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The structure of graphs with forbidden induced C_4, $\overline{C}_4$, C_5, S_3, chair and co-chair

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Abstract

We find the structure of graphs that have no C_4, $\overline{C}_4$, C_5, S_3, chair and co-chair as induced subgraphs. Then we deduce the structure of the graphs having no induced C_4, $\overline{C_4}$, S_3, chair and co-chair and the structure of the graphs G having no induced C_4, $\overline{C_4}$ and such that every induced P_4 of G is contained in an induced C_5 of G.

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1. Introduction

In this paper, graphs are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Two edges of a graph G are said to be *adjacent* if they have a common endpoint and two vertices x and y are said to be *adjacent* if xy is an edge of G. The *neighborhood* of a vertex v in a graph G, denoted by $N_G(v)$, is the set of all vertices adjacent to v and its *degree* is $d_G(v) = |N_G(v)|$. We omit the subscript if the graph is clear from the context. For two set of vertices U and W of a graph G, let $E[U, W]$ denote the set of all edges in the graph G that joins a vertex in U to a vertex in W. A graph is empty if it has no edges. For $A \subseteq V(G)$, $G[A]$ denotes the sub-graph of G induced by A. If $G[A]$ is an empty graph, then A is called a

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stable set. While, if $G[A]$ is a complete graph, then A is called a *clique set*, that is any two distinct vertices in A are adjacent. The *complement graph* of G is denoted by $\overline{G}$ and defined as follows: $V(G) = V(\overline{G})$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. A graph H is called a *forbidden subgraph* of G if H is not (isomorphic to) an induced subgraph of G.

A *cycle* on n vertices is denoted by $C_n = v_1v_2...v_nv_1$ while a *path* on n vertices is denoted by $P_n = v_1v_2...v_n$. A *chair* is any graph on 5 distinct vertices x, y, z, t, v with exactly 5 edges xy, yz, zt and zv. The *co-chair* or $\overline{chair}$ is the complement of a chair. S_3 is the graph on 6 vertices as indicated in Figure 1.

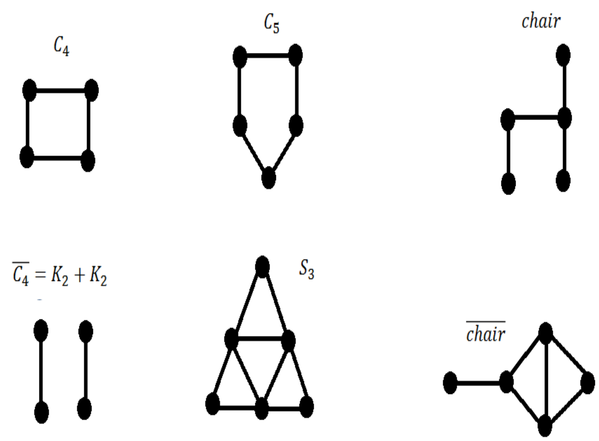

Figure 1. The graphs C_4, C_5, $\bar{C}_4$, S_3, Chair and Co-chair.

Many graphs encountered in the study of graph theory are characterized by configurations or subgraphs they contain. However, there are occasions where it is easier to characterize graphs by sub-graphs or induced sub-graphs they do not contain. For example, trees are the connected graph without (induced) cycles. Bipartite graphs are those without (induced) odd cycles ([1]). Split graphs are those without induced C_4, $\overline{C}_4$ and C_5. Line graphs are characterized by the absence of only nine particular graphs as induced sub-graph (see [2]). Perfect graphs are characterized by C_{2n+1} and $\overline{C}_{2n+1}$ being forbidden, for all $n \geq 2$ (see [3]). The purpose of this paper is to find the structure of graphs such that C_4, $\overline{C}_4$, C_5, S_3 chair and co-chair are forbidden subgraphs. These graphs will be called generalized combs and they are generalization of generalized stars and generalization of combs (See [6, 8]). Seymour's Second Neighborhood Conjecture (see [9]) is proved for orientation of graphs obtained from the complete graph by deleting the edges of a generalized star and for those obtained by deleting the edges of a comb [6, 8]. Generalized stars (also called threshold graphs) are the graphs with C_4, $\overline{C}_4$ and P_4 forbidden. Finding the structure of the generalized comb, might give a clearer vision for an attempt to prove Seymour's conjecture for oriented graphs obtained from the complete graph by deleting the edges of a generalized comb.

2. Preliminary Definitions and Theorems

Definition 1. A graph G is a called a *split graph* if its vertex set is the disjoint union of a stable set S and a clique set K. In this case, G is called an $\{S, K\}$-split graph.

If G is an $\{S, K\}$-split graph and $\forall s \in S$, $\forall x \in K$ we have $sx \in E(G)$, then G is called a *complete split graph*.

If G is an $\{S, K\}$-split graph and $E[S, K]$ forms a perfect matching of G, then G is called a *perfect split graph*.

Theorem 2.1. *(Földes and Hammer [4]) G is a split graph if and only if C_4, $\overline{C}_4$ and C_5 are forbidden subgraphs of G.*

Definition 2. ([5]) A *threshold graph G* can be defined as follows:

1) $V(G) = \bigcup_{i=1}^{n+1}(X_i \cup A_{i-1})$, where the A_i's and X_i's are pair-wisely disjoint sets.

2) $K := \bigcup_{i=1}^{n+1} X_i$ is a clique and the X_i's are nonempty, except possibly X_{n+1}.

3) $S := \bigcup_{i=0}^{n} A_i$ is a stable set and the A_i's are nonempty, except possibly A_0.

4) $\forall\, i, j \in [1, n]$ and $j \leq i$, $G[A_i \cup X_j]$ is a complete split graph.

5) The only edges of G are the edges of the subgraphs mentioned above.

In this case, G is called an $\{S, K\}$-threshold graph.

In fact, threshold graphs are exactly the *generalized stars* defined in [6].

Theorem 2.2. *(Hammer and Chvàtal [5]) G is a threshold graph if and only if C_4, $\overline{C}_4$ and P_4 are forbidden subgraphs of G.*

Theorem 2.3. *([7]) C_4, $\overline{C_4}$ are forbidden subgraphs of a graph G if and only if $V(G)$ is disjoint union of three sets S, K and C such that:*

1) *$G[S \cup K]$ is an $\{S, K\}$-split graph;*

2) *$G[C]$ is empty or isomorphic to the cycle C_5;*

3) *every vertex in C is adjacent to every vertex in K but to no vertex in S.*

3. Main Results

Lemma 3.1. *Suppose that C_4, $\overline{C}_4$, C_5, chair and co-chair are forbidden subgraphs of G. If the path $mbb'm'$ is an induced subgraph of G, then:*

$$N(m) - \{b\} = N(m') - \{b'\}$$

and

$$N(b) - \{m\} = N(b') - \{m'\}.$$

Proof. Since C_4, $\overline{C}_4$ and C_5 are forbidden, then G is an $\{S, K\}$-split graph for some stable set S and a clique set K. Since $mbb'm'$ is an induced subgraph of G, then $m, m' \in S$ and $b, b' \in K$.

Assume that there is $x \in N(m) - \{b\}$ but $x \notin N(m') - \{b'\}$. Since xm is an edge of G and S is stable, then we must have $x \in K$. But K is a clique, then x is adjacent to b and b'. Thus $G[\{x, m, b, b', m'\}]$ is a co-chair. Contradiction. So $N(m) - \{b\} \subseteq N(m') - \{b'\}$. By symmetry, $N(m') - \{b'\} \subseteq N(m) - \{b\}$. Thus $N(m) - \{b\} = N(m') - \{b'\}$.

Assume that there is $x \in N(b) - \{m\}$ but $x \notin N(b') - \{m'\}$. Suppose that $x \in S$. Then $G[\{x, m, b, b', m'\}]$ is a chair. Contradiction. Thus $x \in K$. But K is a clique. Whence $x \in N(b')\{m'\}$. Thus $N(b) - \{m\} \subseteq N(b') - \{m'\}$. By symmetry, $N(b') - \{m'\} \subseteq N(b) - \{m\}$. Therefore $N(b) - \{m\} = N(b') - \{m'\}$. □

Proposition 3.1. *If P_4 is a forbidden subgraph of an $\{S, K\}$-split graph G, then G is an $\{S, K\}$-threshold graph.*

Proof. We prove this by induction on the number of vertices of G. This is clearly true for small graphs. Suppose that P_4 is a forbidden subgraph of an $\{S, K\}$-split graph G. It is clear that G is a threshold graph. We have to prove that G is $\{S, K\}$-threshold graph. Let $x \in K$ be a vertex with minimum degree in G, that is $d_G(x) = min\{d_G(y); y \in K\}$ and $G' := G - x$ be the graph induced by the vertices of G except x (If $K = \phi$, then the statement is true). Then P_4 is a forbidden subgraph of the $\{S, K - \{x\}\}$-split graph G'. By the induction hypothesis, G' is an $\{S, K - \{x\}\}$-threshold graph. We follow the notations in Definition 2. Assume that $\exists a \in S - A_n$ such that $ax \in E(G)$. Let $x_n \in X_n$. Since $d(x_n) \geq d(x)$, then there is $a_n \in A_n$ such that $a_n x_n \in E(G)$ but $a_n x \notin E(G)$. Then axx_na_n is an induced P_4 in G. Contradiction. Thus we may suppose that $N(x) \cap S \subseteq A_n$. If $N(x) \cap A_n = \phi$, then we add x to X_{n+1}. If $N(x) \cap A_n = A_n$, then we add x_n to X_n. Otherwise $\phi \subsetneq N(x) \cap A_n \subsetneq A_n$. In this case we do the following: remove from A_n the element of $N(x) \cap A_n$, create $A_{n+1} = N(x) \cap A_n$, remove the elements of X_{n+1} to the new set X_{n+2} and add x to X_{n+1} (so that the new $X_{n+1} = \{x\}$). Then G is $\{S, K\}$-threshold graph. □

Definition 3. A graph G is called a generalized comb if:

1) $V(G)$ is disjoint union of sets $A_0, ..., A_n, M_1, ..., M_l, X_1, ..., X_{n+1}, Y_2, ..., Y_{l+2}$. Let $Y_1 = X_1$ (These sets are called the sets of the generalized comb G).

2) $S := A \cup M$ is a stable set, where $M = \bigcup_{i=1}^{l} M_i$ and $A = \bigcup_{i=0}^{n} A_i$.

3) $K := X \cup Y$ is a clique, where $X = \bigcup_{i=1}^{n+1} X_i$ and $Y = \bigcup_{i=1}^{l+2} Y_i$.

4) $\forall\, i, j \in [1, n]$ and $j \leq i$, $G[A_i \cup X_j]$ is a complete split graph.

5) $G[A \cup Y]$ is a complete split graph.

6) $\forall\, i \in [1, l]$, $G[Y_i \cup M_i]$ is a perfect split graph or $M_i = \phi$.

7) $\forall\, i, j \in [1, l+1]$ and $i < j$, $G[Y_j \cup M_i]$ is a complete split graph.

8) $X_{n+1}, Y_{l+2}, Y_{l+1}$ and A_0 are the only possibly empty sets among the $X_i's$, $Y_i's$, $A_i's$.

9) The only edges of G are the edges of the subgraphs mentioned above.

In this case, we say that G is an $\{S, K\}$-generalized comb. Note, that we may assume that no two consecutive sets M_i and M_{i+1} are both empty. We use this assumption in the rest.

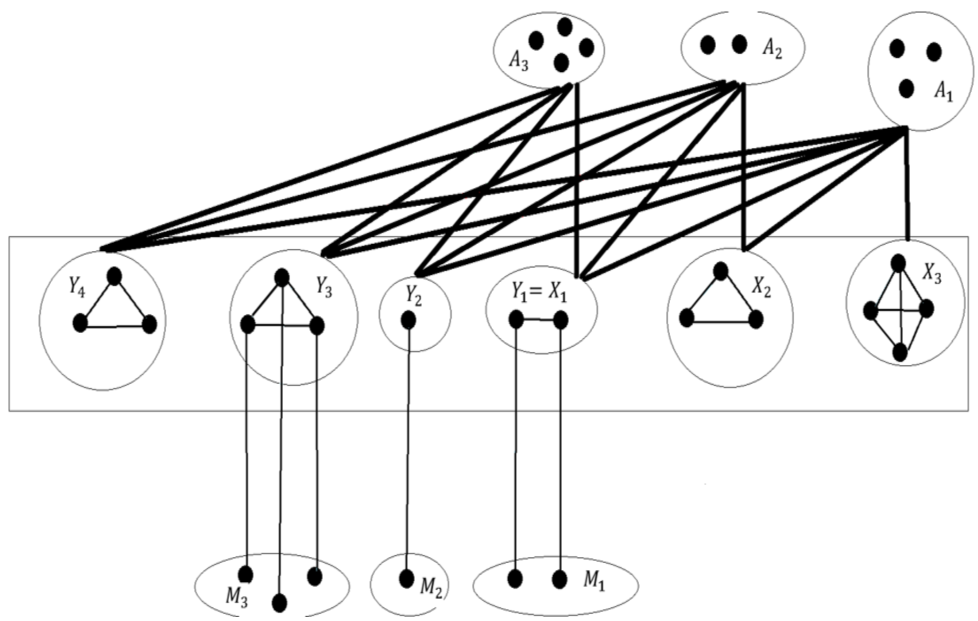

Figure 2. Generalized Comb, with $n = l = 3$, $X_{n+1} = Y_{l+2} = \phi$, $A \cup M$ is stable, $X \cup Y$ is a clique. Any 2 vertices in 2 sets joined by a thick bold edge are adjacent.

It is clear that the *comb* defined in [8] is a particular case of the generalized comb (see Figure 3). Moreover, we have the following:

Lemma 3.2. *Every $\{S, K\}$-threshold graph is an $\{S, K\}$-generalized comb.*

Proof. Let G be an $\{S, K\}$-threshold graph defined as in Definition 2. Following the notations in Definition 3, we take $l = 1$ and $M_l = Y_{l+1} = Y_{l+2} = \phi$. This shows that G is an $\{S, K\}$-generalized comb. □

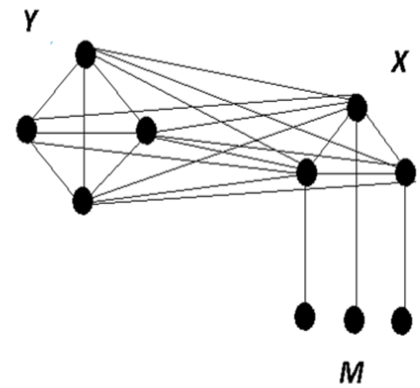

Figure 3. Comb G. $X \cup Y$ is a clique, $G[X \cup M]$ is a perfect split graph, no edges between Y and M.

Theorem 3.1. *If S_3, chair and co-chair are forbidden subgraphs of an $\{S, K\}$-split graph G, then G is an $\{S, K\}$-generalized comb.*

Proof. We prove the statement by induction on the number of vertices. The statement is true for small graphs. Suppose that S_3, chair and co-chair are forbidden subgraphs of an $\{S, K\}$-split graph G. If P_4 is also a forbidden subgraph of G, then G is an $\{S, K\}$-threshold graph, and hence, G is an $\{S, K\}$-generalized comb. So we may suppose that G contains at least one induced path of length four.

Suppose that G has exactly one induced path of length four, say $mbb'm'$. Suppose $N(m) = \{b\}$. Then $N(m') = \{b'\}$. Let $H = G[K \cup S - \{m, m'\}]$. By induction hypothesis, we have H is $\{S - \{m, m'\}, K\}$-generalized comb. But H has no induced P_4, then H is in fact $\{S - \{m, m'\}, K\}$-threshold graph. We use the nation in the definition of threshold graph, in what follows. Assume that $\exists i \geq 2$ such that $b \in X_i$. Let $x \in X_1$ and $a \in A_1$. Then $mbxa$ is induced P_4 in G, a contradiction. So $b \in X_1$. Then also $b' \in X_1$, because b and b' have the same neighborhood in H. Define $Y_2 = \phi$, $M_1 = \{m, m'\}$, $Y_3 = X_1 - \{b, b'\}$ and the new X_1 is the $\{b, b'\}$. Then G is an $\{S, K\}$-generalized comb with $l = 1$ and $Y_{l+1} = \phi$.

Otherwise, G has at least two induced P_4. Let m be a vertex of G such that $d(m) = min\{d(z);$ z is a leaf of an induced P_4 in $G\}$ and let $P = mbb'm'$ be an induced P_4. Note that $d(m) = d(m')$. Let $Q = udd'u'$ be an induced P_4 distinct from P (Note that $m, m', u, u' \in S$ while $b, b', d, d' \in K$). Either $m \notin \{u, u'\}$ or $m' \notin \{u, u'\}$, since $N(m) - \{b\} = N(m) - \{b'\}$ (Lemma 3.1). We may assume without loss of generality that $m \notin \{u, u'\}$ and let $H = G[(S - m') \cup (K - b')]$. By the induction hypothesis, H is an $\{S - m', K - b'\}-$generalized comb.

Suppose first that $m' \in \{u, u'\}$ and assume without loss of generality that $m' = u'$. Assume that $b' \neq d'$. If $b = d$, then by using Lemma 3.1 repeatedly, we can prove easily that $G[\{m', m, u, b, b', d'\}]$ is an S_3, a contradiction. So $b \neq d$. Note that $b' \neq d$, because $u'b' = mb' \in E(G)$, while $u'd \notin E(G)$. By applying Lemma 3.1 repeatedly, we have the following: Since $u'b' = m'b' \in E(G)$, then $ub' \in E(G)$, thus $ub \in E(G)$, whence $u'b \in E(G)$, therefore $m'b \in E(G)$, which is a contradiction. Therefore, $b' = d'$. Note that $b \neq d$, since otherwise, we get $u \in N(b) - \{m\}$, thus by Lemma 3.1, we get $u \in N(b') - \{m'\} = N(d') - \{u'\}$, whence $ud' \in E(G)$, a contradiction. Since $udd'u' = udb'm'$ is an induced path of length four of G, then by Lemma 3.1 also $udbm$ is an induced path of G and thus of H. Then, by the definition of the generalized comb H, $\exists i$; $u, m \in M_i$ (We follow the notations of definition 3.). In this case we add m' to M_i and b' to Y_i. This shows that G is an $\{S, K\}-$generalized comb.

Now, suppose that $m' \notin \{u, u'\}$. Assume that $m \in A$. By definition of the generalized comb H and since $udd'u'$ is an induced P_4 of H, we get that $N_H(u) \subseteq N_H(m)$ and $d' \in N_H(m) - N_H(u)$. So $d_H(u) < d_H(m)$. Assume that $b \notin N_H(u)$. Then $b \notin N(u)$ and thus by Lemma 3.1, we get $b' \notin N(u)$. Therefore, $d_G(u) = d_H(u) < d_H(m) = d_G(m)$, which is a contradiction to the choice of m. Hence, $b \in N_H(u)$ and so, by Lemma 3.1, we get $b, b' \in N(u) \cap N(u')$. Note that $d, d' \in N(m)$ and hence $d, d' \in N(m')$. Thus $G[\{u, d', m', b, m, b'\}]$ is an induced S_3 in G, a contradiction.

So $m \in M$. Let l be the greatest such that $M_l \neq \phi$. Suppose that $m \notin M_l$. Let $m'' \in M_l$ and $b'' \in Y_l$ be its neighbor. $\exists i < l$ such that $m \in M_i$. Then $b''m \in E(G)$ and $N_H(m'') \subseteq N_H(m)$. Let $c \in Y_i$ be the neighbor of m. Let k be the smallest such that $k > i$ and $M_k \neq \phi$ (Note that k exists and $i < k \leq l$, moreover we may assume $k = i + 1$ or $k = i + 2$).

Suppose $b \in N(m'')$. Then also $b' \in N(m'')$. If $b \neq b''$, then $\exists j > k$ such that $b \in Y_j$. Then by using Lemma 3.1, we can prove easily that $G[\{m, m', m'', b, b', c\}]$ is an induced S_3 of G, a contradiction. However, if $b = b'$, then also by using Lemma 3.1, we can observe that $G[\{m, m', m'', b, b', c\}]$ is an induced S_3 in G, a contradiction.

Suppose $b \notin N(m'')$. Then $b' \notin N_H(m) - N_H(m'')$, $b \neq b''$ and $\exists i < j \leq k$ such that $b \in Y_j$. Thus $d(m'') = d_H(m'') < d_H(m) = d_G(m)$, a contradiction is reached if m'' is a leaf of an induced P_4 of G. So, we have m'' is not a leaf of an induced P_4 of G and thus of H and thus $M_k = \{m''\}$ and $j < k$. If $c = b$, then we add b' to Y_i and m' to M_i and thus G is an $\{S, K\}-$generalized comb. So suppose $c \neq b$. Assume there is $mcm'''b'''$ an induced P_4 in H. Then $m''' \in M_i$ and $b''' \in Y_i$. Then by using Lemma 3.1, we can observe that $G[\{m, m', m''', b, b', c\}]$ is an induced S_3 in G, a contradiction. Thus m is not a leaf of an induced P_4 of H, that is $M_i = \{m\}$. By definition of k, we get $M_j = \phi$. Thus $j = i + 1$ and $k = i + 2$. Now, to Y_{i+1} we add c and remove b, while to Y_i we add b and remove c. Then, we can add b' to Y_i and m' to M_i to get that G is an $\{S, K\}-$generalized comb.

Therefore $m \in M_l$. Let $Y_l \cap N(m) = \{c\}$. If $b = c$, then we add b' to Y_l and m' to M_l and thus G is $\{S, K\}$-generalized comb. Now suppose that $b \neq c$. Suppose that c is not the only vertex in Y_l and thus there is an induced path $mcc''m''$ with $c, c'' \in Y_l$ and $m'' \in Y_l$. By using Lemma 3.1, we can prove easily that $G[\{b, b', c, m, m', m''\}]$ is an induced S_3 of G a contradiction. Therefoe c is the only vertex in Y_l. Since $bm \in E(H)$, then $b \in Y_{l+1}$. We do the following: To Y_{l+1} add c and remove b and to Y_l add b and remove c. Then we add b' to Y_l and m' to M_l (as in the case $b = c$) and this shows that G is an $\{S, K\}$-generalized comb. □

Corollary 3.1. *G is a generalized comb if and only if C_4, $\overline{C}_4$, C_5, S_3 chair and co-chair are forbidden subgraphs of G.*

Proof. The necessary condition is obvious by the definition of the generalized comb. For the sufficient condition it is enough to note that the statement C_4, $\overline{C}_4$, C_5, S_3, chair and co-chair are forbidden subgraphs of G is equivalent to the statement that G is a split graph and S_3, chair and co-chair are forbidden subgraphs of G. □

Corollary 3.2. *G is a generalized comb if and only if every induced subgraph of G is a generalized comb.*

Proof. Let G' be an induced subgraph of a generalized comb G. It is clear that G' contains no induced C_4, $\overline{C}_4$, C_5, chair and co-chair. Thus G' is a generalized comb. The sufficient condition is clear. □

Corollary 3.3. *C_4, $\overline{C_4}$, S_3, chair and co-chair are forbidden subgraphs of a graph G if and only if $V(G)$ is disjoint union of three sets S, K and C such that:*

1) *$G[S \cup K]$ is an $\{S, K\}$-generalized comb;*

2) *$G[C]$ is empty or isomorphic to the cycle C_5;*

3) *every vertex in C is adjacent to every vertex in K but to no vertex in S.*

Proof. The sufficient condition is clear by construction of G. We prove the necessary condition. Suppose that C_4, $\overline{C_4}$, S_3, chair and co-chair are forbidden subgraphs of a graph G. Then by Theorem 2.3, $V(G)$ is disjoint union of three sets S, K and C such that:

1) $G[S \cup K]$ is an $\{S, K\}$-split graph;

2) $G[C]$ is empty or isomorphic to the cycle C_5;

3) every vertex in C is adjacent to every vertex in K but to no vertex in S.

Then C_4, $\overline{C_4}$, C_5, S_3 chair and co-chair are forbidden subgraphs of $G[S \cup K]$. Thus $G[S \cup K]$ is an $\{S, K\}$-generalized comb. □

Corollary 3.4. *C_4, $\overline{C_4}$ are forbidden subgraphs of G and every induced P_4 of G is contained in an induced C_5 of G if and only if $V(G)$ is disjoint union of three sets S, K and C such that:*

1) *$G[S \cup K]$ is an $\{S, K\}$-threshold;*

2) *$G[C]$ is empty or isomorphic to the cycle C_5;*

3) *every vertex in C is adjacent to every vertex in K but to no vertex in S.*

Proof. The sufficient condition is clear by construction of G. We prove the necessary condition. Suppose that C_4, $\overline{C_4}$ are forbidden subgraphs of a graph G and every induced P_4 of G is contained in an induced C_5 of G. Then by Theorem 2.3, $V(G)$ is disjoint union of three sets S, K and C such that:

1) $G[S \cup K]$ is an $\{S, K\}$-split graph;

2) $G[C]$ is empty or isomorphic to the cycle C_5;

3) every vertex in C is adjacent to every vertex in K but to no vertex in S.

Then $G[C]$ is the unique induced C_5 of G or G has no induced C_5. Then C_4, $\overline{C_4}$, P_4 are forbidden subgraphs of $G[S \cup K]$. Thus $G[S \cup K]$ is an $\{S, K\}$-threshold graph. □

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