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On maximum cycle packings in polyhedral graphs

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Abstract

This paper addresses upper and lower bounds for the cardinality of a maximum vertex-/edge-disjoint cycle packing in a polyhedral graph G. Bounds on the cardinality of such packings are provided, that depend on the size, the order or the number of faces of G, respectively. Polyhedral graphs are constructed, that attain these bounds.

Keywords: Maximum cycle packing, polyhedral graphs, vertex-disjoint cycles, edge-disjoint cycle

1. Introduction

Packing vertex- or edge-disjoint cycles in graphs is also a widely studied graph-theoretical problem. A large amount of literature can be found concerning conditions that are sufficient for the existence of some number of disjoint cycles which may satisfy further restrictive conditions.

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For examples, we refer to publications [6], [9], [10], [12], [15], [16], [18], [20], [21], [23], [24]. The algorithmic problems concerning cycle packings are typically hard ([5], [11], [20]) and approximation algorithms are described ([11], [17]). Several authors mention practical applications in computational biology ([3], [8], [13]) or the design of optical networks ([1]). In this paper, we investigate maximum cycle packings in polyhedral graphs G. We derive different bounds on the cardinality of such packings depending on the size of G, the order of G and the number of faces of G, respectively. As our main result we show that the bounds are sharp in the sense that we construct corresponding polyhedral graphs attaining these bounds.

2. Preliminaries and basic definitions

In the sequel all graphs G will be finite and undirected with vertex set $V(G)$ and edge set $E(G)$ that contains no loops or multiple edges. We recall some basic notions. If an edge $e \in E(G)$ has two incident vertices u and v we write $e = (u, v)$. For finite sequence $(v_{i_0}, e_0, v_{i_1}, e_1, \ldots, e_{r-1}, v_{i_r})$ of vertices $v_{i_j} \in V(G)$ and pairwise disjoint edges $e_j = (v_{i_j}, v_{i_{j+1}}) \in E(G)$ the subgraph W of G with vertex set $V(W)$ and edge set $E(W)$ is called a *walk* of length r with start vertex v_{i_0} and end vertex v_{i_r}. A *path* $P(v_{i_0}, v_{i_r})$ is a walk in which all vertices v have degree $\delta_W(v) \leq 2$. If $P(v_{i_0}, v_{i_r})$ is closed, i.e. $v_{i_0} = v_{i_r}$, it is called a *cycle*. A graph G is *k-vertex-connected* if for each pair $u, v \in V(G)$ there are k paths $P_i(u, v)$ in G that mutually have no common vertices, except u and v. In addition, G is called *Eulerian* if it is connected and all vertices have even degree. An independent set in G is a subset of $V(G)$ without edges between them. A *vertex-disjoint (edge-disjoint) cycle packing* $\mathcal{C}(G) = \{C_1, C_2, \ldots, C_q\}$ of G is a collection of cycles C_i of G such that all C_i are mutually vertex-disjoint (edge-disjoint). The maximum cardinality of a vertex-disjoint (edge-disjoint) cycle packing of G is denoted by $\nu(G)$ or $\nu'(G)$, respectively. A related packing is called maximum vertex-disjoint (edge-disjoint) cycle packing.

A *planar graph* is a graph G which can be drawn in a plane without any mutual crossings of edges. In a plane drawing an area F that is surrounded by edges of G is called a *face* of G. $E(F)$ are the surrounding edges. The set of faces is denoted by $F(G)$. If G is planar and connected *Euler formula* holds (see [19]), i.e. $n - m + f = 2$, where $n = |V(G)|$ denotes the order of G, $m = |E(G)|$ its size and $f = |F(G)|$ the number of faces, respectively. It is well known (see [2], [22]) that every planar graph has a 4-coloring of its vertices, and in consequence, every planar graph G has an independent set of size at least $|V(G)|/4$.

A graph G, resulting from a stereographic projection of vertices and edges of a convex polyhedron $P \subset \mathbb{R}^3$ into the plane $\mathbb{R}^2$ is called a *polyhedral graph*. The set of polyhedral graphs will be denoted by $\mathcal{P}$. Due to the *Theorem of Steinitz* (see [4]) G is a polyhedral graph if and only if G is planar and 3-connected. The class of polyhedral graphs is a well investigated field in graph theory. The fundamental relation between geometry and graph theory in the class $\mathcal{P}$ has generated a large variety of results concerning different topics. For a comprehensive overview we refer to [14] and [25].

3. Vertex-disjoint cycle packings in polyhedral graphs

In this section we give bounds on the cardinality of maximum vertex-disjoint cycle packings. These bounds depend on n, m or f. It turns out that the provided bounds are sharp, in the sense that there exist polyhedral graphs that attain the bounds. For $n, f \geq 4, m = 6$ or $m \geq 8$ let $\mathcal{PV}_n := \{G \in \mathcal{P} \mid |V(G)| = n\}$ denote the set of polyhedral graphs of size n, $\mathcal{PE}_m := \{G \in \mathcal{P} \mid |E(G)| = m\}$ the set of these graphs of order m and $\mathcal{PF}_f := \{G \in \mathcal{P} \mid |F(G)| = f\}$ the set of polyhedral graphs with f faces, respectively. First, we make the following observation

Lemma 3.1. *For a polyhedral graph G the following holds:*

$$1 \leq \nu(G) \leq \left\lfloor \frac{n}{3} \right\rfloor \leq \left\lfloor \frac{2m}{9} \right\rfloor \leq \left\lfloor \frac{2(f-2)}{3} \right\rfloor.$$

Proof. Obviously, $1 \leq \nu(G)$ holds since $f \geq 1$ for $G \in \mathcal{P}$. By the fact that all cycles in G have length greater or equal to 3, immediately $\nu(G) \leq \left\lfloor \frac{n}{3} \right\rfloor$ follows. Using *Euler formula* and the property that $3n \leq 2m$ is true for $G \in \mathcal{P}$ we get

$$\frac{n}{3} \leq \frac{2m}{9} = \frac{6m - 4m}{9} \leq \frac{2(m-n)}{3} = \frac{2(f-2)}{3}.$$

□

In the following we want to examine, whether these bounds are sharp in the classes $\mathcal{PV}_n, \mathcal{PE}_m$ and $\mathcal{PF}_f$, respectively. In Figure 1 polyhedral graphs $G_1, \ldots, G_{10}$ are drawn, which belong to $\mathcal{PE}_m, m = 6$ or $8 \leq m \leq 16$, to $\mathcal{PV}_n, n \in \{4, 5, 6, 7, 8, 9\}$ and to $\mathcal{PF}_f, f \in \{4, 5, 6, 7\}$. Obviously, $\nu(G_i) = \left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{2m}{9} \right\rfloor, i \in \{1, \ldots, 10\}$ and $\nu(G_i) = \left\lfloor \frac{2(f-2)}{3} \right\rfloor, i \in \{1, 3, 4, 5, 6, 8, 9\}$.

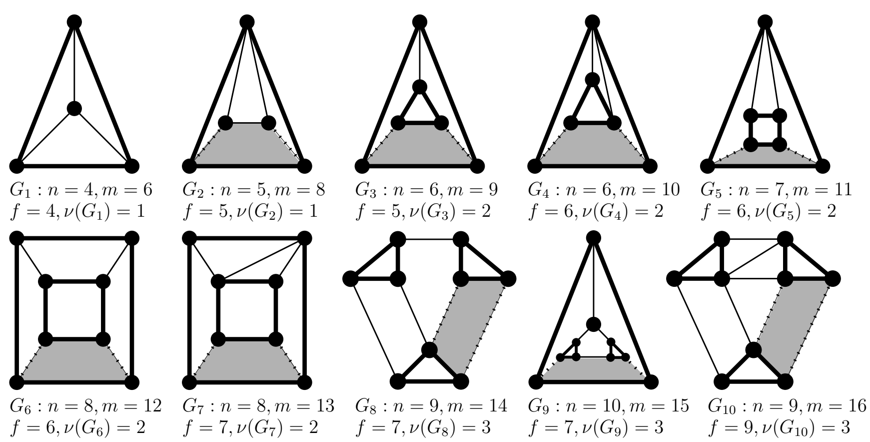

Figure 1. Graphs $G_i, i \in \{1, 2, \ldots, 10\}$, used for induction in Proposition 3.1.

A vertex-disjoint cycle packing in G_i is indicated by bold edges. Moreover, each of the graphs $G_2, G_3, \ldots, G_{10}$ has a face F such that $|E(F)| \geq 4$ (shaded area) and for which two of the edges $e_1, e_2 \in E(F)$ (dotted edges) do not belong to the maximum cycle packing. These graphs are used in order to show

Proposition 3.1. *The following is true:*

(i) *for* $n \geq 4$*, there is* $G \in \mathcal{PV}_n$ *such that* $\nu(G) = \left\lfloor \frac{n}{3} \right\rfloor$,

(ii) *for* $m = 6$ *or* $m \geq 8$*, there is* $G \in \mathcal{PE}_m$ *such that* $\nu(G) = \left\lfloor \frac{2m}{9} \right\rfloor$,

(iii) *for* $f \geq 4$*, there is* $G \in \mathcal{PF}_f$ *with* $\nu(G) = \left\lfloor \frac{2(f-2)}{3} \right\rfloor$.

Proof. Let us use the planar graph T, drawn in Figure 2.

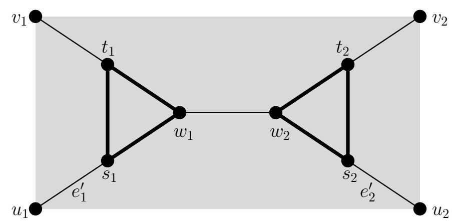

Figure 2. Graph T used for the iterative step in Proposition 3.1.

Now, consider $G \in \mathcal{P}$ such that G contains a face F with $|E(F)| \geq 4$. Let e_1, e_2 denote two non-adjacent edges of F. Thus, we define $G'(e_1, e_2) := G \oplus T$ by identifying the edges $e_1 = (u_1, v_1)$ with the path (u_1, s_1, t_1, v_1) and $e_2 = (u_2, v_2)$ with (u_2, s_2, t_2, v_2), respectively, and embedding T into the interior of the face F. Then, $|V(G'(e_1, e_2))| = |V(G)| + 6$, $|E(G'(e_1, e_2))| = |E(G)| + 9$ and $|F(G'(e_1, e_2))| = |F(G)| + 3$. Clearly, $G'(e_1, e_2) \in \mathcal{P}$, since it is planar and 3-connected. We show not only that $\nu(G) = \left\lfloor \frac{2m}{9} \right\rfloor$, but also that there is always a face F in G such that $|E(F)| \geq 4$ and for which two non adjacent edges $e_1, e_2 \in E(F)$ do not belong to a maximum cycle packing of G.

(i) This assertion is true for $8 \leq m \leq 16$, since each of the graphs $G_2, \ldots, G_{10}$ has a face F such that $|E(F)| \geq 4$ (shaded area) and for which two non adjacent edges $e_1, e_2 \in E(F)$ (dotted edges) do not belong to a maximum cycle packing of G (bold edges). In order to use induction arguments, we assume, that it is true for some $G \in \mathcal{PE}_m$. Let $\nu(G) = \left\lfloor \frac{2m}{9} \right\rfloor$ and $\mathcal{C}(G)$ be a corresponding vertex-disjoint cycle packing. Clearly, $G'(e_1, e_2) \in \mathcal{PE}_{m+9}$, since it is planar and 3-connected. For $C_1 = (s_1, t_1, w_1, s_1)$ and $C_2 = (s_2, t_2, w_2, s_2)$ the set $\mathcal{C}(G'(e_1, e_2)) = \mathcal{C}(G) \cup \{C_1, C_2\}$ is a vertex-disjoint cycle packing of $G'(e_1, e_2)$ with $|\mathcal{C}(G'(e_1, e_2))| = \nu(G) + 2$ which is maximal, since

$$|\mathcal{C}(G'(e_1, e_2))| \leq \nu(G'(e_1, e_2)) \leq \left\lfloor \frac{2(m+9)}{9} \right\rfloor = \nu(G) + 2.$$

Moreover, $e'_1 = (u_1, s_1)$ and $e'_2 = (u_2, s_2)$ are two non adjacent edges of the boundary of the same face $F' \in G'$. Since $\{e'_1, (s_1, w_1), (w_1, w_2), (w_2, s_2), e'_2\} \in E(F')$.

(ii) Using the graphs G_i with $i \in \{2, 3, 5, 6, 8, 9\}$ from Figure 1 the assertion holds for graphs $G \in \mathcal{PV}_n, 5 \leq n \leq 10$. Performing the same induction arguments as in (i), we get (ii).

(iii) The graphs G_i with $i \in \{2,4,7\}$ show that the assertion is true for $G \in \mathcal{PV}_f, 5 \leq f \leq 7$. Again, we perform the same induction arguments as in (i) to get (iii).

□

With respect to the lower bound $\nu(G) \geq 1$ of a polyhedral graph G we remark

Remark 3.1. A *wheel* W_n on $n \geq 4$ vertices is a graph with n vertices $v_1, \ldots, v_n$ with v_1 having degree $n-1$ and all the other vertices having degree 3. The vertex v_1 is adjacent to vertices, and for $i \in \{2, \ldots, n-1\}$, v_i is adjacent to v_{i+1}, and v_n is adjacent to v_2.

- Obviously, $\nu(W_n) = 1$. In [7] it is shown that for 3-connected planar graphs with more than 5 vertices wheels are the only graphs with $\nu(G) = 1$.
- Since W_n is self-dual, $W_n \in \mathcal{PV}_n \cap \mathcal{PF}_n, n \geq 4$, i.e. wheel graphs W_n attain the minimum cardinality of a maximum cycle packing in the classes $\mathcal{PV}_n$ and $\mathcal{PF}_f, n, f \geq 4$, respectively.
- As $|E(W_n)| = 2(n-1)$, W_n is also the graph that minimizes the cardinality of a maximum cycle packing in the set $\mathcal{PE}_m, m \geq 6$ and even m.
- To investigate $\mathcal{PE}_m, m \geq 11$ and odd m we observe, that $W_{\frac{m+1}{2}} \in \mathcal{PE}_{m-1}$. Since v_2, v_3 are adjacent in $W_{\frac{m+1}{2}}$, there are two nonadjacent vertices v_i, v_j, different from v_2, v_3 and a path $P(v_i, v_j) \in W_{\frac{m+1}{2}}$ not containing $\{v_1, v_2, v_3\}$. We now define $G \in \mathcal{PE}_m$ by
$$G = W_{\frac{m+1}{2}} \cup \{(v_i, v_j)\}.$$
Then $C_1 = (v_1, v_2, v_3, v_1)$ and $C_2 = P(v_i, v_j) \cup \{(v_i, v_j)\}$ are two vertex-disjoint cycles in G, i.e. the minimal cardinality in this class is 2.
- In addition, $\nu(G) = 1$ holds for $G \in \mathcal{PE}_9 \cap \mathcal{PV}_5$ with Lemma 3.1.

4. Edge-disjoint cycle packings in polyhedral graphs

In the following section upper and lower bounds for the cardinality of maximum edge-disjoint cycle packings are established. It is shown that in almost all cases they are sharp.

Lemma 4.1. *For $G \in \mathcal{P}$ the following holds:*

(i) $\max\left\{\left\lceil \frac{f}{4} \right\rceil, \left\lceil \frac{m+6}{12} \right\rceil, \left\lceil \frac{n+4}{8} \right\rceil\right\} \leq \nu'(G)$,

(ii) $1 \leq \nu'(G) \leq \min\left\{n-2, \left\lfloor \frac{m}{3} \right\rfloor, \left\lfloor \frac{2(f-2)}{3} \right\rfloor\right\}$.

Proof. (i) Let G^* be the dual graph of a plane drawing of G. G^* is the graph drawn by placing a new vertex inside each face of G and connecting these vertices in G^* whenever the corresponding two faces share an edge in G. As G is 3-connected, G^* is simple and planar and therefore, has an independent set S of vertices of size $|S| \geq \frac{f}{4}$. Hence, $\nu'(G) \geq \left\lceil \frac{|F(G)|}{4} \right\rceil$. Moreover, $f \geq \frac{n+4}{2}$ and $f \geq \frac{m+6}{3}$. By this immediately (i) follows.

(ii) Obviously, $1 \leq \nu'(G)$ holds, since $f \geq 4$ for $G \in \mathcal{P}$. Now, let $c_i = |\{v \in G | \delta_G(v) = i\}|, i \in \{3, 4, 5, \ldots\}$ and $\Delta := \max\{\delta_G(v) | v \in V\}$. By c we denote the number of vertices of odd degree. By the two facts that all cycles in G have at least a length of 3 and there are at least $\frac{1}{2}c$ edges that cannot belong to any maximum cycle packing it follows

$$\nu'(G) \leq \left\lfloor \frac{m - \frac{1}{2}c}{3} \right\rfloor \leq \left\lfloor \frac{m}{3} \right\rfloor \leq n - 2.$$

More sophisticated, we get

$$\begin{aligned} m - \frac{1}{2}c &= \frac{1}{2}\left(\sum_{\substack{i=3,\\ i \text{ odd}}}^{\Delta} ic_i + \sum_{\substack{i=3,\\ i \text{ even}}}^{\Delta} ic_i - \sum_{\substack{i=3,\\ i \text{ odd}}}^{\Delta} c_i \right) \\ &= \frac{1}{2}\left(\sum_{j=1}^{\Delta} (2j+1)c_{2j+1} + \sum_{j=2}^{\Delta} 2jc_{2_j} - \sum_{j=1}^{\Delta} c_{2j+1} \right) \\ &= \frac{1}{2}\left(\sum_{j=1}^{\Delta} 2jc_{2j+1} + \sum_{j=2}^{\Delta} 2jc_{2j} \right) = \sum_{j=1}^{\Delta} jc_{2j+1} + \sum_{j=2}^{\Delta} jc_{2j} \\ &\leq \sum_{i=1}^{\Delta} (i-2)c_i = 2m - 2n = 2(f-2) \end{aligned}$$

from which we conclude $\nu'(G) \leq \left\lfloor \frac{2(f-2)}{3} \right\rfloor$.

□

Remark 4.1. The graphs $G \in \mathcal{PF}_f$ attaining the upper bound $\nu(G) = \left\lfloor \frac{2(f-2)}{3} \right\rfloor$ according to Proposition 3.1, of course, attain the upper bound $\nu'(G) = \left\lfloor \frac{2(f-2)}{3} \right\rfloor$. This follows, since every vertex-disjoint cycle packing of G induces an edge-disjoint cycle packing.

Again, we show that also the two other bounds in Lemma 4.1 are sharp for graphs in $\mathcal{PE}_m$ and $\mathcal{PV}_n$, respectively. More precisely we prove

Proposition 4.1. *The following is true:*

(i) *for $n = 6$ or $n \geq 8$ there is $G \in \mathcal{PV}_n$ with $\nu'(G) = n - 2$,*

(ii) *for $m \in \{8, 11, 12, 13, 14\}$ or $m \geq 16$ there is $G \in \mathcal{PE}_m$ with $\nu'(G) = \left\lfloor \frac{m}{3} \right\rfloor$.*

Proof. For the proof induction arguments are used. For this we first consider the planar graph D, drawn in Figure 3. Obviously, $\delta_D(u) = \delta_D(v) = \delta_D(w) = 2$. For a planar graph G that contains a triangle $C = (\bar{u}, \bar{v}, \bar{w}, \bar{u})$, which is also a face F of G, we define $G'(\bar{u}, \bar{v}, \bar{w}) := G \oplus D$ by identifying the vertices $\{\bar{u}, \bar{v}, \bar{w}\}$ with the vertices $\{u, v, w\}$, and embedding D into the interior of the face F.

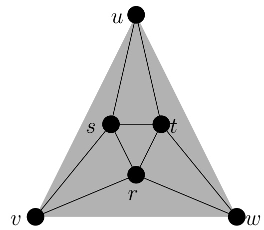

Figure 3. Graph D used for the iterative step in Proposition 4.1.

(i) We will show not only that $\nu'(G) = |V(G)| - 2$, but it also has a maximum edge-disjoint cycle packing $\mathcal{C}$, that contains a cycle $C = (\bar{u}, \bar{v}, \bar{w}, \bar{u})$, which is also a face F of G. The assertion is true for $n \in \{6, 8, 10\}$. The corresponding graphs G_i with $i \in \{3, 7, 9\}$ are listed in Figure 4. In order to use induction arguments, let us assume that it is true for

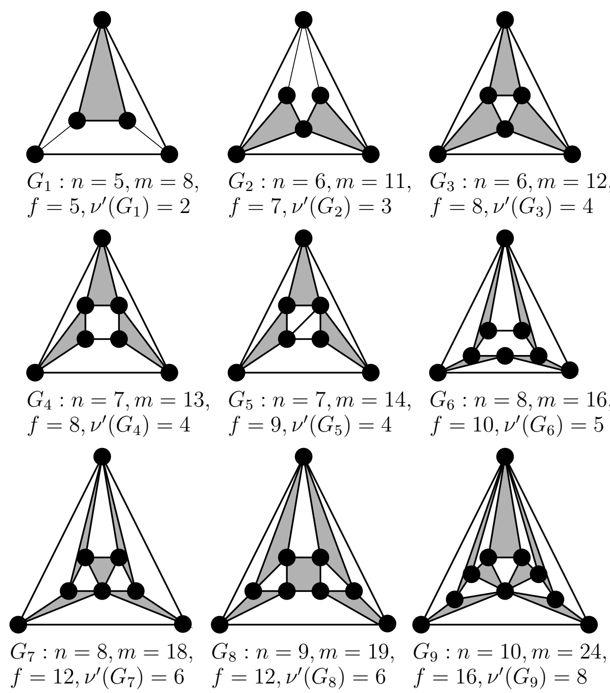

Figure 4. Plane drawings with $\nu'(G) = \min\left\{n-2, \left\lfloor \frac{m}{3} \right\rfloor, \left\lfloor \frac{2(f-2)}{3} \right\rfloor\right\}$.

some $G \in \mathcal{FV}_n, n \geq 11$, i.e. $\nu'(G) = n - 2$, and there is a maximum edge-disjoint cycle packing $\mathcal{C}$ of G such that it contains a cycle $C = (\bar{u}, \bar{v}, \bar{w}, \bar{u})$, which is also a face F of G. Define $G' := G'(\bar{u}, \bar{v}, \bar{w}) = G \oplus D$. Clearly, $G' \in \mathcal{PV}_{n+3}$, since it is planar and 3-connected. Moreover, there is an edge-disjoint cycle packing $\mathcal{C}'$ of G', given by

$$\mathcal{C}' = \mathcal{C} \cup \{u, s, t, u\} \cup \{v, s, r, v\} \cup \{w, r, t, w\},$$

i.e. $\nu'(G') \geq \nu'(G) + 3 = (|V(G)| - 2) + 3 = |V(G')| - 2$. With Lemma 4.1 $\nu'(G') =$

$|V(G')| - 2$ follows. Moreover, each of the three additional cycles is the boundary of a face of G'.

(ii) As before, we show not only that $\nu'(G) = \left\lfloor \frac{|E(G)|}{3} \right\rfloor$, but it also has a maximum edge-disjoint cycle packing $\mathcal{C}$, that contains a cycle $C = (\bar{u}, \bar{v}, \bar{w}, \bar{u})$, which is also a face F of G. This is true for $m \in \{8, 11, 12, 13, 14, 16, 18, 19, 24\}$. Corresponding graphs are listed in Figure 4. In order to use induction arguments, let us assume that it is true for some $G \in \mathcal{PE}_m, m \geq 16$, i.e. $\nu'(G) = \left\lfloor \frac{m}{3} \right\rfloor$, and there is a maximum edge-disjoint cycle packing $\mathcal{C}$ of G such that it contains a cycle $C = (\bar{u}, \bar{v}, \bar{w}, \bar{u})$, which is also a face F of G. Again, set $G' = G'(\bar{u}, \bar{v}, \bar{w}) = G \oplus D$. Clearly, $G' \in \mathcal{PE}_{n+9}$, since it is planar and 3-connected. Moreover, there is a maximum edge-disjoint cycle packing $\mathcal{C}'$ of G', given by

$$\mathcal{C}' = \mathcal{C} \cup \{u, s, t, u\} \cup \{v, s, r, v\} \cup \{w, r, t, w\},$$

i.e. $\nu'(G') \geq \nu'(G) + 3 = \left\lfloor \frac{|E(G)|}{3} \right\rfloor + 3 = \left\lfloor \frac{|E(G)|+9}{3} \right\rfloor = \left\lfloor \frac{|E(G')|}{3} \right\rfloor$. Again, $\nu'(G') = \left\lfloor \frac{|E(G')|}{3} \right\rfloor$ follows. Moreover, each of the three additional cycles is the boundary of a face of G'.

□

Immediately we deduce

Corollary 4.1. *There are infinitely many $n \in \mathbb{N}$ for which there is $G \in \mathcal{PV}_n$ such that*

$$\nu'(G) = n - 2 = \left\lfloor \frac{m}{3} \right\rfloor = \left\lfloor \frac{2(f-2)}{3} \right\rfloor. \tag{1}$$

Proof. An easy calculation shows, that (1) is true for the octahedron $G \in \mathcal{PV}_6 \cap \mathcal{PE}_{12} \cap \mathcal{PF}_8$. Using the construction scheme of the last proposition for induction we get that $G' \in \mathcal{PV}_{|V(G)|+3} \cap \mathcal{PE}_{|E(G)|+9} \cap \mathcal{PF}_{|F(G)|+6}$, from which

$$\nu'(G') = |V(G')| - 2 = \left\lfloor \frac{|E(G')|}{3} \right\rfloor = \left\lfloor \frac{2(|F(G')| - 2)}{3} \right\rfloor$$

follows. □

Remark 4.2. The upper bounds in Proposition 4.1 with respect to m and n are not sharp in the cases $G \in \mathcal{PE}_m, m \in \{6, 9, 10, 15\}$ and $G \in \mathcal{PV}_n, n \in \{4, 5, 7\}$. This is true for $m \in \{6, 9, 10, 15\}$, because according to Lemma 4.1 a necessary condition for graphs $G \in \mathcal{PE}_m, m \in \{6, 9, 15\}$ to attain $\nu'(G) = \left\lfloor \frac{m}{3} \right\rfloor$ is to be Eulerian. A necessary condition for $G \in \mathcal{PE}_{10}$ to attain $\nu'(G) = 3$ is that it has most two vertices of odd degree. But these conditions are not satisfied: to realize this, we first observe that $G \in \mathcal{PE}_6$ implies $|V(G)| = 4$ and $G \in \mathcal{PE}_9$ implies $|V(G)| \in \{5, 6\}$, respectively. If $G \in \mathcal{PE}_{10}$ $|V(G)| = 6$ and for $G \in \mathcal{PE}_{15}$ implies $|V(G)| \in \{7, \ldots, 10\}$. Investigation of all cases show that

- a graph $G \in \mathcal{PE}_{10}$ has at least 4 vertices of odd degree,

- a 3-connected Eulerian graph G with $|V(G)| = 7, |E(G)| = 15$ contains $K_{3,3}$, hence it is not planar,
- all remaining cases lead to graphs which are non-Eulerian.

A similar consideration shows that for the cases $n \in \{4, 5, 7\}$ the bound $n - 2$ cannot be attained by graphs $G \in \mathcal{PV}_n$. Graphs $G \in \mathcal{PE}_m, m \in \{6, 9, 10, 15\}$ satisfying $\nu'(G) = \lfloor \frac{m}{3} \rfloor - 1$ and graphs $G \in \mathcal{PV}_n, n \in \{4, 5, 7\}$ satisfying $\nu'(G) = n - 3$ are listed in Figure 5.

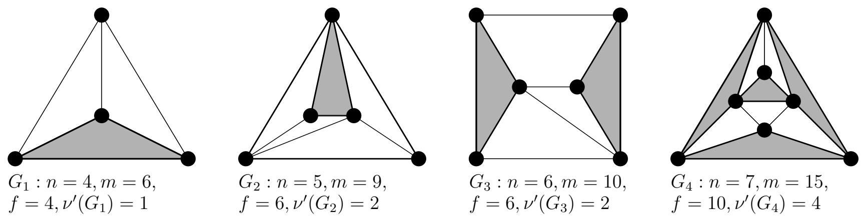

Figure 5. Plane drawings of $G_i \in \mathcal{PE}_m$ for $m \in \{6, 9, 10, 15\}$ with the property $\nu'(G) = \lfloor \frac{m}{3} \rfloor - 1$.

For the lower bounds of the cardinality of maximum cycle packings we proof the following result

Proposition 4.2. *The following is true:*

(i) *for $n \geq 4$ there is $G \in \mathcal{PV}_n$, such that $\nu'(G) = \lceil \frac{n+4}{8} \rceil$,*

(ii) *for $m = 6$ or $m \geq 8$ there is $G \in \mathcal{PE}_m$, such that $\nu'(G) = \lceil \frac{m+6}{12} \rceil$,*

(iii) *for $f \geq 4$ there is $G \in \mathcal{PF}_f$, such that $\nu'(G) = \lceil \frac{f}{4} \rceil$.*

Proof. We first consider the planar graph S, drawn in the Figure 6. For a planar graph G that

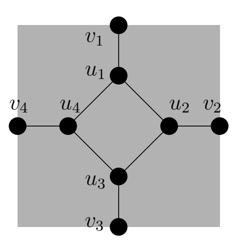

Figure 6. Graph S used for the iterative step in Proposition 4.2.

contains a cycle $C = (e_1, e_2, e_3, e_4)$, which is also a face F of G we define $G'(e_1, e_2, e_3, e_4) := G \oplus S$ by subdividing each of the four edges e_i, identifying the additional vertices with the vertices $\{v_1, v_2, v_3, v_4\}$, and embedding S into the interior of F.

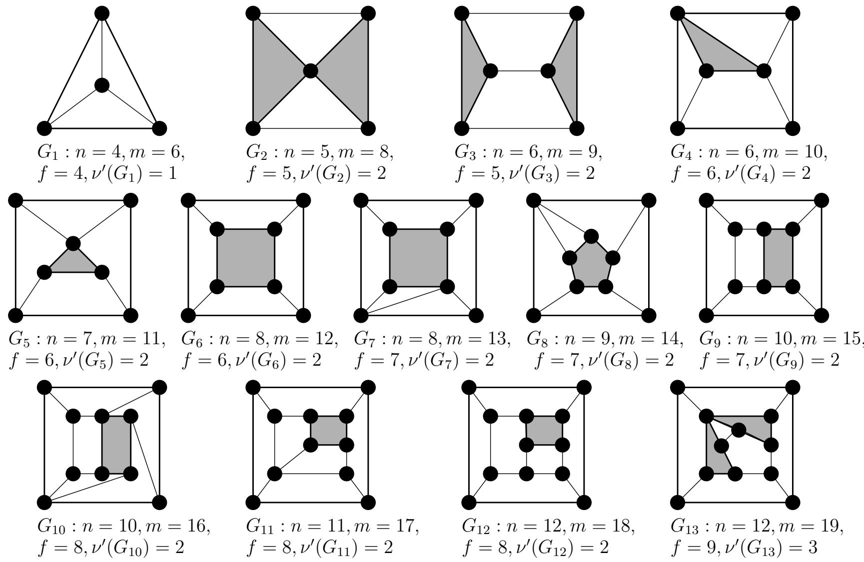

Figure 7. Plane drawings with $\nu'(G) = \max\left\{\left\lceil\frac{n+4}{8}\right\rceil, \left\lceil\frac{m+6}{12}\right\rceil, \left\lceil\frac{f}{4}\right\rceil\right\}$.

(i) We show not only that $\nu'(G)$ attains the bound, but also that in G exists at least one face which is bounded by four edges. The assertion is true for $n \in \{4, 5, \dots, 12\}$. The corresponding graphs G_i with $i \in \{1, 2, 3, 5, 6, 9, 10, 12\}$ are listed in Figure 7.

In order to use induction arguments, let us assume that $\nu'(G) = \left\lceil\frac{n+4}{8}\right\rceil$ is true for some $G \in \mathcal{FV}_n, n \geq 4$, and there is a maximum edge-disjoint cycle packing $\mathcal{C}$ of G such that it contains a cycle C of length 4 which is also a face F of G. Let (e_1, e_2, e_3, e_4) be the boundary of F and set $G' = G'(e_1, e_2, e_3, e_4) = G \oplus S$. Clearly, $G' \in \mathcal{PV}_{n+8}$, since it is planar and 3-connected.
Moreover, there is an edge-disjoint cycle packing $\mathcal{C}'$ of G', given by

$$\mathcal{C}' = \mathcal{C} \cup \{(u_1, u_2, u_3, u_4)\},$$

i.e. $\nu'(G') \geq \nu'(G) + 1 = \left\lceil\frac{|V(G)|+4}{8}\right\rceil + 1 = \left\lceil\frac{|V(G')|+4}{8}\right\rceil$. The additional cycle is, of course, the boundary of a face F' of G'. It remains to show, that $\nu'(G') = \nu'(G) + 1$.

Assume that is not the case. Then $\nu'(G') \geq \nu'(G) + 2$. Let $\mathcal{C}'$ be a corresponding maximum cycle packing. At least two of the cycles in $\mathcal{C}'$ must contain edges of S. By the structure of S exactly two cycles, say $C_1, C_2 \in \mathcal{C}'$, must have this property. Let $v_1, v_2 \in V(C_1)$ and $v_3, v_4 \in V(C_2)$, respectively. With $\delta_S(v_i) = 3, i \in \{1, \dots, 4\}$,

$$E(C) \cap (E(\mathcal{C}') \setminus \{E(C_1), E(C_2)\}) = \emptyset,$$

i.e. $\mathcal{C}' \setminus \{C_1, C_2\} \cup C$ is an edge-disjoint cycle packing of G with cardinality of at least $\nu'(G) + 1$, which contradicts $\nu'(G)$ as cardinality of a maximum cycle packing of G. The embedding of S guarantees that G' has at least one face that is bounded by four edges.

(ii) The proof is similar to (i). In this case we start with graphs $G \in \mathcal{PE}_m, m \geq 8$ (the first thirteen graphs are drawn in Figure 7) and observe that $G' \in \mathcal{PE}_{m+12}$.

(iii) The proof is analogous to (i). We start with graphs $G \in \mathcal{PF}_f, f \geq 4$ (the first four graphs G_i with $i \in \{1, 2, 4, 7\}$ are drawn in Figure 7) and observe that $G' \in \mathcal{PF}_{f+4}$.

□

The following proposition shows that the number of graphs $G \in \mathcal{P}$ with predefined $\nu(G)$ is in general large.

Proposition 4.3. *Let* $k \geq 1$.

(i) *For* n *satisfying* $k + 3 \leq n \leq 8k - 4$ *there is a non-Eulerian* $G \in \mathcal{PV}_n$ *such that* $\nu'(G) = k$*,*

(ii) *for* m *satisfying* $3k + 3 \leq m \leq 12k - 6$ *there is a non-Eulerian* $G \in \mathcal{PE}_m$ *such that* $\nu'(G) = k$*,*

(iii) *for* f *satisfying* $\lceil \frac{3k}{2} \rceil + 2 \leq f \leq 4k$ *there is a non-Eulerian* $G \in \mathcal{PF}_f$ *such that* $\nu'(G) = k$.

Proof. The proof is done by induction. For $k = 1$ the assertion holds with graph G_1 from Figure 7 for (i), (ii) and (iii).

(i) Assume that the assertion holds for $k \geq 1$. We have to show that it is also true for $k+1$, i.e. that for all n with $(k+1)+3 \leq n \leq 8(k+1)-4$ there is non-Eulerian $G \in \mathcal{PV}_n$, with $\nu'(G) = k + 1$. We distinguish between two cases:

(a) Let $k + 4 \leq n \leq 8k - 4$:
Then, for $n' := n - 1$, we get $k + 3 \leq n' \leq 8n - 5$. Hence, there is a non-Eulerian $G' \in \mathcal{PV}_{n'}$ and $\nu'(G') = k$. Let $\mathcal{C}$ be a maximum cycle packing. There must be $e = (u, v) \in E(G')$ such that $e \notin E(\mathcal{C}')$. Let F be the face of G' such that $e \in E(F)$. Define $G := G' \oplus K_{1,3}$ by embedding $K_{1,3}$ into the interior of F in such a way, that u, v is identified with two of the vertices in $K_{1,3}$ and the third vertex of $K_{1,3}$ is identified with an arbitrary vertex $w \in V(F) \setminus \{u, v\}$. Obviously $G \in \mathcal{PV}_n$, G is non-Eulerian and $\nu'(G) = k + 1$.

(b) Let $8k - 4 < n \leq 8k - 4 + 8$:

$$k = \frac{8k}{8} < \frac{n+4}{8} \leq \frac{8k+8}{8} = k + 1,$$

i.e. in these cases $\lceil \frac{n+4}{8} \rceil = k + 1$. With Proposition 4.2, there is $G \in \mathcal{PV}_n$ with $\nu'(G) = k + 1$. Moreover, by construction of G in Proposition 4.2, G is non-Eulerian.

(ii) The proof is performed analogously to (i), but instead of $n' = n - 1$ we here have to consider $m' = m - 3$ and have to distinguish between the cases (a) $3k + 3 \leq m \leq 6(2k - 1)$ and (b) $6(2k - 1) \leq m \leq 6(2(k + 1) - 1)$, respectively.

(iii) We have to show that for $\left\lceil \frac{3(k+1)}{2} \right\rceil + 2 \leq f \leq 4(k+1)$ the assertion holds. First, let k be even, i.e. $k+1 \geq 3$ and $\left\lceil \frac{3(k+1)}{2} \right\rceil + 2 = \left\lceil \frac{3k}{2} \right\rceil + 4$. Again, we distinguish between (a) $\left\lceil \frac{3k}{2} \right\rceil + 4 \leq f \leq 4k$ and (b) $4k < f \leq 4k+4$. The same considerations as in (i) with $f' = f - 2$ instead of $n' = n - 1$ then proves the assertion.

Secondly, if k is odd, we get $\left\lceil \frac{3(k+1)}{2} \right\rceil + 2 = \left\lceil \frac{3k}{2} \right\rceil + 3$. Here, we distinguish between the following two cases: (a) $f = \left\lceil \frac{3k}{2} \right\rceil + 3$, i.e. $f = \frac{3k}{2} + \frac{1}{2} + 3$, from which $k = \frac{2(f-3)}{3} - \frac{1}{3} = \left\lfloor \frac{2(f-3)}{3} \right\rfloor$ follows. Using Remark 4.1 there exists a non-Eulerian $G \in \mathcal{PF}_f$ such that $\nu'(G) = k$. For the remaining cases (b) $\lceil \frac{3k}{2} \rceil + 4 \leq f \leq 4k \leq 4k+4$ the proof is performed as for the even case.

□

Remark 4.3. According to Remark 4.2, for the cases $k = 4$ or $k \geq 6$ in Proposition 4.3

- in (i) even the sharper inequality $k + 2 \leq n \leq 8k - 4$ holds,
- in (ii) even the sharper inequality $3k \leq m \leq 6(2k-1)$ holds.

Using G_4 and G_1 from Figure 4, the construction scheme from Proposition 4.3, moreover, yields that

- for $k \geq 4$ there is $G \in \mathcal{PE}_{3k+1}$ such that $\nu'(G) = k$,
- for $k \geq 2$ there is $G \in \mathcal{PE}_{3k+2}$ such that $\nu'(G) = k$.

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