Electronic Journal of Graph Theory and Applications **7** (1) (2019), 91–103

Electronic Journal of
Graph Theory and Applications

Metric dimension of fullerene graphs

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Abstract

A resolving set W is a set of vertices of a graph $G(V, E)$ such that for every pair of distinct vertices $u, v \in V(G)$, there exists a vertex $w \in W$ satisfying $d(u, w) \neq d(v, w)$. A resolving set with minimum number of vertices is called metric basis of G. The metric dimension of G, denoted by $\dim(G)$, is the minimum cardinality of a resolving set of G. In this paper, we consider $(3, 6)$-fullerene and $(4, 6)$-fullerene graphs and compute the metric dimension for these fullerene graphs. We also give conjecture on the metric dimension of $(3, 6)$-fullerene and $(4, 6)$-fullerene graphs.

Keywords: resolving set, metric dimension, fullerene graph
Mathematics Subject Classification : 05C12
DOI: 10.5614/ejgta.2019.7.1.7

1. Introduction

The metric dimension was initially studied by Slater [16] and Harary and Melter [7]. They characterized the metric dimension of trees. Metric dimension has several applications in robot navigation [9], chemistry [3], sonar [16] and combinatorical optimization [13]. Let G be a molecular graph, that is, a representation of the structural formula of a chemical compound in terms of graph theory. The vertices and edges of G correspond to atoms and chemical bonds, respectively. For $u, v \in V(G)$, the length of a shortest path from u to v is called the *distance* between u and v and is denoted by $d(u, v)$. A graph G is said to be k-*connected* if there does not exist a set of less than k vertices whose removal disconnects the graph G. A *planar* graph G is a graph that can be drawn in such a way that no two edges cross each other. A *cubic* graph G is a graph in which all vertices have degree 3.

Received: 25 July 2017, Revised: 13 September 2018, Accepted: 28 December 2018.

A vertex w of G *resolves* a pair u, v of vertices if $d(w,u) \neq d(w,v)$. Let $W = \{w_1, w_2, \ldots, w_k\} \subset V(G)$. The *metric representation* of a vertex $v \in V(G)$ with respect to W is the k-tuple

$$r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k)).$$

If every pair of distinct vertices of G have a distinct metric representation then the ordered set W is called a *resolving set* of G. A resolving set of minimum cardinality is called the *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $\dim(G)$. If $\dim(G) = k$, then G is said to be k*-dimensional*. Several variations of metric dimension have been discussed in the literature, like resolving dominating sets [2], independent resolving sets [4], local metric sets [11], resolving partitions [5] and strong metric generators [13].

In 1985, Kroto et al. [8] discovered fullerene molecule and since then, scientists took a great interest in the fullerene graphs. A $(k,6)-$*fullerene graph* is a 3-connected cubic plane graph whose faces have sizes k and 6. The only values of k for which a $(k,6)-$fullerene graph exists are 3, 4 and 5. A $(5,6)-$fullerene is an ordinary fullerene and constructed from pentagons and hexagons. Fowler et al. [6] discussed the mathematical properties of $(5,6)-$fullerene.

A $(3,6)$-fullerene graph have attained attention due to the similarity of its structure with ordinary fullerenes. The Eulers formula implies that a $(3,6)$-fullerene graph has exactly four faces of size 3 and $(n/2) - 2$ hexagons. If the triangles in $(3,6)$-fullerene have no common edge then it is called isolated triangular rules (ITR).

A $(4,6)$-fullerene graph is a mathematical model of a boron-nitrogen fullerene. The Eulers formula implies that a $(4,6)$-fullerene graph has exactly six square faces and $(n/2) - 4$ hexagons. If the six quadrangles in $(4,6)$-fullerene don't have common edge, then it is called isolated square rules (ISR).

Ashrafi et al. [1] calculated the topological indices of $(3,6)$- and $(4,6)$-fullerene graphs. Koorepazan-Moftakhar et al. [10] find the automorphism group and fixing number of $(3,6)$- and $(4,6)$-fullerene graphs.

Siddiqui et al. [14, 15] calculated the metric dimension and partition dimension of Nanotubes. Rajan, et al. [12] calculated the metric dimension of enhanced hypercube networks. In this paper, we consider $(3,6)$-fullerene and $(4,6)$-fullerene graphs and compute their metric dimension. We also give conjecture on the metric dimension of $(3,6)$-fullerene and $(4,6)$-fullerene graphs.

2. Metric dimension of (3,6)-fullerene graphs

Let $F_1[n]$, $F_2[n]$, $F_3[n]$ and $F_4[n]$ are the graphs of $(3,6)$-fullerene depicted in Figures **??**-4 with order $8n+4$, $12n+4$, $16n-32$ and $24n$, respectively. In this section, we find the metric dimension of $F_1[n]$, $F_2[n]$, $F_3[n]$ and $F_4[n]$ fullerene graphs.

Theorem 2.1. *The metric dimension of fullerene graph $F_1[n]$ is* 3.

Proof. Let $\{z_1, z_2, z_3\}$ and $\{z_4, z_5, z_6\}$ be the vertex sets of outer triangles of $F_1[n]$. Let $W = \{z_2, z_3, z_5\} \subset V(F_1[n])$. We show that W is a resolving set of $F_1[n]$. For this we give the representation of vertices in $V(F_1[n]) \setminus W$ with respect to W. The representation of vertices z_1, z_4 and z_6 is given by:

$$r(z_1|W) = (1,1,2), \quad r(z_4|W) = (2,2,1), \quad r(z_6|W) = (3,3,1).$$

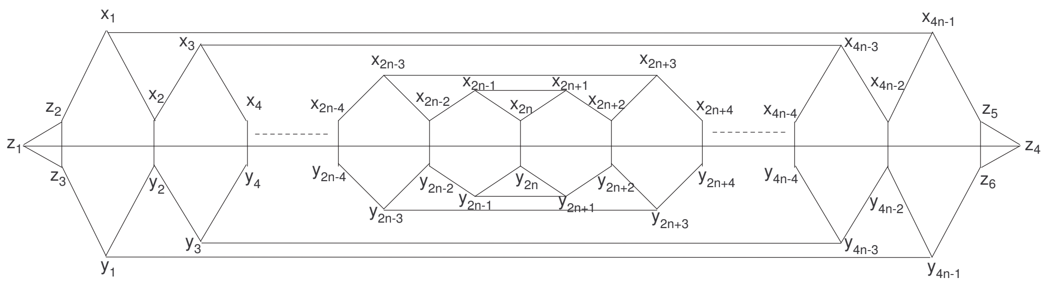

Figure 1. The Graph $F_1[n]$.

The representation of vertices of upper half of the fullerene graph $F_1[n]$ is given by:

$$r(x_i|W) = \begin{cases} (i, i+1, i+1), & \text{if } 1 \leq i \leq 2n-1, \\ (2n, 2n+1, 2n), & \text{if } i = 2n, \\ (4n-i+1, 4n-i+2, 4n-i), & \text{if } 2n+1 \leq i \leq 4n-1. \end{cases}$$

The representation of vertices of lower half of the fullerene graph $F_1[n]$ is given by:

$$r(y_i|W) = \begin{cases} (i+1, i, i+2), & \text{if } 1 \leq i \leq 2n-1, \\ (2n-1, 2n-2, 2n-1), & \text{if } i = 2n, \\ (4n-i+2, 4n-i+1, 4n-i+1), & \text{if } 2n+1 \leq i \leq 4n-1. \end{cases}$$

All the vertices of $F_1[n]$ have different representation with respect to W, this implies that W is a resolving set of $F_1[n]$. Thus the metric dimension of $\dim(F_1[n]) \leq 3$.

On the other hand, we show that $\dim(F_1[n]) \geq 3$ by proving that there is no resolving set W' such that $|W'| = 2$. Let $A = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ be the set of vertices of outer triangles of $F_1[n]$. Suppose on contrary that $\dim(F_1[n]) = 2$ and W' is a resolving set with $|W'| = 2$, then there are following cases:

Case 1. If both vertices of W' are in upper half of the fullerene graph $F_1[n]$, then the representations of pair of vertices z_4, z_6 and z_1, z_3 are the same. Thus W' is not a resolving set of $F_1[n]$.

Case 2. If both vertices of W' are in lower half of the fullerene graph $F_1[n]$, then the representations of pair of vertices z_4, z_5 and z_1, z_2 are the same. Thus W' is not a resolving set of $F_1[n]$.

Case 3. If one vertex of W' is from $\{x_1, x_2, \cdots, x_{4n-1}\}$ and other vertex is from $\{y_1, y_2, \cdots, y_{4n-1}\}$, then the pair of vertices z_1, z_5 or z_2, z_6 have the same representation with respect to W'. Thus W' is not a resolving set of $F_1[n]$.

Case 4. If one vertex from upper half of fullerene graph $F_1[n]$ and one from the set of vertices A in W', then the representation of some vertices in $\{x_1, x_2, \cdots, x_{4n-1}\}$ and $\{y_1, y_2, \cdots, y_{4n-1}\}$ is the same. Therefore W' is not a resolving set in this case.

Case 5. If one vertex from lower half of fullerene graph $F_1[n]$ and one from the set of vertices A in W', then the representation of some vertices in $\{x_1, x_2, \cdots, x_{4n-1}\}$ and $\{y_1, y_2, \cdots, y_{4n-1}\}$ is the same. Therefore W' is not a resolving set in this case.

Case 6. If both vertices of W' belongs to the set of vertices A, then we have the following subcases:

• If $W' = \{z_2, z_5\}$ or $W' = \{z_2, z_6\}$, then the representation of pair of vertices x_1, z_1 or x_1, z_3 are the same. Similarly if $W' = \{z_3, z_5\}$ or $W' = \{z_3, z_6\}$, then the representation of pair of vertices y_1, z_1 or y_1, z_2 are the same.
• All other possible subsets of A, then the representation of remanning pair of vertices of set A is the same.
Thus, in every subcase we get a contradiction.

From above cases, we conclude that there is no resolving set W' containing two vertices of $F_1[n]$. Thus the metric dimension of $F_1[n]$ is 3. This completes the proof.

□

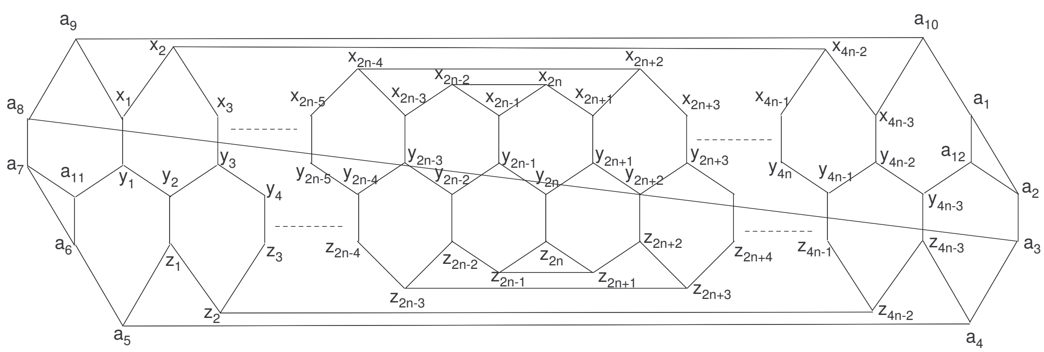

Figure 2. The Graph $F_2[n]$.

Theorem 2.2. *The metric dimension of fullerene graph $F_2[n]$ is* 3.

Proof. Let $\{a_1, a_2, a_{12}\}$ and $\{a_6, a_7, a_{11}\}$ be the vertex sets of outer triangles of $F_2[n]$. Let $W = \{a_1, a_2, a_{11}\} \subset V(F_2[n])$. We show that W is a resolving set of $F_2[n]$. For this purpose, we give the representation of vertices in $V(F_2[n]) \setminus W$ with respect to W. The representation of outer vertices of the fullerene graph $F_2[n]$ is given below:

$$\begin{array}{lll} r(a_3|W) = (2,1,3), & r(a_4|W) = (3,2,3), & r(a_5|W) = (4,3,2), \\ r(a_6|W) = (5,4,1), & r(a_7|W) = (4,3,1), & r(a_8|W) = (3,2,2), \\ r(a_9|W) = (2,3,3), & r(a_{10}|W) = (1,2,4), & r(a_{12}|W) = (1,1,5). \end{array}$$

The representation of vertices of upper half of the fullerene graph $F_2[n]$ is given below:

$$r(x_i|W) = \begin{cases} (i+2, i+3, i+1), & \text{if } 1 \le i \le 2n-2, \\ (2n, 2n+1, 2n), & \text{if } i = 2n-1, \\ (4n-i-1, 4n-i, 4n-i), & \text{if } 2n \le i \le 4n-2, \\ (2,3,5), & \text{if } i = 4n-3. \end{cases}$$

The representation of middle vertices of the fullerene graph $F_2[n]$ for $n = 1$ is given below:

$$r(y_i|W) = \begin{cases} (4n-i, 4n-i, i), & \text{if } i = 2n-11, \\ (4n-i, 4n-i, 4n-i), & \text{if } i = 4n-2. \end{cases}$$

The representation of middle vertices of the fullerene graph $F_2[n]$ for $n \geq 2$ is given below:

$$r(y_i|W) = \begin{cases} (4,5,1), & \text{if } i = 1, \\ (i+3, i+3, i), & \text{if } 2 \leq i \leq 2n-2, \\ (4n-i, 4n-i, i), & \text{if } 2n-1 \leq i \leq 2n, \\ (4n-i, 4n-i, 4n-i+1), & \text{if } 2n+1 \leq i \leq 4n-5, \\ (4n-i, 4n-i, 4n-i+3), & \text{if } 4n-4 \leq i \leq 4n-2. \end{cases}$$

The representation of vertices of lower half of the fullerene graph $F_2[n]$ is given below:

$$r(z_i|W) = \begin{cases} (5,4,3), & \text{if } i = 1, \\ (i+3, i+3, i+2), & \text{if } 2 \leq i \leq 2n-2, \\ (2n+1, 2n+1, 2n+1), & \text{if } i = 2n-1, \\ (4n-i, 4n-i, 4n-i+1), & \text{if } 2n \leq i \leq 4n-3. \end{cases}$$

However all pair of vertices can easily be resolved by the set W. Thus the set W is a resolving set of $F_2[n]$ and $\dim(F_2[n]) \leq 3$. Now, we show that $\dim(F_2[n]) \geq 3$ by showing that there is no resolving set W' such that $|W'| = 2$. Let $A = \{a_1, a_2, a_3, \cdots, a_{12}\}$, $B = \{x_1, x_2, \cdots, x_{4n-3}\}$, $C = \{y_1, y_2, \cdots, y_{4n-3}\}$ and $D = \{z_1, z_2, \cdots, z_{4n-3}\}$ be the sets of vertices of $F_2[n]$. Suppose on contrary that $\dim(F_2[n]) = 2$ and W' is a resolving set with $|W'|$. Then there are following possibilities:

Case 1. The pair of vertices a_1, a_2 and a_7, a_8 have the same distance from the vertices of set B, C and D. Then any subset of B, C and D is not a resolving set of $F_2[n]$.

Case 2. If both vertices of W' are from set of vertices A, then some vertices of A have same representation. Therefore W' is not a resolving set of $F_2[n]$.

Thus in every case we get a contradiction. Thus we conclude that there is no resolving set W' containing two vertices of $F_2[n]$. Thus the metric dimension of $F_2[n]$ is 3. This completes the proof. □

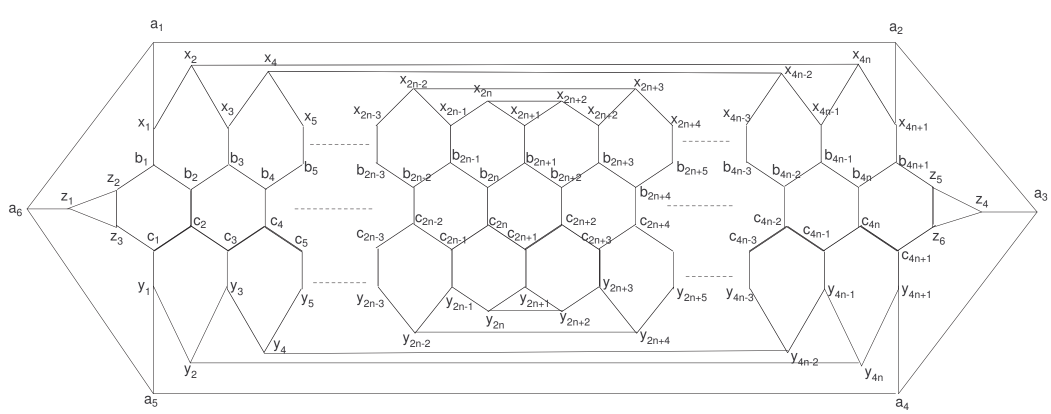

Figure 3. The Graph $F_3[n]$.

Theorem 2.3. *The metric dimension of fullerene graph $F_3[n]$ is* 3.

Proof. Let $\{z_1, z_2, z_3\}$ and $\{z_4, z_5, z_6\}$ be the vertex sets of outer triangles and $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ be the vertices of outer hexagonal of $F_3[n]$. Let $W = \{a_5, z_2, z_5\} \subset V(F_3[n])$. We need to show that W is a resolving set of $F_3[n]$. For this purpose, first we give the representation of vertices in $V(F_3[n]) \setminus W$ with respect to W. The representation of outer vertices of fullerene graph $F_3[n]$ is given by:

$$\begin{aligned} &r(a_1|W) = (2,3,4), \quad r(a_2|W) = (3,4,3), \quad r(a_3|W) = (2,5,2),\\ &r(a_4|W) = (1,4,3), \quad r(a_6|W) = (1,2,5). \end{aligned}$$

The representation of vertices of outer triangles in $F_3[n]$ is given by:

$$\begin{aligned} &r(z_1|W) = (2,1,6), \quad r(z_3|W) = (3,1,7),\\ &r(z_4|W) = (3,6,1), \quad r(z_6|W) = (4,7,1). \end{aligned}$$

The representation of vertices of upper half of the fullerene graph $F_3[n]$ is given by:

$$r(x_i|W) = \begin{cases} (3,2,5), & \text{if } i = 1,\\ (i+2, i+1, i+2), & \text{if } 2 \le i \le 2n,\\ (2n+3, 2n+2, 2n+2), & \text{if } i = 2n+1,\\ (4n-i+5, 4n-i+4, 4n-i+3), & \text{if } 2n+2 \le i \le 4n,\\ (4,5,2), & \text{if } i = 4n+1. \end{cases}$$

The representation of middle vertices of the fullerene graph $F_3[n]$ for $n = 1$ is given by:

$$\begin{aligned} &r(b_1|W) = (4,1,5), \quad r(b_2|W) = (4,2,4), \quad r(b_3|W) = (5,3,3),\\ &r(b_4|W) = (5,4,2), \quad r(b_5|W) = (5,5,1). \end{aligned}$$

The representation of middle vertices of the fullerene graph $F_3[n]$ for $n \ge 2$ is given by:

$$r(b_i|W) = \begin{cases} (4, i, i+5), & \text{if } i \in \{1,2\},\\ (i+2, i, i+3), & \text{if } 3 \le i \le 2n-1,\\ (i+2, i, 4n-i+2), & \text{if } i \in \{2n, 2n+1\},\\ (2n+3, 2n+2, 2n), & \text{if } i = 2n+2,\\ (4n-i+5, 4n-i+5, 4n-i+2), & \text{if } 2n+3 \le i \le 4n-1,\\ (5, 4n-i+7, 4n-i+2), & \text{if } i \in \{4n, 4n+1\}. \end{cases}$$

The representation of middle vertices of the fullerene graph $F_3[n]$ is given by:

$$r(c_i|W) = \begin{cases} (i+1, i+1, i+5), & \text{if } i \in \{1,2\},\\ (i+1, i+1, i+4), & \text{if } 3 \le i \le 2n-1,\\ (i+1, i+1, 4n-i+3), & \text{if } i \in \{2n, 2n+1\},\\ (2n+2, 2n+3, 2n+1), & \text{if } i = 2n+2,\\ (4n-i+4, 4n-i+6, 4n-i+3), & \text{if } 2n+3 \le i \le 4n-1,\\ (4n-i+4, 4n-i+7, 4n-i+3), & \text{if } i \in \{4n, 4n+1\}. \end{cases}$$

The representation of vertices of lower half of the fullerene graph $F_3[n]$ is given by:

$$r(y_i|W) = \begin{cases} (1,3,5), & \text{if } i = 1, \\ (i, i+2, i+3), & \text{if } 2 \leq i \leq 2n, \\ (2n+1, 2n+2, 2n+2), & \text{if } i = 2n+1, \\ (4n-i+3, 4n-i+5, 4n-i+4), & \text{if } 2n+2 \leq i \leq 4n, \\ (2,5,3), & \text{if } i = 4n+1. \end{cases}$$

Therefore the set W resolves all the vertices in $V(F_3[n]) \setminus W$. Thus $\dim(F_3[n]) \leq 3$. Now we show that $\dim(F_3[n]) \geq 3$. For this we show that there does not exists any resolving set W' with two vertices. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $B = \{z_1, z_2, z_3, z_4, z_5, z_6\}$, $C = \{x_1, x_2, \cdots, x_{4n+1}\}$, $D = \{b_1, b_2, \cdots, b_{4n+1}\}$, $E = \{c_1, c_2, \cdots, c_{4n+1}\}$ and $F = \{y_1, y_2, \cdots, y_{4n+1}\}$ be the sets of vertices of $F_3[n]$. Then there are following cases:

Case 1. If both vertices of W' are in the set of vertices C, then the vertices of A and D have the same representation. Therefore W' is not a resolving set of $F_3[n]$.

Case 2. If both vertices of W' are in the set of vertices D, then the vertices of B and C have the same representation. Therefore W' is not a resolving set of $F_3[n]$.

Case 3. The vertices of set A have same distance from the vertices of B. Therefore the resolving set is not the subset of A or B.

Case 4. If both vertices of W' are in the set of vertices F, then the vertices of A and E have the same representation. Therefore W' is not a resolving set of $F_3[n]$.

Case 5. If both vertices of W' are in the set of vertices E, then the vertices of B and F have the same representation. Therefore W' is not a resolving set of $F_3[n]$.

Thus, in every case we get a contradiction. From above cases, we conclude that there is no resolving set W' with exactly two vertices of $F_3[n]$. Thus the metric dimension of $F_3[n]$ is 3. This completes the proof. □

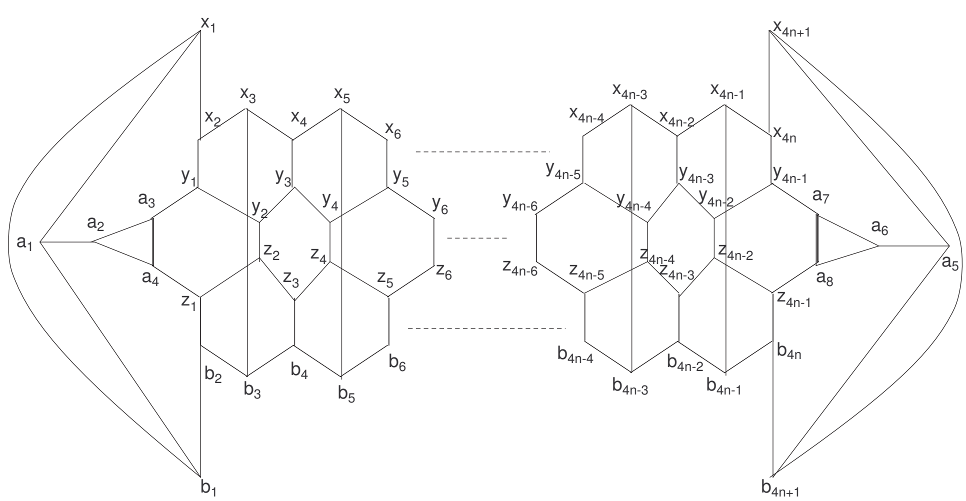

Figure 4. The Graph $F_4[n]$.

Theorem 2.4. *The metric dimension of fullerene graph $F_4[n]$ is 3.*

Proof. Let $W = \{a_3, a_7, a_8\} \subset V(F_4[n])$. We need to show that W is a resolving set of $F_4[n]$. First we give the representation of vertices of $F_4[n] \setminus W$ with respect to W.

$$\begin{array}{lll} r(a_1|W) = (2, 4n+2, 4n+2), & r(a_2|W) = (1, 4n+1, 4n+1), & r(a_4|W) = (1, 4n+1, 4n), \\ r(a_5|W) = (4n+2, 2, 2), & r(a_6|W) = (4n+1, 1, 1). & \end{array}$$

The representation of vertices of upper half of the fullerene graph $F_4[n]$ is given below:

$$r(x_i|W) = \begin{cases} (3, 4n+i, 4n+i+1), & \text{if } i = 1, \\ (i, 4n-i+2, 4n-i+3), & \text{if } 2 \le i \le 4n, \\ (4n+1, 3, 4), & \text{if } i = 4n+1. \end{cases}$$

The representation of middle vertices of the fullerene graph $F_4[n]$ is given below:

$$\begin{array}{ll} r(y_i|W) = (i, 4n-i, 4n-i+1), & 1 \le i \le 4n-1, \\ r(z_i|W) = (i+1, 4n-i+1, 4n-i), & 1 \le i \le 4n-1. \end{array}$$

The representation of vertices of lower half of the fullerene graph $F_4[n]$ is given below:

$$r(b_i|W) = \begin{cases} (3, 4n+2, 4n+1), & \text{if } i = 1, \\ (i+1, 4n-i+3, 4n-i+2), & \text{if } 2 \le i \le 4n, \\ (4n+2, 3, 3), & \text{if } i = 4n+1. \end{cases}$$

All vertices of $V(F_4[n]) \setminus W$ can be resolved with respect to W. Thus W is a resolving set of $F_4[n]$ and $\dim(F_4[n]) \le 3$.

On the other hand, we show that $\dim(F_4[n]) \ge 3$ by proving that there is no resolving set W' with cardinality 2. Suppose on contrary that $\dim(F_4[n]) = 2$ and W' is a resolving set of $F_4[n]$ with $|W'| = 2$. Let $A = \{a_1, a_2, \cdot, a_8\}$, $B = \{x_1, x_2, \cdots, x_{4n+1}\}$, $C = \{y_1, y_2, \cdots, y_{4n+1}\}$, $D = \{z_1, z_2, \cdots, z_{4n+1}\}$ and $E = \{b_1, b_2, \cdots, b_{4n+1}\}$ be the sets of vertices of $F_4[n]$. The pairs of vertices a_1, a_3 and a_5, a_7 have the same distance with the vertices of set B. The pairs of vertices a_2, a_4 and a_6, a_8 have the same distance with the vertices of set C. The pairs of vertices a_2, a_3 and a_6, a_7 have the same distance with the vertices of set D. Similarly, The pairs of vertices a_1, a_4 and a_5, a_8 have the same distance with the vertices of set C. Therefore, there is no a resolving set W' with cardinality 2 of $F_4[n]$. Thus $\dim(F_4[n]) = 3$. This completes the proof. □

3. Metric dimension of (4,6)-fullerene graphs

Suppose $G_1[n]$, $G_2[n]$ and $G_3[n]$ are depicted in Figures 5-7 with order $8n$, $8n+4$ and $12n+12$ respectively. In this subsection we find the metric dimension of $G_1[n]$, $G_2[n]$ and $G_3[n]$ fullerene graphs.

Theorem 3.1. *The metric dimension of fullerene graph $G_1[n]$ is 3 for $n \ge 2$.*

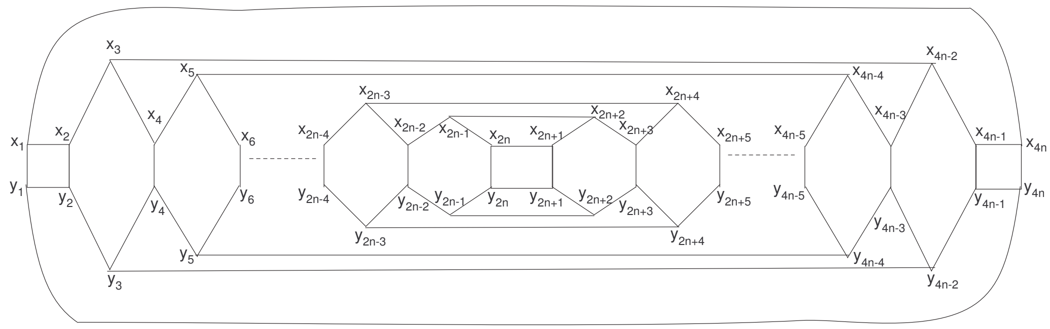

Figure 5. The Graph $G_1[n]$.

Proof. For a set $W = \{x_1, y_1, x_{4n}\} \subset V(G_1[n])$, we need to show that W is a resolving set of $G_1[n]$. First we show that $\dim(G_1[n]) \neq 2$. There are following cases:

Case 1. If both vertices are in the upper half of $G_1[n]$ and the resolving set is $W' = \{x_s, x_t\}$, $1 \leq s \leq t \leq 4n$. Then the representations of x_i and y_{i+1}, $2n + 1 \leq i \leq 4n - 1$ are the same. Similarly the representations of x_{i+1} and y_i, $2 \leq i \leq 2n - 1$ are the same. Therefore the resolving set of $G_1[n]$ is not a subset of $\{x_1, x_2, \cdots, x_{4n}\}$.

Case 2. If both vertices are in the lower half of $G_1[n]$ and the resolving set is $W' = \{y_s, y_t\}$, $1 \leq s \leq t \leq 4n$. Then the representations of x_{i+1} and y_i, $2n + 1 \leq i \leq 4n - 1$ are the same. Similarly the representations of x_i and y_{i+1}, $2 \leq i \leq 2n - 1$ are the same. Therefore the resolving set is not a subset of $\{y_1, y_2, \cdots, y_{4n}\}$.

Case 3. If one vertex belongs to the set of vertices $\{x_1, x_2, \cdots, x_{4n}\}$ and other is in the set of vertices $\{y_1, y_2, \cdots, y_{4n}\}$. Without loss of generality, we can suppose that the resolving set is $W' = \{x_s, y_t\}$, $1 \leq s \leq 4n$ and $1 \leq t \leq 4n$.

If $s = t$, then the representation of pairs of vertices x_{s+1}, x_{s-1} and y_{t-1}, y_{t+1} are the same.
If $s < t$, then the representation of x_i, $i > s$ and y_j, $j < 4n$ are the same.
If $s > t$, then the representation of x_i, $i < 4n$ and y_j, $j > t$ are the same.

Thus, in every subcase we get a contradiction. From above cases, we conclude that there is no resolving set W' with $|W'| = 2$. Thus $\dim(G_1[n]) \geq 3$. Now we show that $\dim(G_1[n] \leq 3$. For this purpose we give the representation of the vertices in $V(G_1[n]) \setminus W$ with respect to W. The representation of vertices of upper half of the fullerene graph $G_1[n]$ is given below:

$$r(x_i|W) = \begin{cases} (i-1, i, i), & \text{if } 2 \leq i \leq 2n, \\ (4n-i+1, 4n-i+2, 4n-i), & \text{if } 2n+1 \leq i \leq 4n-1. \end{cases}$$

The representation of vertices of lower half of the fullerene graph $G_1[n]$ is given below:

$$r(y_i|W) = \begin{cases} (i, i-1, i+1), & \text{if } 2 \leq i \leq 2n, \\ (4n-i+2, 4n-i+1, 4n-i+1), & \text{if } 2n+1 \leq i \leq 4n. \end{cases}$$

This implies that all vertices of $V(G_1[n]) \setminus W$ can be resolved with respect to W. Thus W is a resolving set of $G_1[n]$. Therefore $\dim(G_1[n]) = 3$ for $n \geq 2$. This completes the proof. □

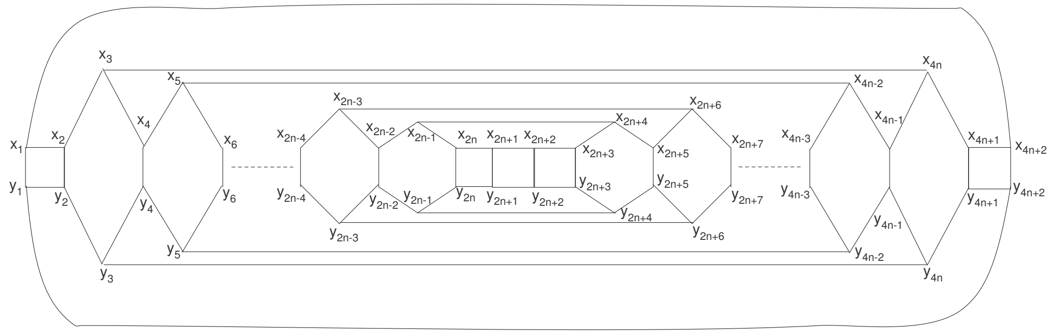

Figure 6. The Graph $G_2[n]$.

Theorem 3.2. *The metric dimension of fullerene graph $G_2[n]$ is 3.*

Proof. Let $W = \{x_1, y_1, x_{4n+2}\} \subset V(G_2[n])$. We show that W is a resolving set of $G_2[n]$. First we give the representation of vertices in $V(G_2[n]) \setminus W$ with respect to W. The representation of vertices of upper half of the fullerene graph $G_2[n]$ is given by:

$$r(x_i|W) = \begin{cases} (i-1, i, i), & \text{if } 2 \le i \le 2n+1, \\ (4n-i+3, 4n-i+4, 4n-i+2), & \text{if } 2n+2 \le i \le 4n+1. \end{cases}$$

The representation of vertices of lower half of the fullerene graph $G_2[n]$ is given by:

$$r(y_i|W) = \begin{cases} (i, i-1, i+1), & \text{if } 2 \le i \le 2n+1, \\ (4n-i+4, 4n-i+3, 4n-i+3), & \text{if } 2n+2 \le i \le 4n+2. \end{cases}$$

This implies that W is a resolving set of $G_2[n]$ and $\dim(G_2[n]) \le 3$. Now we show that $\dim(G_2[n]) \ge 3$ by proving that W' is a resolving set of $G_2[n]$ with $|W'| = 2$. There are following possibilities:

Case 1. If both vertices are in the upper half of $G_2[n]$ and the resolving set is $W' = \{x_s, x_t\}$, $1 \le s \le t \le 4n$. Then the representation of x_i and y_{i+1}, $2n+2 \le i \le 4n-1$ is the same. Similarly the representation of x_{i+1} and y_i, $2 \le i \le 2n$ is the same. Therefore the resolving set of $G_2[n]$ is not a subset of $\{x_1, x_2, \cdots, x_{4n}\}$.

Case 2. If both vertices are in the lower half of $G_1[n]$ and the resolving set is $W' = \{y_s, y_t\}$, $1 \le s \le t \le 4n$. Then the representation of x_{i+1} and y_i, $2n+2 \le i \le 4n-1$ is the same. Similarly the representation of x_i and y_{i+1}, $2 \le i \le 2n$ is the same. Therefore the resolving set of $G_2[n]$ is not a subset of $\{y_1, y_2, \cdots, y_{4n}\}$.

Case 3. If one vertex belongs to the set of vertices $\{x_1, x_2, \cdots, x_{4n}\}$ and other is in the set of vertices $\{y_1, y_2, \cdots, y_{4n}\}$. Without loss of generality, we can suppose that the resolving set of $G_2[n]$ is $W' = \{x_s, y_t\}$, $1 \le s \le 4n$ and $1 \le t \le 4n$.

If $s = t$, then the representation of pair of vertices x_{s+1}, x_{s-1} and y_{t-1}, y_{t+1} is the same.

If $s < t$, then the representation of x_i, $i > s$ and y_j, $j < 4n$ is the same.

If $s > t$, then the representation of x_i, $i < 4n$ and y_j, $j > t$ is the same.

From above cases, we conclude that there is no resolving set W' for $G_2[n]$ with $|W'| = 2$. Thus $\dim(G_2[n]) = 3$. This completes the proof. □

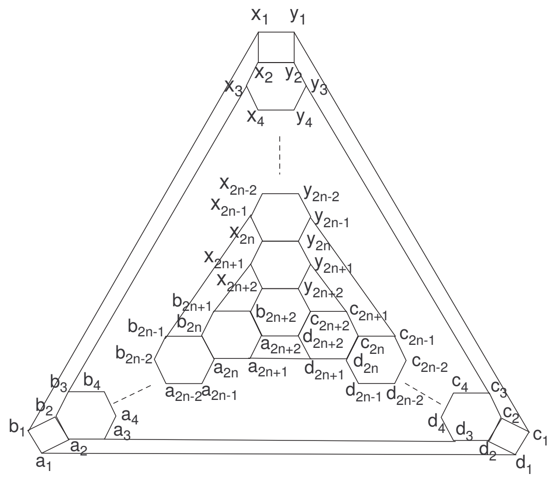

Figure 7. The Graph $G_3[n]$.

Theorem 3.3. *The metric dimension of fullerene graph* $G_3[n]$ *is* 3*.*

Proof. Let $\{x_1, y_1, a_1, b_1, c_1, d_1\}$ be the set of outer vertices of $G_3[n]$. The vertex x_1 and y_1 have the same distance from other vertices and a_1 and b_1 have the same distance from other vertices of $G_3[n]$. Similarly c_1 and d_1 have the same distance from other vertices of $G_3[n]$. Thus any metric basis will contain either x_1 or y_1, a_1 or b_1 and c_1 or d_1. There are 6 boundary vertices in $G_3[n]$. Hence any metric basis of $G_3[n]$ should contain at least 3 nodes of $G_3[n]$. Let $W = \{x_1, a_1, c_1\} \subset V(G_3[n])$, we need to show that W is a resolving set of $G_3[n]$. Then the representation of vertices in $V(G_3[n]) \setminus W$ with respect to W for $n \geq 2$ is given by:

$$
\begin{array}{lll}
r(x_i|W) = (i-1, i+1, i+1), & \text{if} & 2 \leq i \leq 2n+2, \\
r(y_i|W) = (i, i+2, i), & \text{if} & 1 \leq i \leq 2n+2, \\
r(a_i|W) = (i+1, i-1, i+1), & \text{if} & 2 \leq i \leq 2n+2, \\
r(b_i|W) = (i, i, i+2), & \text{if} & 1 \leq i \leq 2n+2, \\
r(c_i|W) = (i+1, i+1, i-1), & \text{if} & 2 \leq i \leq 2n+2, \\
r(d_i|W) = (i+2, i, i), & \text{if} & 1 \leq i \leq 2n+2.
\end{array}
$$

The representation of vertices of $V(G_3[n]) \setminus W$ with respect to W for $n = 1$ is given by:

$$
r(x_i|W) = \begin{cases} (i-1, i+1, i+1), & \text{if } 2 \leq i \leq 2n+1, \\ (i-1, i, i), & \text{if } i = 2n+2. \end{cases}
$$

$$
r(y_i|W) = \begin{cases} (i, i+2, i), & \text{if } 1 \leq i \leq 2n+1, \\ (i, i+1, i), & \text{if } i = 2n+2. \end{cases}
$$

$$
\begin{array}{lll}
r(a_i|W) = (i+1, i-1, i+1), & \text{if} & 2 \leq i \leq 2n+2, \\
r(b_i|W) = (i, i, i+2), & \text{if} & 1 \leq i \leq 2n+2, \\
r(c_i|W) = (i+1, i+1, i-1), & \text{if} & 2 \leq i \leq 2n+2, \\
r(d_i|W) = (i+2, i, i), & \text{if} & 1 \leq i \leq 2n+2.
\end{array}
$$

Hence there are no two vertices having the same representations. Thus $W = \{x_1, c_1, a_1\}$ is a resolving set of $G_3[n]$. Therefore $\dim(G_3[n]) = 3$. This completes the proof. □

4. Conclusion and open problems

In this paper, we considered some $(3,6)$-fullerene and $(4,6)$-fullerene graphs and computed the metric dimension for these fullerene graphs. All $(3,6)$-fullerene and $(4,6)$-fullerene graphs considered in this paper have metric dimension 3. It will be interesting to prove or disprove the following statement:

"All $(3,6)$-fullerene and $(4,6)$-fullerene graphs have metric dimension 3."

Acknowledgement

This research is supported by the Higher Education Commission of Pakistan under grant No. 20-3067/NRPU /R&D/HEC/12

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