Electronic Journal of Graph Theory and Applications **5** (1) (2017), 155–162

Electronic Journal of
Graph Theory and Applications

Restricted size Ramsey number for path of order three versus graph of order five

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Abstract

Let G and H be simple graphs. The Ramsey number $r(G, H)$ for a pair of graphs G and H is the smallest number r such that any red-blue coloring of the edges of K_r contains a red subgraph G or a blue subgraph H. The size Ramsey number $\hat{r}(G, H)$ for a pair of graphs G and H is the smallest number $\hat{r}$ such that there exists a graph F with size $\hat{r}$ satisfying the property that any red-blue coloring of the edges of F contains a red subgraph G or a blue subgraph H. Additionally, if the order of F in the size Ramsey number equals $r(G, H)$, then it is called the restricted size Ramsey number. In 1983, Harary and Miller started to find the (restricted) size Ramsey numbers for pairs of small graphs with orders at most four. Faudree and Sheehan (1983) continued Harary and Miller's works and summarized the complete results on the (restricted) size Ramsey numbers for pairs of small graphs with orders at most four. In 1998, Lortz and Mengenser gave both the size Ramsey numbers and the restricted size Ramsey numbers for pairs of small forests with orders at most five. To continue their works, we investigate the restricted size Ramsey numbers for a path of order three versus any connected graph of order five.

Keywords: restricted size Ramsey number, path, connected graph
Mathematics Subject Classification : 05C55, 05D10, 05C38
DOI: 10.5614/ejgta.2017.5.1.15

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Received: 1 September 2016, Revised: 14 February 2017 Accepted: 29 March 2017.

1. Introduction

A graph G has the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices and edges in G denoted by $v(G)$ and $e(G)$, respectively. Let G and H be graphs. If $H \subseteq G$, then $G - H$ is a graph with the vertex set $V(G)$ and the edge set $E(G) \setminus E(H)$ and $G + H$ is a graph with the vertex set $V(G)$ and the edge set $E(G) \cup E(H)$. For further terminologies in graph, please see [3]. For a pair of graphs G and H, the *Ramsey number* $r(G, H)$ is the smallest number r such that any red-blue coloring of the edges of K_r contains a red subgraph G or a blue subgraph H. The *size Ramsey number* $\hat{r}(G, H)$ is the smallest size of graph F satisfying the property that any red-blue coloring of the edges of F contains a red subgraph G or a blue subgraph H. Furthermore, if the order of F in this case is $r(G, H)$, then it is called the *restricted size Ramsey number* $r^*(G, H)$. In addition, if any red-blue coloring of the edges of F contains a red subgraph G or a blue subgraph H, we say F *arrowing* G and H, and denoted by $F \to (G, H)$.

The concept of the size Ramsey number was introduced by Erdös *et al.* in 1978 [4], while the concept of the restricted size Ramsey number is a direct consequence of the concept of the size Ramsey number and the Ramsey number in graph. Some results on the (restricted) size Ramsey number of graphs can be found in the survey of noncomplete Ramsey theory for graphs given by Burr [1] and a survey of results on the size Ramsey numbers given by Faudree and Schelp [7].

In 1983, Harary and Miller [8] started to find the (restricted) size Ramsey numbers for pairs of small graphs with orders at most four. However, due to the long proof and the tedious works of doing the proof, they did not give the proofs for some of their results. In general, they stated that the exact determination of the size Ramsey number requires rather involved argument even for small graphs. Faudree and Sheehan [5] continued Harary and Miller's works and summarized the complete results on the (restricted) size Ramsey numbers for pairs of small graphs with orders at most four. With the same reason as in Harary and Miller, they also did not give all the proof of their results. In 1998, Lortz and Mengenser [9] gave both the size Ramsey number and the restricted size Ramsey number for pairs of small forests with orders at most five. Similarly, they also omitted the proof of their results. To continue their works, we investigate the restricted size Ramsey numbers for a path P_3 versus all connected graphs of order five. We present the complete proof for this case.

2. Preliminary Results

The complete list of all connected graphs with order five is given in Figure 1.

The Ramsey number for a pair of P_3 and a graph without isolates was given by Chvátal and Harary [2]. We state the result here. This result gives the order of the arrowing graph in finding the restricted size Ramsey number $r^*(P_3, H)$.

Theorem 2.1. *[2] For any graph H with no isolates,*

$$r(P_3, H) = \begin{cases} v(H), & \overline{H}\ has\ 1-factor \\ 2v(H) - 2\beta(\overline{H}) - 1, & otherwise, \end{cases}$$

with $\beta(\overline{H})$ is the maximum number of independent edges in the complement of H.

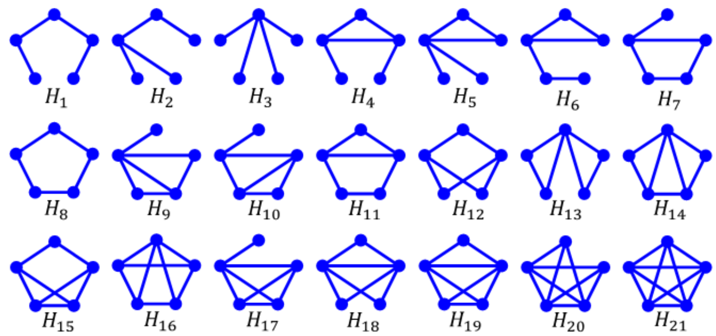

Figure 1. The list of all connected graphs with order 5.

From Theorem 2.1 we have $r(P_3, H_i) = 5$ for $1 \leq i \leq 16$, $r(P_3, H_i) = 7$ for $17 \leq i \leq 20$, and $r(P_3, H_{21}) = 9$. Faudree and Sheehan [6] gave the (restricted) size Ramsey number for P_3 and K_n as stated in Theorem 2.2.

Theorem 2.2. *[6] For a positive integer* $n \geq 2$

$$\hat{r}(P_3, K_n) = r^*(P_3, K_n) = 2(n-1)^2.$$

Lortz and Mengenser [9] gave the size Ramsey number and the restricted size Ramsey number for pairs of small forests with orders at most five. From their results, we have $r^*(P_3, H_1) = 6$, $r^*(P_3, H_2) = 7$, and $r^*(P_3, H_3) = 10$. The last result, namely $r^*(P_3, H_3) = 10$, is from [5]. From Theorem 2.2, we have $r^*(P_3, H_{21}) = 32$. From our previous work in [10], we have $r^*(P_3, H_5) = r^*(P_3, H_9) = r^*(P_3, H_{12}) = r^*(P_3, H_{13}) = r^*(P_3, H_{14}) = r^*(P_3, H_{15}) = r^*(P_3, H_{16}) = 10$, and $r^*(P_3, H_{10}) = r^*(P_3, H_{11}) = 9$. For the remaining graph H_i, we will derive the restricted size Ramsey number $r^*(P_3, H_i)$ here.

Clearly, from the definition of the (restricted) size Ramsey number, we have the monotonicity property as follow. If $F_1' \subseteq F_1$ and $F_2' \subseteq F_2$, then

$$\hat{r}(F_1', F_2') \leq \hat{r}(F_1, F_2), \tag{1}$$

and

$$r^*(F_1', F_2') \leq r^*(F_1, F_2). \tag{2}$$

Note that the monotonicity property of the Ramsey number of graphs has been given by Chvátal and Harary [2].

3. Main Results

In this section we present the "missing values" of the restricted size Ramsey numbers of P_3 versus connected graphs of order five, H_i. The results for $r^*(P_3, H_i)$ for which $r(P_3, H_i) = 5$ are given in Theorems 3.1, 3.2, 3.3, and 3.4. The results related to $r^*(P_3, H_i)$ for which $r(P_3, H_i) = 7$ are given in Theorems 3.5, 3.6, and 3.7.

To prove some of those theorems, we define a graph G_F as in Faudree and Sheehan [6]. Let F be a graph with edges are red-blue colored. Define a graph G_F with $V(G_F) = V(F)$ and $E(G_F)$

consists of red edges in F and edges in $\overline{F}$. It is important to notice that $\overline{G}_F$ is precisely the induced blue subgraph of F. We will give an example to illustrate this definition. Let $F = K_6 - 2K_2$. Then $\overline{F} = 2K_2 \cup 2K_1$. Suppose the edges of F is red-blue colored such that three independent edges are red and the rest is blue. Then $G_F = P_6$ and $\overline{G}_F$ is exactly the induced blue subgraph in the red-blue coloring of F, as given in Figure 2 with red edges in dotted line.

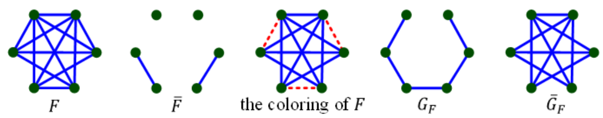

Figure 2. The illustration for G_F.

Theorem 3.1. $r^*(P_3, H_4) = 9$.

Proof. We know that $r(P_3, H_4) = 5$. For the upper bound, consider $F = K_5 - K_2$. Now, consider any red-blue coloring of the edges of F such that there is no red P_3. Then, the graph G_F will be a subgraph of either $P_4 \cup K_1$ or $P_3 \cup K_2$. In both cases, $\overline{G}_F$ contains H_4. Since $\overline{G}_F$ is precisely the induced blue subgraph of F, then $F \rightarrow (P_3, H_4)$ and $r^*(P_3, H_4) \leq 9$.

For the lower bound, we will consider all graphs F with $v(F) = 5$ and $e(F) = 8$. The only graph F satisfying these conditions is either isomorphic to $K_5 - P_3$ or $K_5 - 2P_2$. If $F = K_5 - P_3$, then take a red-blue coloring of the edges of F with no red P_3 such that the graph G_F will be isomorphic to $C_3 \cup P_2$. If $F = K_5 - 2P_2$, then take a red-blue coloring of the edges of F with no red P_3 such that the graph G_F will be isomorphic to $C_4 \cup P_1$. In both cases, the induced blue subgraph of F does not contain H_4. This implies that $F \nrightarrow (P_3, H_4)$ and $r^*(P_3, H_4) \geq 9$. Therefore, the theorem holds. □

Theorem 3.2. $r^*(P_3, H_6) = 8$.

Proof. We know that $r(P_3, H_6) = 5$. For the upper bound, consider $F = K_5 - 2P_2$. Now, consider any red-blue coloring of the edges of F such that there is no red P_3. Then the graph G_F will be a subgraph of either P_5 or $C_4 \cup K_1$. In both cases, $\overline{G}_F$ contains H_6. Since $\overline{G}_F$ is precisely the induced blue subgraph of F, then $F \rightarrow (P_3, H_6)$ and $r^*(P_3, H_6) \leq 8$.

For the lower bound, we will consider all graphs F with $v(F) = 5$ and $e(F) = 7$. Since H_6 contains P_5, F must contain C_5. The only graph F satisfying these conditions is either isomorphic to $C_5 + 2K_2$ or $C_5 + P_3$. If $F = K_5 + 2K_2$, then take a red-blue coloring of the edges of F with no red P_3 such that the blue subgraph is C_5. If $F = K_5 + P_3$, then take a red-blue coloring of the edges of F with no red P_3 such that the blue subgraph is $C_4 \cup K_1$. In both cases, the blue subgraph of F does not contain H_6. This implies that $F \nrightarrow (P_3, H_6)$ and $r^*(P_3, H_6) \geq 8$. Therefore, the theorem holds. □

Theorem 3.3. $r^*(P_3, H_7) = 8$.

Proof. We know that $r(P_3, H_7) = 5$. For the upper bound, consider $F = K_5 - P_3$. Observe that F consists of K_4 and P_3. Now, consider any red-blue coloring of the edges of F with no red P_3.

Then, there are at most two red independent edges. Since F consists of K_4 and P_3, the induced blue subgraph of F must contain H_7. Thus $F \rightarrow (P_3, H_7)$ and $r^*(P_3, H_7) \leq 8$.

For the lower bound, we will consider all graphs F with $v(F) = 5$ and $e(F) = 7$. Since graph F must contain C_4 and P_3, the only graph F satisfying these properties is isomorphic to either $K_5 - P_4$ or $K_5 - (P_3 \cup P_2)$. In both cases, take a red-blue coloring of the edges of F with no red P_3 such that the red edges is two edges from C_4. In this coloring, the induced blue subgraph of F does not contain H_7. This implies $F \nrightarrow (P_3, H_7)$ and $r^*(P_3, H_7) \geq 8$. Therefore, the theorem holds. □

Theorem 3.4. $r^*(P_3, H_8) = 9$.

Proof. We know that $r(P_3, H_8) = 5$. For the upper bound, consider $F = K_5 - K_2$. Now, consider any red-blue coloring of the edges of F such that there is no red P_3. Then, the graph G_F will be a subgraph of either $P_4 \cup K_1$ or $P_3 \cup K_2$. In both cases, $\overline{G}_F$ contains H_8. Since $\overline{G}_F$ is precisely the induced blue subgraph of F, then $F \rightarrow (P_3, H_8)$ and $r^*(P_3, H_8) \leq 9$.

For the lower bound, we will consider all graphs F with $v(F) = 5$ and $v(F) = 8$. The only graph F satisfying these properties is isomorphic to either $K_5 - P_3$ or $K_5 - 2P_2$. If $F = K_5 - P_3$, then take a red-blue coloring of the edges of F with no red P_3 such that the graph G_F is isomorphic to $C_4 \cup K_1$. If $F = K_5 - 2P_2$, then take a red-blue coloring of the edges of F with no red P_3 such that the graph G_F is isomorphic to $C_3 \cup K_2$. In both cases, the induced blue subgraph of F does not contain H_8. This implies that $F \nrightarrow (P_3, H_8)$ and $r^*(P_3, H_8) \geq 9$. Therefore, the theorem holds. □

The next results are $r^*(P_3, H_i)$ for which $r(P_3, H_i) = 7$. Observe that each of H_{17}, H_{19}, and H_{20} contains K_4. To find $r^*(P_3, H_i)$, for $i = 17, 19$, and 20, we will use the following lemma. The idea of this lemma is from the proof of Theorem 2.2 given by Faudree and Sheehan [5].

Lemma 3.1. *Let F be a graph with $v(F) = 7$. If $F \rightarrow (P_3, K_4)$ then $\delta(F) \geq 5$.*

Proof. Let F be a graph with $v(F) = 7$ and $F \rightarrow (P_3, K_4)$. For a contradiction, suppose there exists a vertex $v \in V(F)$ with $d(v) \leq 4$. It means there are at least two vertices $w, x \in V(F)$ not adjacent to v. Now, take a red-blue coloring of the edges of F by giving red to independent edges incident to $V(F) \setminus \{v, w, x\}$ and edge wx (if they exist) and the rest are blue. In this coloring, there is no a red P_3 or a blue K_4, a contradiction to $F \rightarrow (P_3, K_4)$. □

Theorem 3.5. $r^*(P_3, H_{17}) = r^*(P_3, H_{19}) = 18$.

Proof. We know that $r(P_3, H_{17}) = r(P_3, H_{19}) = 7$. Notice that $H_{17} \subseteq H_{19}$. For the upper bound, consider $F = K_7 - 3P_2$. Now, consider any red-blue coloring of the edges of F such that there is no red P_3. Then, the graph G_F will be a subgraph of either P_7 or $C_4 \cup P_3$. In both cases, $\overline{G}_F$ contains H_{19}. Since $\overline{G}_F$ is precisely the induced blue subgraph of F, then $F \rightarrow (P_3, H_{19})$ and $r^*(P_3, H_{19}) \leq 18$. Furthermore, since $H_{17} \subseteq H_{19}$, Equation (2) implies $r^*(P_3, H_{17}) \leq 18$.

For the lower bound, we will consider all graphs F with $v(F) = 7$ and $e(F) = 17$. All graphs F satisfying these conditions will have $\delta(F) \leq 4$. According to Lemma 3.1, $F \nrightarrow (P_3, K_4)$. Since $K_4 \subseteq H_{17} \subseteq H_{19}$, we have $F \nrightarrow (P_3, H_{17})$ and $F \nrightarrow (P_3, H_{19})$. This implies $r^*(P_3, H_{17}) \geq 18$ and $r^*(P_3, H_{19}) \geq 18$. Therefore, the theorem holds. □

Theorem 3.6. $r^*(P_3, H_{20}) = 19$.

Proof. We know that $r(P_3, H_{20}) = 7$. For the upper bound, consider $F = K_7 - 2P_2$. Now, consider any red-blue coloring of the edges of F such that there is no red P_3. Then, the graph G_F will be a subgraph of either $P_6 \cup K_1$, $P_5 \cup K_2$, or $C_4 \cup P_2 \cup K_1$. In all cases, $\overline{G}_F$ contains H_{20}. Since $\overline{G}_F$ is precisely the induced blue subgraph of F, then $F \to (P_3, H_{20})$ and $r^*(P_3, H_{20}) \leq 19$.

For the lower bound, we will consider all graphs F with $v(F) = 7$ and $e(F) = 18$. Since H_{20} contains K_4, according to Lemma 3.1, $\delta(F) \geq 5$. The only graphs satisfying these conditions is $F = K_7 - 3P_2$. To show that $F \nrightarrow (P_3, H_{20})$, take a red-blue coloring of the edges of F with no red P_3 such that the graph G_F is isomorphic to P_7. In this red-blue coloring, the induced blue subgraph of F contains K_4. However, each vertex not in this K_4 is not adjacent to exactly two vertices of this K_4. This implies that the induced blue subgraph of F does not contain H_{20}. As a consequence, $r^*(P_3, H_{20}) \geq 19$. Therefore, the theorem holds. □

In the following, we are going to give the value of $r^*(P_3, H_{18})$. However, we need Lemma 3.2 to do so. Note that H_{18} is a triangle book graph with three sheets. It means H_{18} consists of three triangles with exactly one shared edge.

Lemma 3.2. *Let F be a graph with $v(F) = 7$ and all the edges of F is red-blue colored so that no red P_3. Let G_F be a graph with $V(G_F) = V(F)$ and $E(G_F)$ consists of red edges in F together with $E(\overline{F})$. Then, $F \to (P_3, H_{18})$ if and only if G_F has two non-adjacent vertices u and v with the property $|N(u) \cup N(v)| \leq 2$.*

Proof. Let F be a graph with the properties as given in the Lemma. We define G_F accordingly. Suppose to the contrary $F \to (P_3, H_{18})$ and G_F does not have two non-adjacent vertices u and v with the property $|N(u) \cup N(v)| \leq 2$. It means that for every two non-adjacent vertices $u, v \in V(G_F)$, $|N(u) \cup N(v)| \geq 3$. To construct H_{18} in the $\overline{G}_F$, suppose uv is the shared edge in H_{18}. To have H_{18}, we need to find three independent P_3 which end in u and v. However, it is impossible because $N(u) \cap N(v)$ consists of at most two vertices in $V(\overline{G}_F)$.

Conversely, suppose G_F has two non-adjacent vertices u and v with the property $|N(u) \cup N(v)| \leq 2$. It means $N(u) \cap N(v)$ consists of at least three vertices in $V(\overline{G}_F)$. We can construct H_{18} in the $\overline{G}_F$ by taking uv as the shared edge and adding three independent P_3 which end in u and v with internal vertices are the vertices in $N(u) \cap N(v)$. As a consequence, $F \to (P_3, H_{18})$. Therefore, the lemma holds. □

Theorem 3.7. $r^*(P_3, H_{18}) = 15$.

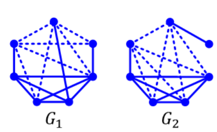

Figure 3. Graphs G_1 and G_2.

Proof. We know that $r(P_3, H_{18}) = 7$. For the upper bound, consider $F = K_7 - K_4$. Now, consider any red-blue coloring of the edges of F such that there is no red P_3. Then, the graph G_F will be a subgraph of either G_1 or G_2 as given with the solid line in Figure 3. In both cases, $\overline{G}_F$ contains H_{18} (the graphs with dotted line in Figure 3). Since $\overline{G}_F$ is precisely the induced blue subgraph of F, then $F \rightarrow (P_3, H_{18})$ and $r^*(P_3, H_{18}) \leq 15$.

For the lower bound, we will consider all graphs F with $v(F) = 7$ and $e(F) = 14$. Notice that F must be connected, since $r(P_3, H_{18}) = 7$. There are 64 non-isomorphic graphs satisfying these properties. Let $\{F\}$ be the collection of these 64 graphs. To show that for all $F \in \{F\}$ satisfy $F \nrightarrow (P_3, H_{18})$, for each $F \in \{F\}$ we construct graph G_F as follows. Starting from $G_F = \overline{F}$. We need to add more independent edges to G_F that representing red edges in the red-blue coloring of F with no red P_3. To do so, connect two vertices with the least degree in G_F. Do the same thing to two vertices with the next least degree, and so on. Using this algorithm, we can add at least two and at most three independent edges to get the final G_F. The complete list of G_F for 64 graphs F is given in Figure 4 with the red edges in dotted line. It is easy to verify that for each $F \in \{F\}$, there is no two non-adjacent vertices u and v with $|N(u) \cup N(v)| \leq 2$ in G_F. Lemma 3.2 implies $F \nrightarrow (P_3, H_{18})$. This implies $r^*(P_3, H_{18}) \geq 15$. Therefore, the theorem holds. □

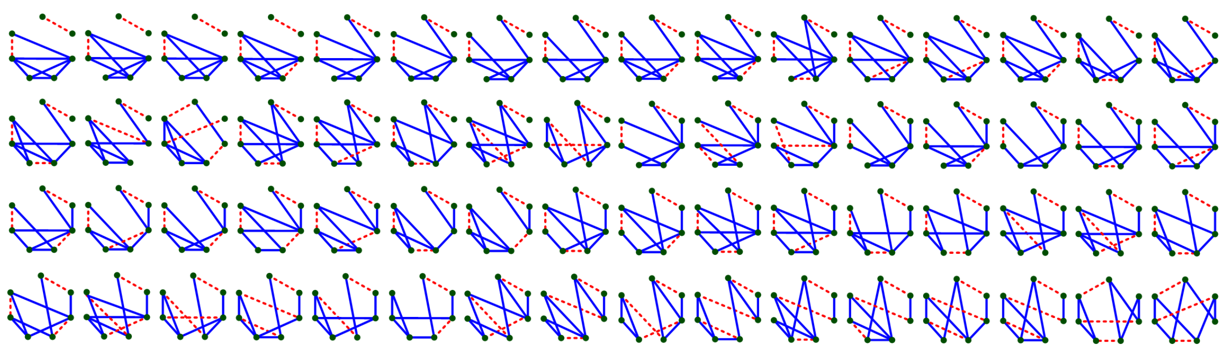

Figure 4. The graphs G_Fs for 64 connected graphs F with $v(F) = 7$ and $e(F) = 14$.

We summarize the restricted size Ramsey number for P_3 versus connected graphs of order five in Table 1.

Table 1. Summary of $r^*(P_3, H)$ with H is a connected graph of order five.

r^*	H_1	H_2	H_3	H_4	H_5	H_6	H_7
P_3	6 [9]	7 [9]	10 [9]	9 Th. 3.1	10 [10]	8 Th. 3.2	8 Th. 3.3
r^*	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}
P_3	9 Th. 3.4	10 [10]	9 [10]	9 [10]	10 [10]	10 [10]	10 [10]
r^*	H_{15}	H_{16}	H_{17}	H_{18}	H_{19}	H_{20}	H_{21}
P_3	10 [10]	10 [10]	18 Th. 3.5	15 Th. 3.7	18 Th. 3.5	19 Th. 3.6	32 [5]

Acknowledgement

This research was supported by the Research Grant "Program Penelitian Unggulan Perguruan Tinggi" the Ministry of Research, Technology, and Higher Education, Indonesia.

References

[1] S.A. Burr, A survey of noncomplete Ramsey theory for graphs, *Ann. New york Acad. Sci.* **328** (1979), 58–75.

[2] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III: small off-diagonal number, *Pacific Journal of Mathematics* **41** (2) (1972), 335–345.

[3] R. Diestel. *Graph Theory*, Springer-Verlag Heidelberg, New York, 4 edition, 2005.

[4] P. Erdös, R.J. Faudree, C.C. Rousseau, and R.H. Schelp, The size Ramsey number, *Periodica Mathematica Hungarica*, Volume **9** (1978), Issue 1-2, 145–161.

[5] R.J. Faudree and J. Sheehan, Size Ramsey numbers for small-order graphs, *J. Graph Theory* **7** (1983), 53–55.

[6] R.J. Faudree and J. Sheehan, Size Ramsey numbers involving stars, *Disc. Math.* **46** (1983), 151–157.

[7] R.J. Faudree and R. Schelp, A survey of results on the size Ramsey numbers, *Paul Erdös and His Mathematics* II **10** (2002), 291–309.

[8] F. Harary and Z. Miller, Generalized Ramsey theory VIII: the size Ramsey number of small graphs, *Studies in Pure Mathetematics - To the Memory of Paul Turán*, Birkhaüser (1983), 271–283.

[9] R. Lortz and I. Mengersen, Size Ramsey results for paths versus stars, *Australas. J. Combin.* **18** (1998), 3–12.

[10] D.R. Silaban, E.T. Baskoro, and S. Uttunggadewa, On the restricted size Ramsey number for path of order three versus any connected graph, *submitted*.