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On the distance domination number of bipartite graphs

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Abstract

Let G be a graph and k be a positive integer. A vertex set D is called a k-distance dominating set of G if each vertex of G is either in D or at a maximum distance k from some vertex of D. k-distance domination number of G is the minimum cardinality among all k-distance dominating sets of G. In this note we give upper bounds on the k-distance domination number of a connected bipartite graph, and improve some results have been given like Theorems 2.1 and 2.7 in [Tian and Xu, A note on distance domination of graphs, Australasian Journal of Combinatorics, 43 (2009), 181-190].

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1. Introduction

We refer the reader to [9] for terminology and notation on graph theory not given here. In a simple graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, the *order* and the *size* of G is denoted by $n = |V(G)|$ and $m = |E(G)|$, respectively. The *open neighborhood* of a vertex v is defined as $N(v) := \{u \in V : uv \in E\}$, and the set $N[v] = N(v) \cup \{v\}$ is called the *closed*

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neighborhood of v. Similarly, the set $N(S) = \cup_{v\in S} N(v)$ is called the *open neighborhood* of a set $S \subseteq V$ and the set $N[S] = N(S) \cup S$ is the *closed neighborhood* of S. For a vertex $v \in V$, the *degree* of v is $\deg_G(v) = \deg(v) = |N(v)|$. $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the *minimum degree* and *maximum degree*, respectively, among all vertices of G. For a vertex $v \in V$, the set $N_k(v) = \{u : d(u, v) \leq k \text{ and } u \neq v\}$ is called the *open k-neighborhood* of v. In the other words, $N_k(v)$ is the set of all vertices in within distance k of v. The set $N_k[v] = N_k(v) \cup \{v\}$ is said to be the *closed k-neighborhood* of v.
A set $D \subseteq V$ is a *dominating set* if every vertex in $V - D$ has a neighbor in D. The minimum cardinality among all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. A vertex set $K \subseteq V$ is a *k-distance dominating set* if every vertex in $V - K$ is within distance k of some vertex in K. In the other words, if $K \subseteq V$ is a k-distance dominating set of G, then $N_k[K] = V$. The k-distance domination number of G, $\gamma_k(G)$, is the minimum cardinality among all k-distance dominating sets in G, for further see, [3, 4, 5, 8]. The kth power graph of G is a new graph with $V(G^k) = V(G)$ and two vertices $x, y \in V(G^k)$ are adjacent in G^k if $d_G(x, y) \leq k$. Note that $\gamma_k(G)$ equals to $\gamma(G^k)$, where G^k is the kth power graph of G, see [2, 4, 6, 7].

2. Previous known results

Tian and Xu [7] studied k-distance domination number in graphs. They have proved the following results.

Theorem 2.1 (Tian and Xu [7], Theorem 2.1). *Let $V = \{1, 2, \cdots, n\}$ be the vertex set of a connected graph G. Then $\gamma_k(G) \leq \min_{(p_1,p_2,\cdots,p_n)\in(0,1)^n} \sum_{i=1}^{n} \Big(p_i + (1 - p_i) \prod_{j\in N_k(i)} (1 - p_j)\Big)$ where $p_i \in (0, 1)$ is the probability of existence of the vertex i in a random subset of V.*

Then they considered connected bipartite graph.

Lemma 2.1 (Tian and Xu [7], Lemma 2.5). *Let G be a connected bipartite graph with bipartition V_1 and V_2, where $|V_j| = n_j$ and $\delta_j = \min\{\deg(v) : v \in V_j\}$, for $j = 1, 2$.*
For any vertex $v \in V_1$ with $N_k[v] \neq V$,

$$|N_k(v) \cap V_1| \geq (\lceil k/6\rceil - 1)(\delta_2 + 1), \tag{1}$$

$$|N_k(v) \cap V_2| \geq \lceil k/6\rceil(\delta_1 + 1) - 1. \tag{2}$$

Similarly, for any vertex $v \in V_2$ with $N_k[v] \neq V$,

$$|N_k(v) \cap V_1| \geq \lceil k/6\rceil(\delta_2 + 1) - 1, \tag{3}$$

$$|N_k(v) \cap V_2| \geq (\lceil k/6\rceil - 1)(\delta_1 + 1). \tag{4}$$

Let G be a connected bipartite graph. It is said to be *perfect* if $\delta_1\delta_2 > 1$, $n_2[M(\delta_2 + 1) - 1] > n_1[(M - 1)(\delta_1 + 1) + 1]$ and $n_1[M(\delta_1 + 1) - 1] > n_2[(M - 1)(\delta_2 + 1) + 1]$, where $M = \lceil k/6\rceil$. A simple calculation shows that a connected bipartite graph is perfect if and only if $n_1 - n_2\delta_2 < M[n_1(\delta_1 + 1) - n_2(\delta_2 + 1)] < n_1\delta_1 - n_2$. As a consequence of Lemma 2.1 and Theorem 2.1, Tian and Xu obtained the following.

Theorem 2.2 (Tian and Xu [7], Theorem 2.7)**.** *Let G be a perfect bipartite graph and*

$$0 < p_1 = \frac{[(M-1)(\delta_1+1)+1]\ln u - [M(\delta_1+1)-1]\ln v}{(2M-1)(\delta_1\delta_2-1)} < 1$$

$$0 < p_2 = \frac{[(M-1)(\delta_2+1)+1]\ln v - [M(\delta_2+1)-1]\ln u}{(2M-1)(\delta_1\delta_2-1)} < 1,$$

where $u = \frac{n_2[M(\delta_2+1)-1]-n_1[(M-1)(\delta_1+1)+1]}{n_1(2M-1)(\delta_1\delta_2-1)}$ and $v = \frac{n_1[M(\delta_1+1)-1]-n_2[(M-1)(\delta_2+1)+1]}{n_2(2M-1)(\delta_1\delta_2-1)}$. Then

$$\gamma_k(G) \le h(p_1,p_2) \le \min_{0<p<1} h(p,p) \le \frac{n(1+\ln[(2M-1)(\delta+1)])}{(2M-1)(\delta+1)},$$

where $M = \lceil k/6 \rceil$.

In this manuscript we improve Theorem 2.2 via improving the Lemma 2.1.

3. Main results

In order to improve Theorem 2.2, we first improve Lemma 2.1.

Lemma 3.1. *Let G be a connected bipartite graph with bipartition V_1 and V_2, where $|V_j| = n_j$ and $\delta_j = \min\{\deg(v) : v \in V_j\}$, for $j = 1, 2$. Then*
(i) *For any vertex $v \in V_1$ with $N_k[v] \neq V$,*

$$|N_k(v) \cap V_1| \ge \lceil (k-1)/4 \rceil \max\{2, \delta_2\} + 2\lfloor k/4 \rfloor - \lfloor k/2 \rfloor, \tag{5}$$

$$|N_k(v) \cap V_2| \ge \delta_1 + (\lceil k/4 \rceil - 1)\max\{2, \delta_1\} + \lfloor (k-1)/2 \rfloor - 2\lfloor (k-1)/4 \rfloor. \tag{6}$$

Furthermore, (5) and (6), improve (1) and (2), repectively.
(ii) *For any vertex $v \in V_2$ with $N_k[v] \neq V$,*

$$|N_k(v) \cap V_1| \ge \lceil k/4 \rceil \max\{2, \delta_2\} + \lfloor (k-1)/2 \rfloor - 2\lfloor (k-1)/4 \rfloor, \tag{7}$$

$$|N_k(v) \cap V_2| \ge \lceil (k-1)/4 \rceil \max\{2, \delta_1\} + 2\lfloor k/4 \rfloor - \lfloor k/2 \rfloor. \tag{8}$$

Furthermore, (7) and (8) improve (3) and (4), respectively.

Proof. Let G be a connected bipartite graph with bipartition V_1 and V_2, where $|V_j| = n_j$ and $\delta_j = \min\{\deg(v) : v \in V_j\}$, for $j = 1, 2$. For any vertex v and any integer l with $1 \le l \le k$, let $X_l(v) = \{u \in V | d(v,u) = l\}$. It is obvious that $N_k(v) = X_1(v) \cup X_2(v) \cup \cdots \cup X_k(v)$. Furthermore, $X_1(v)$, $X_2(v)$,...,and $\ldots, X_k(v)$ are pairly disjoint.
(i) Let $v \in V_1$ be a vertex with $N_k[v] \neq V$. Observe that $X_1(v) \cup X_3(v) \cup \cdots \cup X_{2\lfloor (k+1)/2 \rfloor - 1}(v) \subseteq V_2$, $X_2(v) \cup X_4(v) \cup \cdots \cup X_{2\lfloor k/2 \rfloor}(v) \subseteq V_1$, and

$$N_k(v) \cap V_1 = \bigcup_{m=1}^{\lfloor k/2 \rfloor} X_{2m}(v), \quad N_k(v) \cap V_2 = \bigcup_{m=1}^{\lfloor (k+1)/2 \rfloor} X_{2m-1}(v).$$

Thus, $|N_k(v) \cap V_1| = \sum_{m=1}^{\lfloor k/2 \rfloor} |X_{2m}(v)|$ and $|N_k(v) \cap V_2| = \sum_{m=1}^{\lfloor (k+1)/2 \rfloor} |X_{2m-1}(v)|$. Since $N_k[v] \neq V$, there exists a vertex u such that $d(v, u) > k$. Therefore, there exists a path, $P := vx_1x_2 \ldots u$ of length of at least $k+1$. For $l = 1, 2, \cdots, k$, $X_l(v) \neq \emptyset$, because $x_l \in X_l(v)$. Moreover, if l is odd, then $\deg(x_l) \geq \max\{2, \delta_2\}$, because $x_l \in V_2$; while if l is even, then $\deg(x_l) \geq \max\{2, \delta_1\}$, because $x_l \in V_1$. We continue with two following claims.
Claim 1. $|X_2(v)| \geq \max\{2, \delta_2\} - 1 \geq \delta_2 - 1$.
To see this, note that since $x_1 \in X_1(v) \subseteq V_2$, we have $|X_2(v)| = \deg(x_1) - 1$. Since $\deg(x_1) \geq \max\{2, \delta_1\}$, we find that $|X_2(v)| \geq \max\{2, \delta_2\} - 1$, as desired.
Claim 2. For $2 \leq l \leq k-1$, $|X_{l-1}(v)| + |X_{l+1}(v)| \geq \deg(x_l)$.
To see this, note that for $2 \leq l \leq k-1$, we have $N_1(x_l) = N(x_l) \subseteq X_{l-1}(v) \cup X_{l+1}(v)$, since $x_l \in X_l(v)$.

By Claim 2, $|X_{4m}(v)| + |X_{4m+2}(v)| \geq \deg(x_{4m+1})$ for every $m = 1, 2, ..., \lfloor \frac{\lfloor k/2 \rfloor - 1}{2} \rfloor$. To compute $|N_k(v) \cap V_1|$, we discuss on $\frac{\lfloor k/2 \rfloor - 1}{2}$ which may be an integer or not.

First we assume that $\frac{\lfloor k/2 \rfloor - 1}{2}$ is an integer. Hence,

$$\begin{aligned}
|N_k(v) \cap V_1| &= \sum_{m=1}^{\lfloor k/2 \rfloor} |X_{2m}(v)| = |X_2(v)| + \sum_{m=2}^{\lfloor k/2 \rfloor} |X_{2m}(v)| \\
&= |X_2(v)| + \sum_{m'=1}^{(\lfloor k/2 \rfloor - 1)/2} (|X_{4m'}(v)| + |X_{4m'+2}(v)|) \\
&\geq \max\{2, \delta_2\} - 1 + \sum_{m'=1}^{(\lfloor k/2 \rfloor - 1)/2} \max\{2, \delta_2\} \quad (\textit{by Claims } 1 \textit{ and } 2).
\end{aligned}$$

Thus, $|N_k(v) \cap V_1| \geq (\lfloor k/2 \rfloor + 1) \max\{2, \delta_2\}/2 - 1$ and a simple calculation shows that $(\lfloor k/2 \rfloor + 1) \max\{2, \delta_2\}/2 - 1 = \lceil (k-1)/4 \rceil \max\{2, \delta_2\} + 2\lfloor k/4 \rfloor - \lfloor k/2 \rfloor$, as desired.

Next we assume that $\frac{\lfloor k/2 \rfloor - 1}{2}$ is not an integer. Hence,

$$\begin{aligned}
|N_k(v) \cap V_1| &= \sum_{m=1}^{\lfloor k/2 \rfloor} |X_{2m}(v)| = |X_2(v)| + \sum_{m=2}^{\lfloor k/2 \rfloor - 1} |X_{2m}(v)| + |X_{2\lfloor k/2 \rfloor}| \\
&= |X_2(v)| + \sum_{m'=1}^{(\lfloor k/2 \rfloor - 2)/2} (|X_{4m'}(v)| + |X_{4m'+2}(v)|) + |X_{2\lfloor k/2 \rfloor}| \\
&\geq \max\{2, \delta_2\} - 1 + \sum_{m'=1}^{(\lfloor k/2 \rfloor - 2)/2} \max\{2, \delta_2\} + 1 \quad (\textit{by Claims } 1 \textit{ and } 2).
\end{aligned}$$

Therefore, we have $|N_k(v) \cap V_1| \geq \lfloor k/2 \rfloor \max\{2, \delta_2\}/2$ and a simple calculation shows that $\lfloor k/2 \rfloor \max\{2, \delta_2\}/2 = \lceil (k-1)/4 \rceil \delta_2 + 2\lfloor k/4 \rfloor - \lfloor k/2 \rfloor$, as desired.

Consequently, inequality (5) holds. We next prove inequality (6). Since $\deg(v) \geq \delta_1$ and $N(v) = X_1(v) \subseteq V_2$, we find that $|X_1(v)| \geq \delta_1$.

From Claim 2, we can easily see that $|X_{4m-1}(v)| + |X_{4m+1}(v)| \geq \deg(x_{4m}) \geq \max\{2, \delta_1\}$ for every $m = 1, 2, ..., \lfloor\frac{k-1}{4}\rfloor$. We discuss on $\frac{\lfloor(k+1)/2\rfloor}{2}$ which may be an integer or not.

First we assume that $\frac{\lfloor(k+1)/2\rfloor}{2}$ is an integer. Hence,

$$\begin{aligned}|N_k(v) \cap V_2| &= \sum_{m=1}^{\lfloor(k+1)/2\rfloor} |X_{2m-1}(v)| = |X_1(v)| + \sum_{m=2}^{\lfloor(k+1)/2\rfloor} |X_{2m-1}(v)| \\ &= |X_1(v)| + \sum_{m'=1}^{\lfloor(k+1)/2\rfloor/2-1} (|X_{4m'-1}(v)| + |X_{4m'+1}(v)|) + |X_{2\lfloor(k+1)/2\rfloor-1}(v)| \\ &\geq \delta_1 + \sum_{m'=1}^{\lfloor(k+1)/4\rfloor-1} \max\{2, \delta_1\} + 1 \quad (by\ Claim\ 2).\end{aligned}$$

Thus, $|N_k(v) \cap V_2| \geq \delta_1 + (\lfloor(k+1)/4\rfloor - 1)\max\{2, \delta_1\} + 1$. Now a simple calculation shows that $\delta_1 + (\lfloor(k+1)/4\rfloor - 1)\max\{2, \delta_1\} + 1 = \delta_1 + (\lceil k/4\rceil - 1)\max\{2, \delta_1\} + \lfloor(k-1)/2\rfloor - 2\lfloor(k-1)/4\rfloor$ as desired.

Next we assume that $\frac{\lfloor(k+1)/2\rfloor}{2}$ is not an integer. Hence,

$$\begin{aligned}|N_k(v) \cap V_2| &= \sum_{m=1}^{\lfloor(k+1)/2\rfloor} |X_{2m-1}(v)| = |X_1(v)| + \sum_{m=2}^{\lfloor(k+1)/2\rfloor} |X_{2m-1}(v)| \\ &= |X_1(v)| + \sum_{m'=1}^{(\lfloor(k+1)/2\rfloor-1)/2} (|X_{4m'-1}(v)| + |X_{4m'+1}(v)|) \\ &\geq \delta_1 + \sum_{m'=1}^{\lfloor(k-1)/4\rfloor} \max\{2, \delta_1\} \quad (by\ Claim\ 2).\end{aligned}$$

Thus, $|N_k(v) \cap V_2| \geq \delta_1 + \lfloor(k-1)/4\rfloor \max\{2, \delta_1\}$. Now a simple calculation shows that

$$\delta_1 + \lfloor(k-1)/4\rfloor \max\{2, \delta_1\} = \delta_1 + (\lceil k/4\rceil - 1)\max\{2, \delta_1\} + \lfloor(k-1)/2\rfloor - 2\lfloor(k-1)/4\rfloor$$

as desired.

We next show that inequality 5 is an improvement of inequality 1. We will show that:

$$\lceil\frac{k-1}{4}\rceil \max\{2, \delta_2\} + 2\lfloor\frac{k}{4}\rfloor - \lfloor\frac{k}{2}\rfloor \geq (\lceil\frac{k}{6}\rceil - 1)(\delta_2 + 1)$$

It is obvious that if $\delta_2 = 1$, then the left side of the above inequality is $2\lceil\frac{k-1}{4}\rceil + 2\lfloor\frac{k}{4}\rfloor - \lfloor\frac{k}{2}\rfloor = \lfloor\frac{k}{2}\rfloor$ and the right side is $2(\lceil\frac{k}{6}\rceil - 1)$, and clearly $2\lceil\frac{k-1}{4}\rceil + 2\lfloor\frac{k}{4}\rfloor - \lfloor\frac{k}{2}\rfloor = \lfloor\frac{k}{2}\rfloor \geq 2(\lceil\frac{k}{6}\rceil - 1)$ for $k \geq 1$. Thus assume that $\delta_2 \geq 2$. We show that

$$(\lceil\frac{k-1}{4}\rceil - \lceil\frac{k}{6}\rceil + 1)\delta_2 \geq \lceil\frac{k}{6}\rceil - 1 - 2\lfloor\frac{k}{4}\rfloor + \lfloor\frac{k}{2}\rfloor$$

for $k \geq 1$. Let $L = (\lceil \frac{k-1}{4} \rceil - \lceil \frac{k}{6} \rceil + 1)\delta_2$ and $R = \lceil \frac{k}{6} \rceil - 1 - 2\lfloor \frac{k}{4} \rfloor + \lfloor \frac{k}{2} \rfloor$. Now, we show that $L \geq R$. Let $k = 12p + q$, where $1 \leq q \leq 12$. Then

$$L = (\lceil \frac{k-1}{4} \rceil - \lceil \frac{k}{6} \rceil + 1)\delta_2 = p\delta_2 + (\lceil \frac{q-1}{4} \rceil - \lceil \frac{q}{6} \rceil + 1)\delta_2.$$

$$R = \lceil \frac{k}{6} \rceil - 1 - 2\lfloor \frac{k}{4} \rfloor + \lfloor \frac{k}{2} \rfloor = 2p + \lceil \frac{q}{6} \rceil - 1 - 2\lfloor \frac{q}{4} \rfloor + \lfloor \frac{q}{2} \rfloor.$$

Since $\delta_2 \geq 2$, we have $p\delta_2 \geq 2p$. So we need to show that $(\lceil \frac{q-1}{4} \rceil - \lceil \frac{q}{6} \rceil + 1)\delta_2 \geq \lceil \frac{q}{6} \rceil - 1 - 2\lfloor \frac{q}{4} \rfloor + \lfloor \frac{q}{2} \rfloor$. Since $1 \leq q \leq 12$, we show this by Table 1.

q	1	2	3	4	5	6	7	8	9	10	11	12
$(\lceil \frac{q-1}{4} \rceil - \lceil \frac{q}{6} \rceil + 1)\delta_2.$	0	δ_2	δ_2	δ_2	δ_2	$2\delta_2$	δ_2	δ_2	δ_2	$2\delta_2$	$2\delta_2$	$2\delta_2$
$\lceil \frac{q}{6} \rceil - 1 - 2\lfloor \frac{q}{4} \rfloor + \lfloor \frac{q}{2} \rfloor$	0	1	1	0	0	1	2	1	1	2	2	1

Table 1.

Thus, inequality (5) is an improvement of inequality (1). Next, we show that inequality (6) is an improvement of inequality (2). We will show that :

$$\delta_1 + (\lceil k/4 \rceil - 1)\max\{2, \delta_1\} + \lfloor (k-1)/2 \rfloor - 2\lfloor (k-1)/4 \rfloor \geq \lceil k/6 \rceil(\delta_1 + 1) - 1$$

If $\delta_1 = 1$, then the above inequality becomes $1 + 2(\lceil k/4 \rceil - 1) + \lfloor (k-1)/2 \rfloor - 2\lfloor (k-1)/4 \rfloor = \lceil k/2 \rceil \geq 2\lceil \frac{k}{6} \rceil - 1$ which is valid for any $k \geq 1$. Thus we assume that $\delta_1 \geq 2$. It is sufficient to show that

$$(\lceil \frac{k}{4} \rceil - \lceil \frac{k}{6} \rceil)\delta_1 \geq \lceil \frac{k}{6} \rceil - 1 - \lfloor \frac{k-1}{2} \rfloor + 2\lfloor \frac{k-1}{4} \rfloor$$

for $k \geq 1$. Let $L = (\lceil \frac{k}{4} \rceil - \lceil \frac{k}{6} \rceil)\delta_1$ and $R = \lceil \frac{k}{6} \rceil - 1 - \lfloor \frac{k-1}{2} \rfloor + 2\lfloor \frac{k-1}{4} \rfloor$. Thus, we need to show that $L \geq R$. Let $k = 12p + q$, where $1 \leq q \leq 12$. Hence,

$$L = (\lceil \frac{k}{4} \rceil - \lceil \frac{k}{6} \rceil)\delta_1 = p\delta_1 + (\lceil \frac{q}{4} \rceil - \lceil \frac{q}{6} \rceil)\delta_1.$$

$$R = \lceil \frac{k}{6} \rceil - 1 - \lfloor \frac{k-1}{2} \rfloor + 2\lfloor \frac{k-1}{4} \rfloor = 2p + \lceil \frac{q}{6} \rceil - 1 - \lfloor \frac{q-1}{2} \rfloor + 2\lfloor \frac{q-1}{4} \rfloor.$$

q	1	2	3	4	5	6	7	8	9	10	11	12
$(\lceil \frac{q}{4} \rceil - \lceil \frac{q}{6} \rceil)\delta_1$	0	0	0	0	δ_1	δ_1	0	0	δ_1	δ_1	δ_1	δ_1
$\lceil \frac{q}{6} \rceil - 1 - \lfloor \frac{q-1}{2} \rfloor + 2\lfloor \frac{q-1}{4} \rfloor$	0	0	-1	-1	0	0	0	0	1	1	0	0

Table 2.

Since $\delta_1 \geq 2$, we have $p\delta_1 \geq 2p$. Therefore, it is sufficient to show that $(\lceil \frac{q}{4} \rceil - \lceil \frac{q}{6} \rceil)\delta_1 \geq \lceil \frac{q}{6} \rceil - 1 - \lfloor \frac{q-1}{2} \rfloor + 2\lfloor \frac{q-1}{4} \rfloor$. We do this in Table 2, since $1 \leq q \leq 12$. Thus (6) is an improvement of (2).

The proof of part (ii), (i.e. (7) and (8)) is similar and straightforward, and therefore is omitted. □

Theorem 3.1. *If G is a bipartite graph and k is a positive integer, then*

$$\gamma_k(G) \le \min_{(p_1,p_2)\in(0,1)^2} h^*(p_1,p_2),$$

where $$h^*(p_1,p_2) = n_1p_1 + n_1e^{-p_1(A_{11}+1)-p_2A_{12}} + n_2p_2 + n_2e^{-p_1A_{21}-p_2(A_{22}+1)}$$

$$\begin{aligned}
A_{11} &= \lceil (k-1)/4\rceil \max\{2,\delta_2\} + 2\lfloor k/4\rfloor - \lfloor k/2\rfloor\\
A_{12} &= \delta_1 + (\lceil k/4\rceil - 1)\max\{2,\delta_1\} + \lfloor (k-1)/2\rfloor - 2\lfloor (k-1)/4\rfloor\\
A_{21} &= \delta_2 + (\lceil k/4\rceil - 1)\max\{2,\delta_2\} + \lfloor (k-1)/2\rfloor - 2\lfloor (k-1)/4\rfloor\\
A_{22} &= \lceil (k-1)/4\rceil \max\{2,\delta_1\} + 2\lfloor k/4\rfloor - \lfloor k/2\rfloor
\end{aligned}$$

This bound improve the given bound in Theorem 2.2.

Proof. By Theorem 2.1, we have

$$\gamma_k(G) \le \min_{(p_1,p_2)\in(0,1)^2}\left(\sum_{v\in V_1}\left[p_1 + (1-p_1)^{|N_k(v)\cap V_1|+1}(1-p_2)^{|N_k(v)\cap V_2|}\right] + \sum_{v\in V_2}\left[p_2 + (1-p_1)^{|N_k(v)\cap V_1|}(1-p_2)^{|N_k(v)\cap V_2|+1}\right]\right).$$

By Lemma 3.1, we have

$$\begin{aligned}
\gamma_k(G) &\le \min_{(p_1,p_2)\in(0,1)^2}\left(\sum_{v\in V_1}\left[p_1 + (1-p_1)^{A_{11}+1}(1-p_2)^{A_{12}}\right] + \sum_{v\in V_2}\left[p_2 + (1-p_1)^{A_{21}}(1-p_2)^{A_{22}+1}\right]\right)\\
&\le \min_{(p_1,p_2)\in(0,1)^2}\left(\left[n_1p_1 + n_1(1-p_1)^{A_{11}+1}(1-p_2)^{A_{12}}\right] + \left[n_2p_2 + n_2(1-p_1)^{A_{21}}(1-p_2)^{A_{22}+1}\right]\right)\\
&\le \min_{(p_1,p_2)\in(0,1)^2}\left(n_1p_1 + n_1e^{-p_1(A_{11}+1)-p_2A_{12}} + n_2p_2 + n_2e^{-p_1A_{21}-p_2(A_{22}+1)}\right).
\end{aligned}$$

That is $\gamma_k(G) \le \min_{(p_1,p_2)\in(0,1)^2} h^*(p_1,p_2)$. To show that our bound is an improvement of the bound given in Theorem 2.2, note that by Lemma 3.1 one can easily see that $h^*(p_1,p_2) \le h(p_1,p_2)$, since $\exp(-x)$ is a decreasing function. □

Example 3.1. *It remains to show that there are perfect graphs that our bound is better than the older one. For this purpose, let G be a connected bipartite graph with $n_1 = n_2 = \frac{n}{2}$, $\delta_1 = \delta_2 = \delta \ge 2$, and $k = 4m+1$ with $m = 1,2,3,\cdots$. We can easily see that the graph is perfect. Now we have $A_{11} = A_{22} = m\delta, A_{12} = A_{21} = (m+1)\delta$ and*

$$h^*(p_1,p_2) = \frac{n}{2}[p_1 + p_2 + e^{-p_1(m\delta+1)-p_2(m+1)\delta} + e^{-p_1(m+1)\delta-p_2(m\delta+1)}].$$

By using of calculus method, we see that the unique minimum of h^* occurs at

$$p_1 = p_2 = \frac{\ln[(2m+1)\delta+1]}{(2m+1)\delta+1},$$

since $0 < p_1 = p_2 < 1$, we have $\min\limits_{(p_1,p_2)\in(0,1)^2} h^*(p_1,p_2) = n(\dfrac{1+\ln[(2m+1)\delta+1]}{(2m+1)\delta+1})$. By calculus, we note that the function $f(x) = \frac{1+\ln x}{x}$ is decreasing on interval $(1,\infty)$ and also we have $(2m+1)\delta+1 \geq (2\lceil k/6\rceil - 1)(\delta+1)$, thus the new bound refinements the bound in Theorem 2.2.

3.1. Minimizing $h^*(p_1, p_2)$

In this part of paper we wish to minimize $h^*(p_1,p_2)$. For this purpose, we consider two different cases and we use calculation methods.

3.1.1. k is even

In this case we will show that either h^* hasn't local extremum or it has infinitely local minimum on $(0,1)^2$. However h^* has local minimum on closed unit square $[0,1]^2$, thus we extend the domain of h^* into $[0,1]^2$.

Before introducing our main results, we explain an observation in calculus :

Observation 3.1. *Consider the function $f(x) = \frac{a+\ln x}{x}$ where $x > 0$ and $a > 0$. f has a unique maximum in $x = e^{1-a} \leq e$ thus $f(x) \leq f(e^{1-a}) = e^{a-1}$. Now, if $a < 1$, then $f(x) < 1$ for all $x > 0$.*

Our main result in this states is :

Theorem 3.2. *If k is an even integer, $\delta_1, \delta_2 \geq 2$ and $T = \max\{\frac{nA_{12}}{n_2}, \frac{nA_{21}}{n_1}\}$, in each of three cases*
(i) $\frac{nA_{12}}{n_2} = \frac{nA_{21}}{n_1}$
(ii) $\frac{nA_{12}}{n_2} < \frac{nA_{21}}{n_1}$ and $\frac{1}{A_{21}} \ln \frac{nA_{21}}{n_1} < 1$
(iii) $\frac{nA_{12}}{n_2} > \frac{nA_{21}}{n_1}$ and $\frac{1}{A_{12}} \ln \frac{nA_{12}}{n_2} < 1$
we have $\inf\limits_{(p_1,p_2)\in(0,1)^2} h^*(p_1,p_2) = \min\limits_{(p_1,p_2)\in[0,1]^2} h^*(p_1,p_2) = n(\frac{1+\ln T}{T})$.

Proof. If we assume that $k \overset{4}{\equiv} 0$, then

$$A_{11} = k\delta_2/4, \quad A_{12} = k\delta_1/4+1, \quad A_{21} = k\delta_2/4+1, \quad A_{22} = k\delta_1/4$$

and if $k \overset{4}{\equiv} 2$, then

$$A_{11} = (k+2)\delta_2/4-1, \quad A_{12} = (k+2)\delta_1/4, \quad A_{21} = (k+2)\delta_2/4, \quad A_{22} = (k+2)\delta_1/4-1.$$

Thus, in both cases we have $A_{11}+1 = A_{21}$ and $A_{22}+1 = A_{12}$, and therefore,

$$h^*(p_1,p_2) = n_1p_1 + n_2p_2 + ne^{-p_1A_{21}-p_2A_{12}}.$$

To minimize $h^*(p_1,p_2)$, using partial differential, we have $h^*_{p_1} = n_1 - nA_{21}e^{-p_1A_{21}-p_2A_{12}}$ and $h^*_{p_2} = n_2 - nA_{12}e^{-p_1A_{21}-p_2A_{12}}$. From $h^*_{p_1} = 0$, we obtain that $e^{-p_1A_{21}-p_2A_{12}} = \frac{n_1}{nA_{21}}$, and so $p_1A_{21} + p_2A_{12} = \ln\frac{nA_{21}}{n_1}$. Likewise, from $h^*_{p_2} = 0$, we obtain $p_1A_{21} + p_2A_{12} = \ln\frac{nA_{12}}{n_2}$.
(i) If $\frac{A_{21}}{n_1} = \frac{A_{12}}{n_2}$, then for all (p_1,p_2) with $p_1A_{21} + p_2A_{12} = \ln\frac{nA_{12}}{n_2} = \ln\frac{nA_{21}}{n_1}$, we have:

$$h^*(p_1,p_2) = n_1p_1 + n_2p_2 + ne^{-p_1A_{21}-p_2A_{12}} = \frac{n_1}{A_{21}}(p_1A_{21}+p_2A_{12}) + ne^{-p_1A_{21}-p_2A_{12}}$$
$$= \frac{n_1}{A_{21}}\ln\frac{nA_{21}}{n_1} + ne^{-\ln\frac{nA_{21}}{n_1}} = \frac{n_1}{A_{21}}(1+\ln\frac{nA_{21}}{n_1}).$$

Therefore, h^* is constant for all (p_1,p_2) with $p_1A_{21}+p_2A_{12} = \ln\frac{nA_{12}}{n_2} = \ln\frac{nA_{21}}{n_1}$ (See Figure 1). Note that two points $(0,\frac{1}{A_{12}}\ln\frac{nA_{12}}{n_2})$ and $(\frac{1}{A_{21}}\ln\frac{nA_{21}}{n_1},0)$ are located on the line $p_1A_{21}+p_2A_{12} = \ln\frac{nA_{12}}{n_2} = \ln\frac{nA_{21}}{n_1}$ and by Observation 3.1, we have $0 < \min\{\frac{1}{A_{12}}\ln\frac{nA_{12}}{n_2}, \frac{1}{A_{21}}\ln\frac{nA_{21}}{n_1}\} < 1$ because $0 < \min\{\ln\frac{n}{n_2}, \ln\frac{n}{n_1}\} < 1$.

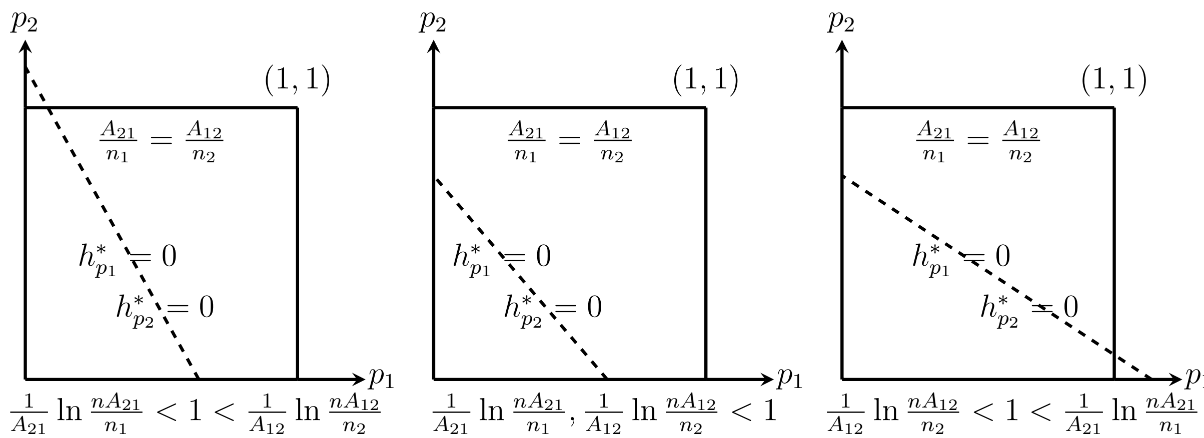

Figure 1. $\frac{A_{21}}{n_1} = \frac{A_{12}}{n_2}$

Thus the minimum of $h^*(p_1,p_2)$ is $\frac{n_1}{A_{21}}(1+\ln\frac{nA_{21}}{n_1})$, and note that it happens for every pairs $(p_1,p_2) \in (0,1)^2$, satisfying $h^*_{p_1} = h^*_{p_2} = 0$. Now letting $T = \frac{nA_{21}}{n_1} = \frac{nA_{12}}{n_2}$, we obtain that $\min\limits_{p_1,p_2} h^*(p_1,p_2) = n(\frac{1+\ln T}{T})$, as desired.

If $\frac{A_{21}}{n_1} \neq \frac{A_{12}}{n_2}$, then $p_1A_{21}+p_2A_{12} = \ln\frac{nA_{21}}{n_1}$ and $p_1A_{21}+p_2A_{12} = \ln\frac{nA_{12}}{n_2}$ are two distinct parallel lines in the p_1p_2-coordinate system. Thus, h^* has no extremum in $(0,1)^2$ but it has an infimum value in $(0,1)^2$. For this purpose we seek the extremum of h^* in $[0,1]^2$. Observe that the line $p_1A_{21}+p_2A_{12} = \ln\frac{nA_{21}}{n_1}$ intersects the p_1-axis in $M_1 = (\frac{1}{A_{21}}\ln\frac{nA_{21}}{n_1},0)$ and p_2-axis in $N_1 = (0,\frac{1}{A_{12}}\ln\frac{nA_{21}}{n_1})$. Similarly, the line $p_1A_{21}+p_2A_{12} = \ln\frac{nA_{12}}{n_2}$ intersects the p_1-axis in $M_2 = (\frac{1}{A_{21}}\ln\frac{nA_{12}}{n_2},0)$ and p_2-axis in $N_2 = (0,\frac{1}{A_{12}}\ln\frac{nA_{12}}{n_2})$. Moreover, let $Q_1 = (1,0)$ and $Q_2 = (0,1)$.
(ii) $\frac{nA_{12}}{n_2} < \frac{nA_{21}}{n_1}$ and $\frac{1}{A_{21}}\ln\frac{nA_{21}}{n_1} < 1$ we prove that the minimum of h^* occurs in M_1. For each point (p_1,p_2) in unit square $[0,1]^2$ there is a unique point (p'_1,p_2) on segments M_1N_1 or N_1Q_2

(dotted segments in Figure 2) such that $h^*(p_1', p_2) \leq h^*(p_1, p_2)$. Hence, the minimum of h^* occurs on $M_1N_1 \cup N_1Q_2$. Also, there is a unique point (p_1, p_2') on segments M_2N_2 or M_2Q_1 (dashed segments in Figure 2) such that $h^*(p_1, p_2') \leq h^*(p_1, p_2)$. Therefore, the minimum of h^* occurs on $M_2N_2 \cup M_2Q_1$. This two sets of points intersect in one point, M_1. Hence, we have $h(M_1) \leq h^*(p_1, p_2)$ and $h^*(M_1) = h(\frac{1}{A_{21}} \ln \frac{nA_{21}}{n_1}, 0) = \frac{n_1}{A_{21}}(1 + \ln \frac{nA_{21}}{n_1})$.

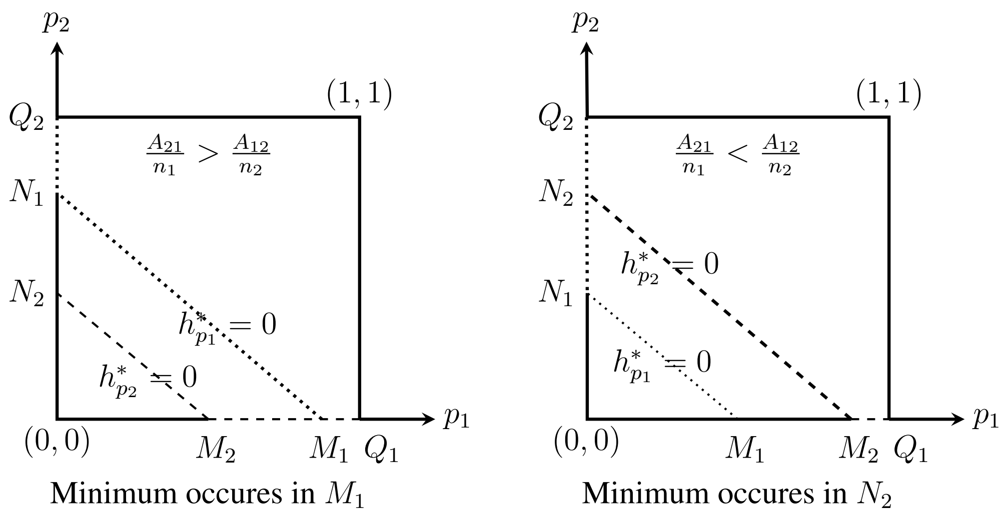

Figure 2. $\frac{A_{21}}{n_1} \neq \frac{A_{12}}{n_2}$

(iii) If $\frac{nA_{12}}{n_2} > \frac{nA_{21}}{n_1}$ and $\frac{1}{A_{12}} \ln \frac{nA_{12}}{n_2} < 1$, then we prove that the minimum of h^* occurs in N_2. For each point (p_1, p_2) in unit square $[0,1]^2$, there is a unique point (p_1', p_2) on segments M_1N_1 or N_1Q_2 such that $h^*(p_1', p_2) \leq h^*(p_1, p_2)$. Hence, the minimum of h^* occurs on $M_1N_1 \cup N_1Q_2$. Also, there is a unique point (p_1, p_2') on segments M_2N_2 or M_2Q_1 such that $h^*(p_1, p_2') \leq h^*(p_1, p_2)$. Therefore, the minimum of h^* occurs on $M_2N_2 \cup M_2Q_1$. This two sets of points intersect in one point, N_2, that is, $h^*(N_2) \leq h^*(p_1, p_2)$ and $h^*(N_2) = h^*(0, \frac{1}{A_{12}} \ln \frac{nA_{12}}{n_2}) = \frac{n_2}{A_{12}}(1 + \ln \frac{nA_{12}}{n_2})$.

In each of three cases, if we set $T = \max\{\frac{nA_{12}}{n_2}, \frac{nA_{21}}{n_1}\}$, then we have :

$$\min_{p_1, p_2} h^*(p_1, p_2) = n(\frac{1 + \ln T}{T}).$$

□

We now pose a problem.

Problem 3.1. *Minimize h^* if $\delta_1 = 1$ or $\delta_2 = 1$.*

3.1.2. k is odd

We assume that k is an odd integer and we wish to minimize $h^*(p_1,p_2)$. For this purpose, we use calculus methodes.

$$\begin{cases} h_{p_1} = n_1 - n_1(A_{11}+1)e^{-p_1(A_{11}+1)-p_2A_{12}} - n_2A_{21}e^{-p_1A_{21}-p_2(A_{22}+1)} \\ h_{p_2} = -n_1A_{12}e^{-p_1(A_{11}+1)-p_2A_{12}} + n_2 - n_2(A_{22}+1)e^{-p_1A_{21}-p_2(A_{22}+1)} \end{cases}$$

$$\begin{cases} h_{p_1} = 0 \\ h_{p_2} = 0 \end{cases} \Longrightarrow \begin{cases} n_1(A_{11}+1)e^{-p_1(A_{11}+1)-p_2A_{12}} + n_2A_{21}e^{-p_1A_{21}-p_2(A_{22}+1)} = n_1 \\ n_1A_{12}e^{-p_1(A_{11}+1)-p_2A_{12}} + n_2(A_{22}+1)e^{-p_1A_{21}-p_2(A_{22}+1)} = n_2 \end{cases}$$

Therefore, we have:

$$\begin{cases} e^{-p_1A_{21}-p_2(A_{22}+1)} = \dfrac{n_2(A_{11}+1) - n_1A_{12}}{n_2[A_{12}A_{21} - (A_{11}+1)(A_{22}+1)]} \\ e^{-p_1(A_{11}+1)-p_2A_{12}} = \dfrac{n_1(A_{22}+1) - n_2A_{21}}{n_1[A_{12}A_{21} - (A_{11}+1)(A_{22}+1)]} \end{cases}$$

Let $E_1 = \dfrac{n_2(A_{11}+1) - n_1A_{12}}{n_2[A_{12}A_{21} - (A_{11}+1)(A_{22}+1)]}$, $\quad E_2 = \dfrac{n_1(A_{22}+1) - n_2A_{21}}{n_1[A_{12}A_{21} - (A_{11}+1)(A_{22}+1)]}$.
If $E_1 > 0$ and $E_2 > 0$, then we have a linear equations system

$$\begin{cases} p_1A_{21} + p_2(A_{22}+1) = -\ln E_1 \\ p_1(A_{11}+1) + p_2A_{12} = -\ln E_2 \end{cases}$$

with a unique answer and we set :

$$\begin{cases} P_1 = \dfrac{(A_{22}+1)\ln E_2 - A_{12}\ln E_1}{A_{12}A_{21} - (A_{11}+1)(A_{22}+1)} \\ P_2 = \dfrac{(A_{11}+1)\ln E_1 - A_{21}\ln E_2}{A_{12}A_{21} - (A_{11}+1)(A_{22}+1)} \end{cases}$$

Definition 3.1. *A connected bipartite graph G is called 4-perfect if $E_1 > 0$, $E_2 > 0$ where $E_1 = \dfrac{n_2(A_{11}+1) - n_1A_{12}}{n_2[A_{12}A_{21} - (A_{11}+1)(A_{22}+1)]}$ and $E_2 = \dfrac{n_1(A_{22}+1) - n_2A_{21}}{n_1[A_{12}A_{21} - (A_{11}+1)(A_{22}+1)]}$.*

So we get the following.

Corollary 3.1. *If G is a 4-perfect graph, $0 < P_1 < 1$ and $0 < P_2 < 1$, then*

$$\min_{(p_1,p_2)\in(0,1)^2} h^*(p_1,p_2) = h^*(P_1,P_2) = n_1[E_2 + P_1] + n_2[E_1 + P_2].$$

Note that Corollary 3.1 improves Theorem 2.2 if G is both perfect and 4-perfect. It remains to show that there are perfect graphs that are 4-perfect as well. For this purpose, we consider the graph introduced in Example 3.1.

Example 3.2. *Let $n_1 = n_2 = \frac{n}{2}$, $\delta_1 = \delta_2 = \delta$, and $k = 4m+1$. Thus,*

$$E_1 = E_2 = \frac{1}{(2m+1)\delta + 1}\ , P_1 = P_2 = \frac{\ln[(2m+1)\delta + 1]}{(2m+1)\delta + 1}$$

Since $E_1, E_2 > 0$, G is 4-perfect. It is also easy to see that G is perfect.

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