path(r,v)$. Now, by the construction of the BFS-tree of G, there is a vertex c' (in the same level as c) such that $c'a, c'b \in E(T)$. Now, let r' be the last intersection point of the paths (starting from the root r) $path(r,c')$ and $path(r,c)$. By the construction of BFS-tree and chordality of the graph G, since the cycle $\mathcal{C}$ based on the four vertices r', c, b, c' is of length greater than 3, we conclude that $c'c \in E(G)$. Thus, it implies that the graph G contains a 4-clique $abcc'$. This contradicts the fact that the graph G is K_4-free. This completes our argument by the contradiction.

Thus, the triangle-free graph $\tilde{G}$ obtained from $\hat{G}$ by deleting the vertex v has $n-1$ vertices and $m-n+1$ edges, as required. □

Remark 3.1. The following figure illustrates the process of obtaining the triangle-free graph $\tilde{G}$ from the original graph G for a sample K_4-free connected chordal graph G.

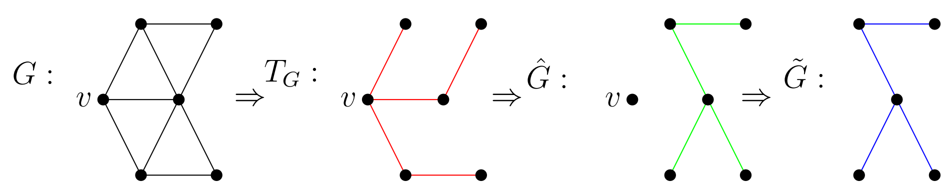

Figure 1. The construction of the triangle-free graph $\tilde{G}$.

Now, we are ready to give a proof of our main theorem.

Proof of Theorem 1.1. We first note that by Corollary 2.1, every connected chordal graph always has a clique root -1. Now, the rest of the proof is similar to the proof of Proposition 3.1. □

Next we give an algebraic proof of the *Turan's Graph Theorem* [4] for K_4 - free graphs which is indeed Corollary 1.1.

Proof of Corollary 1.1. We first note that since we want to prove an upper bound for the maximum possible number of edges, without loss of generality we can assume that the graph G is chordal. By the proof of Theorem 1.1, we already know that if G is a given K_4 - free chordal graph with n vertices and m edges, then the following *quadratic* equation

$$Q(G,x) = 1 + (n-1)x + (m-n+1)x^2,$$

has only real zeros. Hence, it's *discriminant* is nonnegative and therefore we have the following inequality:

$$(n-1)^2 - 4(m-n+1) \geq 0,$$

which is equivalent to

$$m \leq \left(\frac{n+1}{2}\right)^2 - 1. \tag{10}$$

On the other hand, we have the inequality

$$\left(\frac{n+1}{2}\right)^2 - 1 \leq \frac{n^2}{3}, \tag{11}$$

which is equivalent to the obvious inequality $(n-3)^2 \geq 0$. Thus, the inequalities (10) and (11) immediately imply the Turan's inequality for K_4 - free graphs. □

4. Open problems and questions

We already showed that the class of K_4 - free connected chordal graphs has only clique roots. Now, one might ask whether the class of K_5 - free connected chordal graphs has the same property or not. Unfortunately, this is not true in general. For example, the connected graph K_4^+ (a complete graph K_4 plus one edge) has only two clique roots. Indeed, we have

$$C(K_4^+) = 1 + 5x + 7x^2 + 4x^3 + x^4 = (1+x)(1+4x+3x^2+x^3).$$

Since the cubic polynomial $\phi(x) = 1 + 4x + 3x^2 + x^3$ has the first derivative $\phi(x)' = 3(x+1)^2 + 1$ which is always *positive*, by the first derivative criteria, $\phi(x) = 1 + 4x + 3x^2 + x^3$ has exactly *one* real root. Thus, we come up with the following first open question.

Problem 1. *Which subclasses of K_5 - free chordal graphs have only clique roots?*

Recall that the class of 3 - trees are those graphs which can be constructed *recursively* by starting with a complete graph K_4, and then *repeatedly* adding vertices in such a way that each added vertex has exactly *three* neighbors that form a clique (triangle).
By the above definition, it is not hard to see that the class of 3-trees is a subclass of K_5-free chordal graphs. Next, we come up with the following conjecture.

Conjecture 1. *The class of 3-trees has only clique roots.*

Considering the fact that any connected chordal graph has a clique root -1 and the recursive definition of of chordal graphs, we made the following stronger conjecture.

Conjecture 2. *The class of K_5-free connected chordal graphs with a clique root -1 of multiplicity 2 has only clique roots.*

Considering the above discussions, we also come up with the following open question.

Problem 2. *Which subclasses of $K_{\omega+3}$-free connected graphs ($\omega > 0$) have only clique roots?*

We call an i-clique an *isolated clique* if the intersection of all neighborhoods of it's vertices is an *empty* set. In particular, an isolated 1-clique is called an isolated vertex.
The interesting point is that we strongly believe that the following much stronger conjecture is true.

Conjecture 3. *The class of $K_{\omega+3}$-free graphs without isolated ω-cliques has only clique roots.*

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