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Perfect 3-colorings of the cubic graphs of order 10

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Abstract

Perfect coloring is a generalization of the notion of completely regular codes, given by Delsarte. A perfect m-coloring of a graph G with m colors is a partition of the vertex set of G into m parts $A_1, A_2, \cdots, A_m$ such that, for all $i, j \in \{1, \cdots, m\}$, every vertex of A_i is adjacent to the same number of vertices, namely, a_{ij} vertices, of A_j . The matrix $A = (a_{ij})_{i,j\in\{1,\cdots,m\}}$ is called the parameter matrix. We study the perfect 3-colorings (also known as the equitable partitions into three parts) of the cubic graphs of order 10. In particular, we classify all the realizable parameter matrices of perfect 3-colorings for the cubic graphs of order 10.

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1. Introduction

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see[10]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been done

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on enumerating the parameter matrices of some Johnson graphs, including $J(4,2)$, $J(5,2)$, $J(6,2)$, $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(v,3)$ (v odd) (see [1, 3, 4, 9]).

Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix (see [6, 7, 8]). In this paper all graphs are assumed simple, connected and undirected. First we give some basic definitions and concepts. Let $G = (V, E)$ be a graph. Two vertices $u, v \in V(G)$ are adjacent if there exists an edge $e = \{u, v\} \in E(G)$ to which they are both incident. The adjacent will be shown $u \leftrightarrow v$.

A cubic graph is a 3-regular graph. In [5], it is shown that the number of connected cubic graphs with 10 vertices is 19. Each graph is described by a drawing as shown in Figure 1.

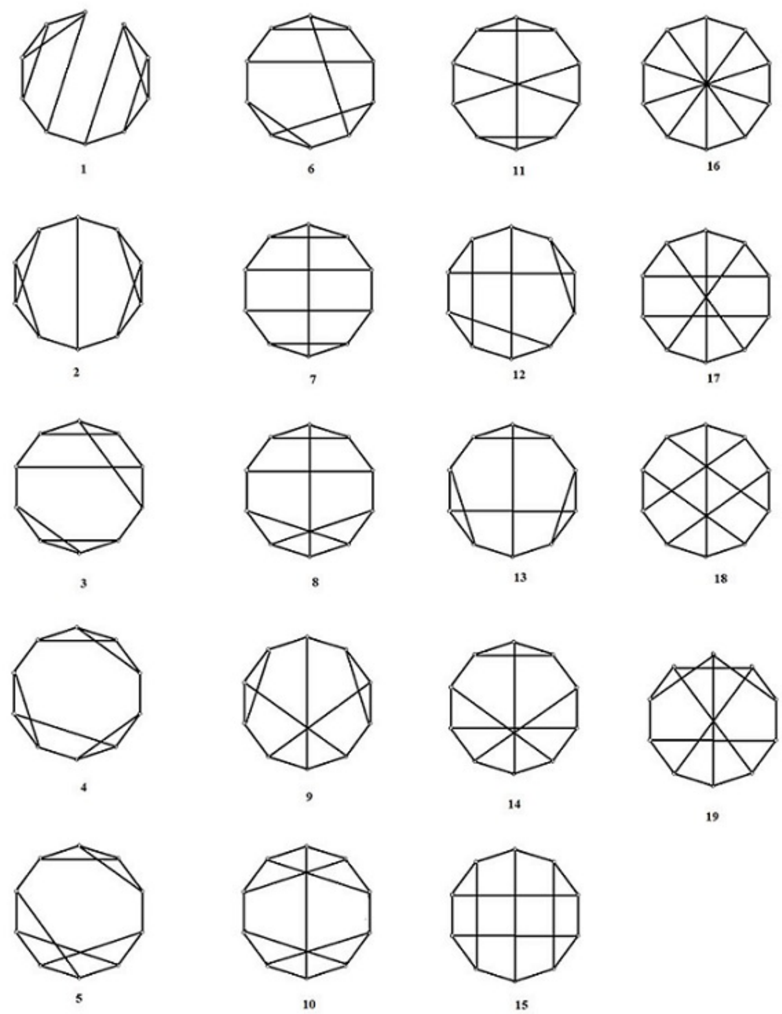

Figure 1. Connected cubic graphs of order 10.

Definition 1.1. For a graph G and a positive integer m, a mapping $T : V(G) \to \{1, \cdots, m\}$ is called a perfect m-coloring with matrix $A = (a_{ij})_{i,j \in \{1,\cdots,m\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the *parameter matrix* of a perfect coloring. In the case $m = 3$, we call the first color *white*, the second color *black*, and the third color *red*. In this paper, we generally show a parameter matrix by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Remark 1.1. In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e. we identify the perfect 3-coloring with the matrices

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}, \qquad \begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix},$$

obtained by switching the colors with the original coloring.

2. Preliminaries and Analysis

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of a cubic connected graph of order 10 with a given parameter matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

The simplest necessary condition for the existence of perfect 3-colorings of a cubic connected graph with the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is:

$$a + b + c = d + e + f = g + h + i = 3.$$

Also, it is clear that we cannot have $b = c = 0$, $d = f = 0$, or $g = h = 0$, since the graph is connected. In addition, $b = 0$, $c = 0$, $f = 0$ if $d = 0$, $g = 0$, $h = 0$, respectively.

The number θ is called an eigenvalue of a graph G, if θ is an eigenvalue of the adjacency matrix of this graph. The number θ is called an eigenvalue of a perfect coloring T into three colors with the matrix A, if θ is an eigenvalue of A. The following lemma demonstrates the connection between the introduced notions.

Lemma 2.1. ***[10]*** *If T is a perfect coloring of a graph G in m colors, then any eigenvalue of T is an eigenvalue of G.*

Now, without lost of generality, we can assume that $|W| \leq |B| \leq |R|$. The following proposition gives us the size of each class of color.

Proposition 2.1. *Let T be a perfect 3-coloring of a graph G with the matrix* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

1. *If $b, c, f \neq 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

2. *If $b = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

3. *If $c = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.$$

4. *If $f = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.$$

Proof. **(1):** Consider the 3-partite graph obtained by removing the edges uv such that u and v are the same color. By counting the number of edges between parts, we can easily obtain $|W|b = |B|d$, $|W|c = |R|g$, and $|B|f = |R|h$. Now, we can conclude the desired result from $|W| + |B| + |R| = |V(G)|$.
The proof of **(2), (3), (4)** is similar to **(1).** □

In the next lemma, under the condition $|W| = 1$, we enumerate all matrices that can be a parameter matrix for a cubic connected graph.

Lemma 2.2. *Let G be a cubic connected graph of order* 10*. If T is a perfect 3-coloring with the matrix A, and $|W| = 1$, then A should be the following matrix:*

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Proof. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a parameter matrix with $|W| = 1$. Consider the white vertex. It is clear that none of its adjacent vertices are white; i.e. $a = 0$. Therefore, we have two cases below.

(1) The adjacent vertices of the white vertex are the same color. If they are black, then $b = 3$ and $c = 0$. From $c = 0$, we get $g = 0$. Also, since the graph is connected, we have $f, h \neq 0$. Hence we obtain the following matrices:

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

If the adjacent vertices of the white vertex are red, then $c = 3$, $b = 0$. From $b = 0$, we get $d = 0$. Also, since the graph is connected, we have $f, h \neq 0$. Hence we obtain the following matrices:

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Finally, by using Remark 1.1 and the fact that $|W| \leq |B| \leq |R|$, it is obvious that there are only six matrices in (1), as shown A_1, A_2, A_3, A_4, A_5, A_6.

$$A_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

(2) The adjacent vertices of the white vertex are different colors. It immediately gives that $b, c \neq 0$. Also, it can be seen that $d = g = 1$. An easy computation as in (1), shows that there are only five matrices that can be a parameter matrix in this case, as shown A_7, A_8, A_9, A_{10}, A_{11}.

$$A_7 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, A_9 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, A_{10} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

By using Proposition 2.1, it is obvious that just the matrix $A := A_2$ can be a parameter.

□

Lemma 2.3. *Let G be a cubic connected graph of order 10. If T is a perfect 3-coloring with the matrix A, and $|W| = |B| = 2$, $|R| = 6$, then A should be the following matrix*

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

Proof. First, suppose that $b, c \neq 0$. As $|W| = 2$, by Proposition 2.1, it follows that $\frac{b}{d} + \frac{c}{g} = 4$. Therefore $b = c = 2, d = g = 1$ and we get a contradiction with $b + c \leq 3$.
Second, suppose that $b = 0$ and then $d = 0$. As $|R| = 4$, by Proposition 2.1, we have $\frac{g}{c} + \frac{h}{f} = \frac{2}{3}$.
Therefore $c = f = 3$, $g = h = 1$, and consequently $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$.
Finally, suppose that $c = 0$ and then $g = 0$. As $|B| = 2$, by Proposition 2.1, it follows that $\frac{d}{b} + \frac{f}{h} = 4$. Therefore $b = f = 2$, $d = h = 1$, or $b = 3$, $d = f = h = 1$ or $b = 3$,$d = 1$, $f = h = 2$. Hence $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$, or $A = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, or $A = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

By using the Proposition 2.1, it can be seen that only the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ can be a parameter. □

Lemma 2.4. *Let G be a cubic connected graph of order 10. Then G has no perfect 3-coloring T with the matrix that $|W| = 2, |B| = 3$, $|R| = 5$.*

Proof. If T is a perfect 3-coloring with the similar proving Lemma2.3, A should be one of the following matrices:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

By using the Proposition 2.1, it can be seen that no matrix can be a parameter. □

Lemma 2.5. *Let G be a cubic connected graph of order 10. If T is a perfect 3-coloring with the matrix A, and also if $|W| = 2, |B| = 4$, $|R| = 4$, then A should be one of the following matrices:*

$$\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix},\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix},\begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix}.$$

Proof. If T is a perfect 3-coloring with the similar proving Lemma2.3, then A should be one of the following matrices:

$$\begin{bmatrix}2&0&1\\0&1&2\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&1&2\\2&1&0\end{bmatrix},\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix},$$

$$\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix},\begin{bmatrix}2&1&0\\1&1&1\\0&2&1\end{bmatrix},\begin{bmatrix}1&2&0\\2&0&1\\0&2&1\end{bmatrix},\begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix}.$$

By using the Proposition 2.1, it can be seen that the following matrices should be parameter:

$$\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix},\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix},\begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix}.$$

□

Lemma 2.6. *Let G be a cubic connected graph of order* 10*. Then G has no perfect 3-coloring T with the matrix that $|W| = 3, |B| = 3, |R| = 4$.*

Proof. If T is a perfect 3-coloring with the similar proving Lemma2.3, then A should be one of the following matrices:

$$\begin{bmatrix}0&1&2\\3&0&0\\1&0&2\end{bmatrix},\begin{bmatrix}0&2&1\\1&2&0\\3&0&0\end{bmatrix},\begin{bmatrix}2&0&1\\0&1&2\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix},$$

$$\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix},\begin{bmatrix}1&0&2\\0&1&2\\2&1&0\end{bmatrix},\begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix},\begin{bmatrix}2&1&0\\2&0&1\\0&3&0\end{bmatrix}.$$

By using Proposition 2.1, it can be seen that no matrix can be a parameter. □

By using Lemmas 2.2, 2.3 and 2.5, it can be seen that only the following matrices can be parameter ones.

$$\begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix},\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix},\begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix},\begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix}.$$

By Remark 1.1, it is clear that the matrix $\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix}$ is the same as the matrix $\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix}$ and the matrix $\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix}$ is the same as the matrix $\begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix}$ up to renaming the colors. Therefore, if T is a perfect 3-coloring with the matrix A , then A should be one of the following matrices:

$$A_1 = \begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix}, A_2 = \begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix}, A_3 = \begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix}, A_4 = \begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix}.$$

The next theorem can be useful to find the eigenvalues of a parameter matrix.

Theorem 2.1. *Let* $A = \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$ *be a parameter matrix of a* k*-regular graph. Then the eigenvalues of* A *are*

$$\lambda_{1,2} = \frac{\mathrm{tr}\,(A) - k}{2} \pm \sqrt{\left(\frac{\mathrm{tr}\,(A) - k}{2}\right)^2 - \frac{\det(A)}{k}} \qquad , \qquad \lambda_3 = k.$$

Proof. By using the condition $a + b + c = d + e + f = g + h + i = k$, it is clear that one of the eigenvalues is k. Therefore $\det(A) = k\lambda_1\lambda_2$. From $\lambda_2 = \mathrm{tr}\,(A) - \lambda_1 - k$, we get

$$\det(A) = k\lambda_1(\mathrm{tr}\,(A) - \lambda_1 - k) = -k\lambda_1^2 + k(\mathrm{tr}\,(A) - k)\lambda_1.$$

By solving the equation $\lambda^2 + (k - \mathrm{tr}\,(A))\lambda + \frac{\det(A)}{k} = 0$, we obtain

$$\lambda_{1,2} = \frac{\mathrm{tr}\,(A) - k}{2} \pm \sqrt{\left(\frac{\mathrm{tr}\,(A) - k}{2}\right)^2 - \frac{\det(A)}{k}}.$$

□

3. Perfect 3-colorings of the cubic connected graphs of order 10

In this section, we enumerate the parameter matrices of all perfect 3-colorings of the cubic connected graphs of order 10.

Theorem 3.1. *The parameter matrices of cubic graphs of order* 10 *are listed in the following table.*

graphs	*matrix* A_1	*matrix* A_2	*matrix*A_3	*matrix* A_4
1	$\surd$	$\times$	$\surd$	$\times$
2	$\surd$	$\times$	$\surd$	$\surd$
3	$\times$	$\times$	$\times$	$\times$
4	$\surd$	$\times$	$\times$	$\times$
5	$\times$	$\times$	$\times$	$\times$
6	$\surd$	$\times$	$\surd$	$\times$
7	$\times$	$\times$	$\times$	$\times$
8	$\times$	$\times$	$\times$	$\times$
9	$\times$	$\times$	$\surd$	$\surd$
10	$\surd$	$\times$	$\surd$	$\times$
11	$\times$	$\times$	$\times$	$\times$
12	$\times$	$\times$	$\times$	$\times$
13	$\times$	$\times$	$\times$	$\surd$
14	$\times$	$\times$	$\times$	$\surd$
15	$\times$	$\times$	$\times$	$\times$
16	$\times$	$\times$	$\times$	$\times$
17	$\times$	$\times$	$\times$	$\times$
18	$\times$	$\times$	$\surd$	$\surd$
19	$\times$	$\times$	$\surd$	$\surd$

Table 1

Proof. As it has been shown in Section 3, only matrices A_1, A_2, A_3 and A_4 can be parameter matrices. With consideration of cubic graphs eigenvalues and using Theorem 2.1, it can be seen that the connected cubic graphs with 10 vertices can have perfect 3-coloring with matrices A_1, A_2, A_3 and A_4 which is represented by Table 2.

graphs	matrix A_1	matrix A_2	matrixA_3	matrix A_4
1	$\surd$	$\surd$	$\surd$	$\surd$
2	$\surd$	$\surd$	$\surd$	$\surd$
4	$\surd$	$\surd$	$\times$	$\times$
5	$\surd$	$\surd$	$\surd$	$\surd$
6	$\surd$	$\surd$	$\surd$	$\surd$
9	$\surd$	$\surd$	$\surd$	$\surd$
10	$\surd$	$\surd$	$\surd$	$\surd$
13	$\times$	$\times$	$\surd$	$\surd$
14	$\times$	$\times$	$\surd$	$\surd$
18	$\surd$	$\surd$	$\surd$	$\surd$
19	$\times$	$\times$	$\surd$	$\surd$

Table 2

The vertices of cubic graphs are labeled clockwise with $a_1, a_2, ..., a_{10}$, respectively. The graph 1 has perfect 3-colorings with the matrices A_1 and A_3. Consider two mappings T_1 and T_2 as follows:

$$T_1(a_1) = T_1(a_{10}) = 1, T_1(a_4) = T_1(a_7) = 2,$$
$$T_1(a_2) = T_1(a_3) = T_1(a_5) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3.$$
$$T_2(a_5) = T_2(a_6) = 1, T_2(a_2) = T_2(a_3) = T_2(a_8) = T_2(a_9) = 2,$$
$$T_2(a_1) = T_2(a_4) = T_2(a_7) = T_2(a_3) = 3.$$

It is clear that T_1 and T_2 are perfect 3-coloring with the matrices A_1 and A_3, respectively.

The graph 2 has perfect 3-colorings with the matrices A_1, A_3 and A_4. Consider three mappings T_1, T_2 and T_3 as follows:

$$T_1(a_2) = T_1(a_7) = 1, T_1(a_5) = T_1(a_{10}) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3.$$
$$T_2(a_1) = T_2(a_6) = 1, T_2(a_3) = T_2(a_4) = T_2(a_8) = T_2(a_9) = 2,$$
$$T_2(a_2) = T_2(a_5) = T_2(a_7) = T_2(a_{10}) = 3.$$
$$T_3(a_1) = 1, T_3(a_2) = T_3(a_6) = T_3(a_{10}) = 2,$$
$$T_3(a_3) = T_3(a_4) = T_3(a_5) = T_3(a_7) = T_3(a_8) = T_3(a_9) = 3.$$

It is clear that T_1, T_2 and T_3 are perfect 3-coloring with the matrices A_1, A_3 and A_4, respectively.

The graph 4 has perfect 3-colorings with the matrix A_1. Consider the mapping T_1 as follows:

$$T_1(a_5) = T_1(a_{10}) = 1, T_1(a_2) = T_1(a_7) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3.$$

It is clear that T_1 is a perfect 3-coloring with the matrix A_1.

The graph 6 has perfect 3-colorings with the matrices A_1 and A_3. Consider two mappings T_1 and T_2 as follows:

$$T_1(a_5) = T_1(a_9) = 1, T_1(a_7) = T_1(a_2) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_{10}) = 3.$$
$$T_2(a_3) = T_2(a_4) = 1, T_2(a_1) = T_2(a_6) = 2 = T_2(a_8) = T_2(a_{10}) = 2,$$
$$T_2(a_2) = T_2(a_5) = T_2(a_7) = T_2(a_9) = 3.$$

It is clear that T_1 and T_2 are perfect 3-coloring with the matrices A_1 and A_3, respectively.

The graph 9 has perfect 3-colorings with the matrices A_3 and A_4. Consider two mappings T_1 and T_2 as follows:

$$T_1(a_1) = T_1(a_6) = 1, T_1(a_3) = T_1(a_4) = T_1(a_8) = T_1(a_9) = 2,$$
$$T_1(a_2) = T_1(a_5) = T_1(a_7) = T_1(a_{10}) = 3.$$
$$T_2(a_1) = 1, T_2(a_2) = T_2(a_6) = 2 = T_2(a_{10}) = 2,$$
$$T_2(a_3) = T_2(a_4) = T_2(a_5) = T_2(a_7) = T_2(a_8) = T_2(a_9) = 3.$$

It is clear that T_1 and T_2 are perfect 3-coloring with the matrices A_3 and A_4, respectively.

The graph 10 has perfect 3-colorings with the matrices A_1 and A_3. Consider two mappings T_1 and T_2 as follows:

$$T_1(a_2) = T_1(a_5) = 1, T_1(a_7) = T_1(a_{10}) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3.$$
$$T_2(a_1) = T_2(a_6) = 1, T_2(a_3) = T_4(a_6) = 2 = T_2(a_8) = T_2(a_9) = 2,$$
$$T_2(a_2) = T_2(a_5) = T_2(a_7) = T_2(a_{10}) = 3.$$

It is clear that T_1 and T_2 are perfect 3-coloring with the matrices A_1 and A_3, respectively.

The graph 13 has perfect 3-colorings with the matrix A_4. Consider a mapping T_1 as follows:

$$T_1(a_6) = 1, T_1(a_1) = T_1(a_5) = T_1(a_7) = 2,$$
$$T_1(a_2) = T_1(a_3) = T_1(a_4) = T_1(a_7) = T_1(a_8) = T_1(a_9) = 3.$$

It is clear that T_1 is a perfect 3-coloring with the matrix A_4.

The graph 14 has perfect 3-colorings with the matrix A_4. Consider a mapping T_1 as follows:

$$T_1(a_6) = 1, T_1(a_1) = T_1(a_5) = T_1(a_7) = 2,$$
$$T_1(a_2) = T_1(a_3) = T_1(a_4) = T_1(a_7) = T_1(a_8) = T_1(a_9) = 3.$$

It is clear that T_1 is a perfect 3-coloring with the matrix A_4.

The graph 18 has perfect 3-colorings with the matrices A_3 and A_4. Consider two mappings T_1 and T_2 as follows:

$$T_1(a_1) = T_1(a_6) = 1, T_1(a_3) = T_1(a_4) = T_1(a_8) = T_1(a_9) = 2,$$
$$T_1(a_2) = T_1(a_5) = T_1(a_7) = T_1(a_{10}) = 3.$$
$$T_2(a_1) = 1, T_2(a_2) = T_2(a_6) = 2 = T_2(a_{10}) = 2,$$
$$T_2(a_3) = T_2(a_4) = T_2(a_5) = T_2(a_7) = T_2(a_8) = T_2(a_9) = 3.$$

It is clear that T_1 and T_2 are perfect 3-coloring with the matrices A_3 and A_4, respectively.

The graph 19 has perfect 3-colorings with the matrices A_3 and A_4. Consider two mappings T_1 and T_2 as follows:

$$T_1(a_2) = T_1(a_9) = 1, T_1(a_1) = T_1(a_4) = T_1(a_6) = T_1(a_8) = 2,$$
$$T_1(a_3) = T_1(a_5) = T_1(a_7) = T_1(a_9) = 3.$$
$$T_2(a_1) = 1, T_2(a_3) = T_2(a_6) = 2 = T_2(a_9) = 2,$$
$$T_2(a_2) = T_2(a_4) = T_2(a_5) = T_2(a_7) = T_2(a_8) = T_2(a_{10}) = 3.$$

It is clear that T_1 and T_2 are perfect 3-coloring with the matrices A_3 and A_4, respectively. There are no perfect 3-colorings with the matrices A_2 and A_4 for graph 1.

Contrary to our claim, suppose that T is a perfect 3-coloring with the matrix A_2 for graph 1. According to the matrix A_2, each vertex with white color has a neighbor with white color, so the two vertices with white color are adjacent. In the case that $a_1 \leftrightarrow a_2$, $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$, $a_3 \leftrightarrow a_4$ by symmetry $a_7 \leftrightarrow a_8$, $a_7 \leftrightarrow a_9$, $a_8 \leftrightarrow a_{10}$ and $a_9 \leftrightarrow a_{10}$, they have less than four adjacent vertices. These vertices are red color, which is a contradiction. So $a_5 \leftrightarrow a_6$, $a_4 \leftrightarrow a_5$ and its symmetric $a_6 \leftrightarrow a_7$ will be remain that are white color. In the case that $a_4 \leftrightarrow a_5$, the neighbors of a_4 and a_5 are red color and vertex a_1 that is their neighbor's is also red color has two neighbors with red color which it is not possible. If a_5 and a_6 are white color, adjacent vertices are red color and other vertices are black color, so each black color is adjacent to another black color vertex, which is a contradiction. So we conclude the graph 1 has no perfect 3-coloring with matrix A_2.

Contrary to our claim, suppose that T is a perfect 3-coloring with the matrix A_4 for graph 1. According to the matrix A_4, each vertex with white color has three adjacent with black color. If a_1 is white color, then a_2, a_3, a_5 are black color, which is a contradiction with the second row of matrix A_4. If a_2 is white color, then according to the matrix A_4, the vertices a_1, a_3, a_4 are black color, which is a contradiction with the second row of matrix A_4. If a_3 is white color, then according to the matrix A_4, the vertices a_1, a_2, a_4 are black color, which is a contradiction with the second row of matrix A_4. If a_4 is white color, then according to the matrix A_4, the vertices a_2, a_3, a_5 are black color, which is a contradiction with the second row of matrix A_4. If a_5 is white color, then a_3 is a vertex that is black color and has three red color neighbors, which is a counteraction with the second row of matrix A_4. According to the symmetric, the vertices $a_6, a_7, a_8, a_9, a_{10}$ can not be white color. Therefore the graph 1 has no perfect 3-coloring with matrix A_4.

As it is stated in the before paragraphs, the graph 1 has no perfect 3-coloring with matrices A_2 and A_4.

About other graphs in Figure 1, similarly, we can get the same result as in Table 1. □

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