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On imbalances in multipartite multidigraphs

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Abstract

A k-partite r-digraph (multipartite multidigraph) (or briefly MMD)($k \geq 3$, $r \geq 1$) is the result of assigning a direction to each edge of a k-partite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. Let $D(X_1, X_2, \cdots, X_k)$ be a k-partite r-digraph with parts $X_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$, $1 \leq i \leq k$. Let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$ be respectively the outdegree and indegree of a vertex x_{ij} in X_i. Define $a_{x_{ij}}$ (or simply a_{ij}) as $a_{ij} = d^+_{x_{ij}} - d^-_{x_{ij}}$ as the imbalance of the vertex x_{ij}, $1 \leq j \leq n_i$. In this paper, we characterize the imbalances of k-partite r-digraphs and give a constructive and existence criteria for sequences of integers to be the imbalances of some k-partite r-digraph. Also, we show the existence of a k-partite r-digraph with the given imbalance set.

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1. Introduction

A digraph without loops and without multi-arcs is called a simple digraph. The imbalance of a vertex v_i in a digraph as a_{v_i} (or simply a_i)= $d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. For undefined terms, we refer to [12].

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The following result [5] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 1. *A sequence of integers $A = [a_1, a_2, \cdots, a_n]$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ is an imbalance sequence if and only if it has sum zero and satisfies*

$$\sum_{i=1}^{k} a_i \leq k(n-k),$$

for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Corollary 2. *A sequence of integers $A = [a_1, a_2, \cdots, a_n]$ with $a_1 \leq a_2 \leq \cdots \leq a_n$ is an imbalance sequence of a simple digraph if and only if for $1 \leq k < n$*

$$\sum_{i=1}^{k} a_i \geq k(k-n),$$

with equality when $k = n$.

Various results for imbalances in simple digraphs, bipartite and multipartite digraphs can be seen in [6, 8, 9, 27] while as characterization of imbalances in multidigraphs, bipartite multitournaments and tripartite multidigraphs can be found in [7, 10, 11].

An r-digraph ($r \geq 1$) is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. Clearly 1-digraph is an oriented graph and complete 1-digraph is a tournament. Various results on scores and marks on tournaments and digraphs can be seen in [2, 13, 14, 15, 16, 17, 18, 24, 25] and on scores and degrees in hypertournaments can be found in [1, 3, 4, 19, 20, 21, 22, 23, 26]. The definition of an imbalance of a vertex in r-digraph is same as in a simple digraph. The following characterization for imbalances in multidigraphs can be found in Pirzada et al. [7].

Theorem 3. *A sequence $A = [a_1, a_2, \cdots, a_p]$ of integers in non-decreasing order is the imbalance sequence of an r-digraph if and only if*

$$\sum_{i=1}^{k} a_i \geq rk(k-n),$$

for $1 \leq k \leq n$ with equality when $k = n$.

2. Imbalance sequences in multipartite multidigraphs

A k-partite r-digraph ($k \geq 3$, $r \geq 1$) (or briefly MMD) is the result of assigning a direction to each edge of a k-partite multidigraph which is without loops and having at most r edges between every pair of vertices, one vertex from each part. We note that k-partite 1-digraph is an oriented k-partite graph and complete k-partite 1-digraph is a k-partite tournament. The case when $k = 1$ gives multidigraph and when $k = 2$ gives bipartite multidigraph, and these have been studied in [9]. Thus throughout this paper the MMD under discussion will have three or more than three parts. Let $D(X_1, X_2, \cdots, X_k)$ (or briefly D) be an MMD with parts $X_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$, $1 \leq i \leq k$. For any vertex x_{ij} in D, let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$ denote the outdegree and indegree. Define $a_{x_{ij}}$ (or simply a_{ij})= $d^+_{x_{ij}} - d^-_{x_{ij}}$ as the imbalance of the vertiex x_{ij}, where $1 \leq j \leq n_i$. Note that we have $1 \leq i \leq k$ and $1 \leq j \leq n_i$, unless otherwise stated.

Let D be an MMD and $x_{ij} \in X_i$, $x_{mn} \in X_m$ and $x_{st} \in X_s$, where $i \neq m$, $m \neq s$ and $s \neq i$. If there are f arcs directed from vertex x_{ij} to x_{mn} and g arcs directed from x_{mn} to x_{ij}, with $0 \leq f \leq r$, $0 \leq g \leq r$ and $0 \leq f + g \leq r$, it is denoted by $x_{ij}(f - -g)x_{mn}$. An r-triple is an induced r-subdigraph of three vertices with one vertex from distinct part and is of the form $x_{ij}(f_1 - -f_2)x_{mn}(g_1 - -g_2)x_{st}(h_1 - -h_2)x_{mn}$, where $0 \leq f_1, f_2, g_1, g_2, h_1, h_2, f_1 + f_2, g_1 + g_2, h_1 + h_2 \leq r$. Also an oriented triple (or 1-triple) in D is an induced 1-subdigraph of three vertices, with one vertex each from distinct part. An oriented triple is said to be transitive if it is of the form $x_{ij}(1 - -0)x_{mn}(1 - -0)x_{st}(0 - -1)x_{ij}$, or $x_{ij}(1 - -0)x_{mn}(0 - -1)x_{st}(0 - -0)x_{ij}$, or $x_{ij}(1 - -0)x_{mn}(0 - -0)x_{st}(0 - -1)x_{ij}$, or $x_{ij}(1 - -0)x_{mn}(0 - -0)x_{st}(0 - -0)x_{ij}$, or $x_{ij}(0 - -0)x_{mn}(0 - -0)x_{st}(0 - -0)x_{ij}$, otherwise it is intransitive. An MMD D is said to be transitive if each of its oriented triple is transitive. In particular, an r-triple C in an MMD D is transitive if every oriented triple of C is transitive.

We have the following observation.

Theorem 4. *Let D and D' be two MMD with the same imbalance sequences. Then D can be transformed to D' by successively transforming*
(i) appropriate triples in one of the following ways.
Either (a) by changing an intransitive oriented triple(cyclic triple) $x_{ij}(1 - -0)x_{mn}(1 - -0)x_{st}(1 - -0)x_{ij}$ to a transitive oriented triple $x_{ij}(0 - -0)x_{mn}(0 - -0)x_{st}(0 - -0)x_{ij}$ which has the same imbalance sequences, or vice versa,
or (b) by changing an intransitive oriented triple $x_{ij}(1 - -0)x_{mn}(1 - -0)x_{st}(0 - -0)x_{ij}$ to a transitive oriented triple $x_{ij}(0 - -0)x_{mn}(0 - -0)x_{st}(0 - -1)x_{ij}$ which has the same imbalance sequences, or vice versa.
or (ii) by changing the symmetric arcs $x_{ij}(1 - -1)x_{mn}$ to $x_{ij}(0 - -0)x_{mn}$, which has the same imbalance sequences, or vice versa.

Proof. This can be easily established. □

As a consequence of Theorem 3, we observe that among all MMD's having same imbalance sequences those with least number of arcs are transitive in nature.

Corollary 5. *Among all MMD's with given imbalance sequences, those with the fewest arcs are transitive.*

In an MMD, a vertex with indegree zero is called a transmitter. Evidently in a transitive oriented MMD with imbalance sequences $A_i = [a_{i1}, a_{i2}, \cdots, a_{in_i}]$, any of the vertices with imbalances a_{in_i}, can act as a transmitter.

The next result provides a useful recursive test of checking whether the sequences of integers are the imbalance sequences of an MMD.

The following result is a combinatorial criterion for determining whether k sequences of integers are realizable as imbalances.

Theorem 6. *The k sequences of integers $A_i = [a_{i1}, a_{i2}, \cdots, a_{in_i}]$, $1 \leq i \leq k$, in non-decreasing order are imbalance sequences of an MMD if and only if*

$$\begin{aligned} &\sum_{j=1}^{m_1} a_{1j} + \sum_{j=1}^{m_2} a_{2j} + \cdots + \sum_{j=1}^{m_k} a_{kj} \\ &\leq r\left[m_1 \prod_{j=1, j\neq 1}^{k} (n_j - m_j) + m_2 \prod_{j=1, j\neq 2}^{k} (n_j - m_j) + \cdots + m_k \prod_{j=1, j\neq k}^{k} (n_j - m_j)\right] \end{aligned} \qquad (1)$$

or briefly

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} a_{ij} \leq r\left[\sum_{i=1}^{k} m_i \left(\prod_{j=1, j\neq i}^{k} (n_j - m_j)\right)\right],$$

for all sets of k integers m_i, $1 \leq m_i \leq n_i$, with equality when $m_i = n_i$, for all i.

Proof. The necessity of the condition follows from the fact that a directed multipartite multi subgraph of an MMD induced by m_i vertices for all i, $1 \leq i \leq k$, $1 \leq m_i \leq n_i$ has a sum of imbalances zero and these vertices can gather at most

$$r\left[m_1 \prod_{j=1, j\neq 1}^{k} (n_j - m_j) + m_2 \prod_{j=1, j\neq 2}^{k} (n_j - m_j) + \cdots + m_k \prod_{j=1, j\neq k}^{k} (n_j - m_j)\right]$$

imbalances from the remaining $n_i - m_i$ vertices.

For sufficiency, assume that $A_i = [a_{i1}, a_{i2}, \cdots, a_{in_i}]$ be k sequences of integers in non-decreasing order satisfying conditions (1) but are not imbalance sequences of any MMD. Let these sequences be chosen in such a way that n_i, are the smallest possible and a_{11} is the least for the choice of n_i.

We consider the following two cases.
Case (i). Suppose equality in (1) holds for $m_j \le n_j$, for $1 \le j \le k$ so that

$$\sum_{j=1}^{m_1} a_{1j} + \sum_{j=1}^{m_2} a_{2j} + \cdots + \sum_{j=1}^{m_k} a_{kj}$$
$$= r\left[m_1 \prod_{j=1, j\neq 1}^{k} (n_j - m_j) + m_2 \prod_{j=1, j\neq 2}^{k} (n_j - m_j) + \cdots + m_k \prod_{j=1, j\neq k}^{k} (n_j - m_j)\right].$$

Consider the following k sequences.

$$A'_1 = \left[a'_{1j}\right]_{j=1}^{m_1} = \left[a_{1j} - r \prod_{t=1, t\neq 1}^{k} (n_t - m_t)\right]_{j=1}^{m_1},$$

$$A'_2 = \left[a'_{2j}\right]_{j=1}^{m_2} = \left[a_{2j} - r \prod_{t=1, t\neq 2}^{k} (n_t - m_t)\right]_{j=1}^{m_2},$$

$$\vdots$$

$$A'_k = \left[a'_{kj}\right]_{j=1}^{m_k} = \left[a_{kj} - r \prod_{t=1, t\neq k}^{k} (n_t - m_t)\right]_{j=1}^{m_k}$$

For $1 \le r_i < m_i$ for all $1 \le i \le k$, we have

$$\sum_{i=1}^{k}\sum_{j=1}^{r_i} a'_{ij} = \sum_{j=1}^{r_1} a'_{1j} + \sum_{j=1}^{r_2} a'_{2j} + \cdots + \sum_{j=1}^{r_k} a'_{kj}$$
$$= \sum_{j=1}^{r_1}\left(a_{1j} - r \prod_{t=1, t\neq 1}^{k} (n_t - m_t)\right) + \sum_{j=1}^{r_2}\left(a_{2j} - r \prod_{t=1, t\neq 2}^{k} (n_t - m_t)\right)$$
$$+ \cdots + \sum_{j=1}^{r_k}\left(a_{kj} - r \prod_{t=1, t\neq k}^{k} (n_t - m_t)\right)$$
$$= \left(\sum_{j=1}^{r_1} a_{1j} + \sum_{j=1}^{r_2} a_{2j} + \cdots + \sum_{j=1}^{r_k} a_{kj}\right)$$
$$- r\left(r_1 \prod_{t=1, t\neq 1}^{k} (n_t - m_t) + r_2 \prod_{t=1, t\neq 2}^{k} (n_t - m_t) + \cdots + r_k \prod_{t=1, t\neq k}^{k} (n_t - m_t)\right)$$

$$\leq r\left(r_1\prod_{j=1,j\neq 1}^{k}(n_j-r_j)+r_2\prod_{j=1,j\neq 2}^{k}(n_j-r_j)+\cdots+r_k\prod_{j=1,j\neq k}^{k}(n_j-r_j)\right)$$
$$-r\left(r_1\prod_{t=1,t\neq 1}^{k}(n_t-m_t)+r_2\prod_{t=1,t\neq 2}^{k}(n_t-m_t)+\cdots+r_k\prod_{t=1,t\neq k}^{k}(n_t-m_t)\right)$$
$$\leq r\left(r_1\prod_{t=1,t\neq 1}^{k}(m_t-r_t)+r_2\prod_{t=1,t\neq 2}^{k}(m_t-r_t)+\cdots+r_k\prod_{t=1,t\neq k}^{k}(m_t-r_t)\right)$$
$$=r\sum_{i=1}^{k}r_i\left(\prod_{t=1,t\neq i}^{k}(m_t-r_t)\right)$$

and

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i}a'_{ij}=\sum_{j=1}^{m_1}a'_{1j}+\sum_{j=1}^{m_2}a'_{2j}+\cdots+\sum_{j=1}^{m_k}a'_{kj}$$
$$=\sum_{j=1}^{m_1}\left(a_{1j}-r\prod_{t=1,t\neq 1}^{k}(n_t-m_t)\right)+\sum_{j=1}^{m_2}\left(a_{2j}-r\prod_{t=1,t\neq 2}^{k}(n_t-m_t)\right)$$
$$+\cdots+\sum_{j=1}^{m_k}\left(a_{kj}-r\prod_{t=1,t\neq k}^{k}(n_t-m_t)\right)$$
$$=\left(\sum_{j=1}^{m_1}a_{1j}+\sum_{j=1}^{m_2}a_{2j}+\cdots+\sum_{j=1}^{m_k}a_{kj}\right)$$
$$-r\left(m_1\prod_{t=1,t\neq 1}^{k}(n_t-m_t)+m_2\prod_{t=1,t\neq 2}^{k}(n_t-m_t)+\cdots+m_k\prod_{t=1,t\neq k}^{k}(n_t-m_t)\right)$$
$$=r\left(m_1\prod_{j=1,j\neq 1}^{k}(n_j-m_j)+m_2\prod_{j=1,j\neq 2}^{k}(n_j-m_j)+\cdots+m_k\prod_{j=1,j\neq k}^{k}(n_j-m_j)\right)$$
$$-r\left(m_1\prod_{t=1,t\neq 1}^{k}(n_t-m_t)+m_2\prod_{t=1,t\neq 2}^{k}(n_t-m_t)+\cdots+m_k\prod_{t=1,t\neq k}^{k}(n_t-m_t)\right)$$
$$=r\sum_{i=1}^{k}m_i\left(\prod_{j=1,j\neq i}^{k}(n_j-m_j)\right)-r\sum_{i=1}^{k}m_i\left(\prod_{t=1,t\neq i}^{k}(n_t-m_t)\right)$$
$$=0.$$

Thus the k sequences $A'_i=\left[a'_{ij}\right]_{j=1}^{m_i}$, for all $1\leq i\leq k$ satisfy (1) and by the minimality of n_i the sequences A'_i are the imbalance sequences of some MMD $D'(V'_1,V'_2,\cdots,V'_k,E')$.

Now consider the following k sequences.

$$A_1'' == \left[a_{1(m_1+j)} + r\prod_{t=1,t\neq 1}^{k} m_t\right]_{j=1}^{n_1}$$

$$A_2'' = \left[a_{2(m_2+j)} + r\prod_{t=1,t\neq 2}^{k} m_t\right]_{j=1}^{m_2},$$

$$\vdots$$

$$A_k'' = \left[a_{k(m_k+j)} + r\prod_{t=1,t\neq k}^{k} m_t\right]_{j=1}^{m_k}$$

For all $1 \leq i \leq k$ and $1 \leq r_i \leq n_i - m_i$, we have

$$\sum_{i=1}^{k}\sum_{j=1}^{r_i} a''_{ij} = \sum_{j=1}^{r_1} a''_{1j} + \sum_{j=1}^{r_2} a''_{2j} + \cdots + \sum_{j=1}^{r_k} a''_{kj}$$
$$= \sum_{j=1}^{r_1}\left(a_{1(m_1+j)} + r\prod_{t=1,t\neq 1}^{k} m_t\right) + \sum_{j=1}^{r_2}\left(a_{2(m_2+j)} + r\prod_{t=1,t\neq 2}^{k} m_t\right)$$
$$+ \cdots + \sum_{j=1}^{r_k}\left(a_{k(m_k+j)} + r\prod_{t=1,t\neq k}^{k} m_t\right)$$

$$= \sum_{j=1}^{r_1}\left(a_{1(m_1+j)} + r\prod_{t=1,t\neq 1}^{k} m_t\right) + \sum_{j=1}^{r_2}\left(a_{2(m_2+j)} + r\prod_{t=1,t\neq 2}^{k} m_t\right)$$
$$+ \cdots + \sum_{j=1}^{r_k}\left(a_{k(m_k+j)} + r\prod_{t=1,t\neq k}^{k} m_t\right)$$
$$= \left(\sum_{j=1}^{m_1+r_1} a_{1j} + \sum_{j=1}^{m_2+r_2} a_{2j} + \cdots + \sum_{j=1}^{m_k+r_k} a_{kj}\right)$$
$$- \left(\sum_{j=1}^{m_1} a_{1j} + \sum_{j=1}^{m_2} a_{2j} + \cdots + \sum_{j=1}^{m_k} a_{kj}\right)$$
$$+ r\left(r_1\prod_{t=1,t\neq 1}^{k} m_t + r_2\prod_{t=1,t\neq 2}^{k} m_t + \cdots + r_k\prod_{t=1,t\neq k}^{k} m_t\right)$$

Therefore

$$\sum_{i=1}^{k}\sum_{j=1}^{r_i} a''_{ij}$$
$$\leq r(m_1+r_1)\prod_{j=1,j\neq 1}^{k}(n_t-(m_t+r_t)) + r(m_2+r_2)\prod_{j=1,j\neq 2}^{k}(n_t-(m_t+r_t))$$
$$+\cdots + r(m_k+r_k)\prod_{j=1,j\neq k}^{k}(n_t-(m_t+r_t))$$
$$-r\left(m_1\prod_{t=1,t\neq 1}^{k}(n_t-m_t) + m_2\prod_{t=1,t\neq 2}^{k}(n_t-m_t)+\cdots+m_k\prod_{t=1,t\neq k}^{k}(n_t-m_t)\right)$$
$$+r\left(r_1\prod_{t=1,t\neq 1}^{k}m_t + r_2\prod_{t=1,t\neq 2}^{k}m_t+\cdots+r_k\prod_{t=1,t\neq k}^{k}m_t\right)$$
$$\leq r\left(r_1\prod_{t=1,t\neq 1}^{k}(n_t-m_t-r_t) + r_2\prod_{t=1,t\neq 2}^{k}(n_t-m_t-r_t)+\cdots+r_k\prod_{t=1,t\neq k}^{k}(n_t-m_t-r_t)\right)$$
$$= r\sum_{i=1}^{k} r_i\left(\prod_{t=1,t\neq i}^{k}(n_t-m_t-r_t)\right)$$

with equality when $r_i = n_i - m_i$ for all i, $1 \leq i \leq k$. Therefore by the minimality of n_i, the k sequences A''_i form the imbalance sequences of some MMD $D''(V''_1, V''_2, \cdots, V''_k, E'')$.

Construct an MMD $D(V_1, V_2, \cdots, V_k, E)$ as follows.

Let $V_1 = V'_1 \cup V''_1$, $V_2 = V'_2 \cup V''_2$, $\cdots$, $V_k = V'_k \cup V''_k$ and $V'_i \cap V''_j = \phi$ for all i, j. The arc set E contains (a) all the arcs between V'_i and V'_j, $i \neq j$, (b) all arcs between V''_i and V''_j, $i \neq j$ and (c) r arcs from each vertex of V'_i to every vertex of V''_j.

Then $D(V_1, V_2, \cdots, V_k)$ has imbalance sequences A_i, which is a contradiction. Clearly this contradicts the given assumption.

Case (ii). Assume that the strict inequality holds in (1) for $m_i \neq n_i$. Let $A_1 = [a_{11} - 1, a_{12}, \cdots, a_{1(n_1-1)}, a_{1n_1} + 1]$ and Let $A_j = [a_{j1}, a_{j2}, \cdots, a_{jn_j}]$ for all j, $2 \leq j \leq k$, so that A_i satisfy the conditions (1). Therefore by the minimality of a_{11}, the sequences A_i are the imbalance sequences of some MMD $D_1(Y_1, Y_2, \cdots, Y_k)$. Let $a_{x_{11}} = a_{11} - 1$ and $a_{x_{1n_1}} = a_{1n_1} + 1$. Since $a_{x_{1n_1}} > a_{x_{11}} + 1$, there exists a vertex x_{jp} in Y_j, $2 \leq j \leq k$, $1 \leq p \leq n_j$, such that $x_{1n_1}(0--0)x_{jp}(1--0)x_{11}$, or $x_{1n_1}(1--0)x_{jp}(0--0)x_{11}$, or $x_{1n_1}(1--0)x_{jp}(1--0)x_{11}$, or $x_{1n_1}(0--0)x_{jp}(0--0)x_{11}$ in $D_1(Y_1, Y_2, \cdots, Y_k)$, and if these are changed to $x_{1n_1}(0--1)x_{jp}(0--0)x_{11}$, or $x_{1n_1}(0--0)x_{jp}(0--1)x_{11}$, or $x_{1n_1}(0--0)x_{jp}(0--0)x_{11}$, or $x_{1n_1}(0--1)x_{jp}(0--1)x_{11}$ respectively, the result is an MMD with imbalance sequences A_i, which is again a contradiction. This completes the proof. □

3. Imbalance sets in multipartite multidigraphs

The set of distinct imbalances of the vertices in an MMD is called its imbalance set. Now we give the existence of an MMD with a given imbalance set.

Theorem 8. *Let $P = \{p_1, p_2, \cdots, p_m\}$ and $Q = \{-q_1, -q_2, \cdots, -q_n\}$, where $p_1, p_2, \cdots, p_m$, $q_1, q_2, \cdots, q_n$ are positive integers with $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_n$ and the greatest common divisor $(p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n) = t$, where $1 \leq t \leq r$. Then there exists an MMD with imbalance set $P \cup Q$.*

Proof. Since $(p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n) = t$ there exist positive integers $f_1, f_2, \cdots, f_m$, g_1, $g_2, \cdots, g_n$ with $f_1 < f_2 < \cdots < f_m$ and $g_1 < g_2 < \cdots < g_n$ such that $p_i = tf_i$ for $1 \leq i \leq m$ and $q_j = tg_j$ for $1 \leq j \leq n$. First assume that k is even, say $k = 2s$ where $s \geq 1$. Construct an MMD $D(X_1, X_2, \cdots, X_k)$ as follows. Let

$$X_1 = X_{11} \cup X_{12} \cup \cdots \cup X_{1m} \cup X_1^1 \cup X_1^2 \cup \cdots \cup X_1^n,$$

$$X_2 = X_{21} \cup X_{22} \cup \cdots \cup X_{2m} \cup X_2^1 \cup X_2^2 \cup \cdots \cup X_2^n,$$

$$X_3 = X_{31} \cup X_{32} \cup \cdots \cup X_{3m} \cup X_3^1 \cup X_3^2 \cup \cdots \cup X_3^n,$$

$$\cdots$$

$$X_{2s} = X_{2s1} \cup X_{2s2} \cup \cdots \cup X_{2sm} \cup X_{2s}^1 \cup X_{2s}^2 \cup \cdots \cup X_{2s}^n,$$

with $X_{ij} \cap X_{uv} = \emptyset$, $X_{ij} \cap X_u^v = \emptyset$, $X_i^j \cap X_u^v = \emptyset$, $|X_{1i}| = g_1$ for all i, $1 \leq i \leq m$, $|X_1^i| = g_i$ for all i, $1 \leq i \leq n$,$|X_{2i}| = f_i$ for all i, $1 \leq i \leq m$, $|X_2^i| = f_1$ for all i, $1 \leq i \leq n$, $|X_{i1}| = g_1$ for all odd i, $3 \leq i \leq 2s-1$, and $|X_{i1}| = f_1$ for all even i, $4 \leq i \leq 2s$. Let there be t arcs directed from each vertex of X_{1i} to every vertex of X_{2i} for all $1 \leq i \leq m$; t arcs directed from each vertex of X_1^i to every vertex of X_2^i $1 \leq i \leq n$ and t arcs directed from each vertex of X_{i1} to every vertex of $X_{(i+1)1}$ for all odd i, $3 \leq i \leq 2s-1$ so that we obtain $2s$-partite r-digraph MMD with imbalance of vertices as follows.

For $1 \leq i \leq m$,

$$a_{x_{1i}} = t|X_{2i}| - 0 = tf_i = p_i, \; for \; all \; x_{1i} \in X_{1i};$$

for $1 \leq j \leq n$,

$$a_{x_1^i} = t|X_2^i| - 0 = tf_1 = p_1, \; for \; all \; x_1^i \in X_1^i;$$

for $1 \leq j \leq m$,

$$a_{x_{2i}} = 0 - t|X_{1i}| = -tg_1 = -q_1, \; for \; all \; x_{2i} \in X_{2i};$$

for $1 \leq j \leq n$,

$$a_{x_2^i} = 0 - t|X_1^i| = -tg_i = -q_i, \; for \; all \; x_2^i \in X_2^i;$$

for all odd i, $3 \leq i \leq 2s-1$

$$a_{x_{i1}} = t|X_{(i+1)1}| - 0 = tf_1 = p_1, \; for \; all \; x_{i1} \in X_{i1};$$

and for even i, $4 \leq i \leq 2s$,

$$a_{x_{i1}} = 0 - t|X_{(i-1)1}| = -tg_1 = -q_1, \ for \ all \ x_{i1} \in X_{i1}.$$

Therefore imbalance set of $D(X_1, X_2, \cdots, X_k)$ is $P \cup Q$.

By using the same argument as above, it can be shown that the result holds for odd k also. Hence the proof is complete. □

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