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Domination number of the non-commuting graph of finite groups

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Abstract

Let G be a non-abelian group. The *non-commuting graph* of group G, shown by Γ_G, is a graph with the vertex set $G \setminus Z(G)$, where $Z(G)$ is the center of group G. Also two distinct vertices of a and b are adjacent whenever $ab \neq ba$. A set $S \subseteq V(\Gamma)$ of vertices in a graph Γ is a *dominating set* if every vertex $v \in V(\Gamma)$ is an element of S or adjacent to an element of S. The *domination number* of a graph Γ denoted by $\gamma(\Gamma)$, is the minimum size of a dominating set of Γ. Here, we study some properties of the non-commuting graph of some finite groups. In this paper, we show that $\gamma(\Gamma_G) < \frac{|G|-|Z(G)|}{2}$. Also we charactrize all of groups G of order n with $t = |Z(G)|$, in which $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) \in \{n-t+1, n-t, n-t-1, n-t-2\}$.

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1. Introduction

Let G be a non-abelian group and $Z(G)$ be the center of G. Associate a graph Γ_G with G as follows: Take $G \setminus Z(G)$ as the vertices of Γ_G. Two vertices a and b are adjacent if $ab \neq ba$. This graph is called the *non-commuting graph of* G. Let Γ be a simple graph. A subset $S \subseteq V(\Gamma)$ is called a *dominating set* if each vertex $v \in V(\Gamma) \setminus S$ has at least one neighbor in S. The size of a smallest dominating set of Γ is called *domination number* of Γ and is denoted by $\gamma(\Gamma)$.

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Erdös considered the non-commuting graph in 1975 for the first time. In 2004, Abdollahi, Akbari and Maimani studied some properties of the non-commuting graph of a group. For more results, see [1],[2], [6], [5] and [12].

Before starting, let us introduce some necessary notation and definitions. For every graph Γ, we denote the set of the vertices of Γ by $V(\Gamma)$. The minimum degree of a graph Γ denoted by $\delta(\Gamma)$. The *complete graph*, *path* and *cycle* on n vertices are denoted by K_n, P_n and C_n, respectively. The open neighborhood of a vertex v in Γ is the set $N_\Gamma(v)$ of vertices joined to v by an edge. The closed neighborhood of v is the set $N_\Gamma[v] = N_\Gamma(v) \cup \{v\}$. The complement of Γ denoted by $\overline{\Gamma}$. If u and v are vertices in Γ, then $d(u, v)$ denotes the length of the shortest path between u and v. A graph Γ is *connected* if there is a path between each pair of the vertices of Γ. The maximum value of $d(u, v)$ between all pair of the vertices of connected graph Γ is called the *diameter* of Γ and denoted by $diam(\Gamma)$. Let Γ_1 and Γ_2 be two graphs.The *corona* $\Gamma = \Gamma_1 o \Gamma_2$ is the graph formed from one copy of Γ_1 and $|V(\Gamma_1)|$ copies of Γ_2 such that the ith vertex of Γ_1 is adjacent to every vertex in the ith copy of Γ_2.

For each $x \in G$, $C_G(x) = \{g \in G \mid gx = xg\}$. We denote the *symmetric group* and the *alternating group* on n letters by S_n and A_n, respectively. Also $Q_8 = \langle A\,,\, B \mid A^4 = 1\,, A^2 = B^2, B^{-1}AB = A^{-1}\rangle$ and $D_{2n} = \langle a\,,\, b \mid a^n = b^2 = (ab)^2 = 1\rangle$ are the *quaternion group* with 8 elements and the *dihedral group* of order $2n$, respectively.

In this paper, we study the domination number of the non-commuting graphs. In particular, we charactrize all groups G of order n with $|Z(G)| = t$, in which $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) \in \{n - t + 1, n - t, n - t - 1, n - t - 2\}$.

2. Preliminaries

In this section, we provide some useful results which will be applied in the next section.

Theorem 2.1. *(Ore) [10] Let Γ be a graph with no isolated vertices. Then $\gamma(\Gamma) \leqslant \frac{n}{2}$.*

Theorem 2.2. *[7] For a graph Γ with even order n and no isolated vertices, $\gamma(\Gamma) = \frac{n}{2}$ if and only if the components of Γ are the cycle C_4 or the Corona HoK_1 for any connected graph H.*

Theorem 2.3. *[1], 2.1. For any non-abelian group G, diam(Γ_G)=2. In particular, Γ_G is connected. Also the girth of Γ_G equal 3.*

Theorem 2.4. *[11] Let Γ be a graph of order n. Then the following holds.*

i) $\gamma(\Gamma) + \gamma(\overline{\Gamma}) \leqslant n + 1$.

ii) $\gamma(\Gamma)\gamma(\overline{\Gamma}) \leqslant n$.

Lemma 2.1. *Let G be a finite non-abelian group. Then $\delta(\Gamma_G) \geqslant 3$.*

Proof. Suppose that $deg_{\Gamma_G}(v) \leqslant 2$, for some $v \in V(\Gamma_G)$. Since $deg_{\Gamma_G}(v) \geqslant \frac{|G|}{2}$, then $|G| \leqslant 4$. So, G is an abelian group, which is a contradiction. □

Remark 2.1. Here, we get figures of the non-commuting graph of some groups. These figures are useful in proving some theorems in the third section. (See Figures 1, 2, 3 and 4.)

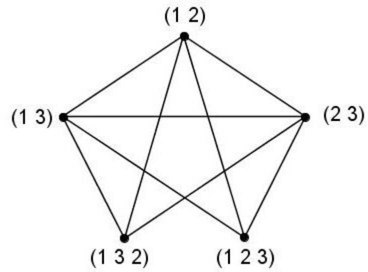

Figure 1. The non-commuting graph of S_3.

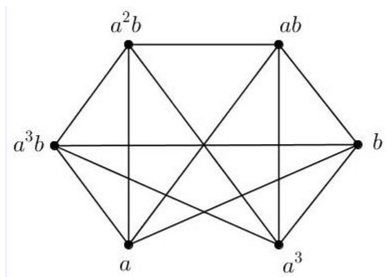

Figure 2. The non-commuting graph of D_8.

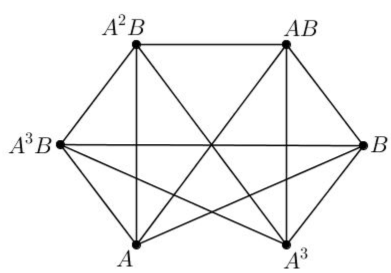

Figure 3. The non-commuting graph of Q_8.

Lemma 2.2. *Let G be a non-abelian group of odd order. Then, the graph $\overline{\Gamma}_G$ contains no isolated vertex.*

Proof. Assume to the contrary, a is an isolated vertex of $\overline{\Gamma}_G$. Then, for each $x \in G \setminus Z(G)$ we have $ax \neq xa$. Hence $o(a) = 2$, which is a contradiction. □

Lemma 2.3. *Let G be a non-abelian group and $|Z(G)| = 1$. Then, the vertices of degree one in $\overline{\Gamma}_G$ occure only at the edges. Furthermore, if $deg_{\overline{\Gamma}_G} a = 1$ then $o(a) = 3$.*

Proof. Let $a \in V(\overline{\Gamma}_G)$, $deg_{\overline{\Gamma}_G} a = 1$ and a be adjacent to b. Then, $C_G(a) = \{1, a, b\}$ and so $b = a^{-1}$. Hence, $C_G(b) = \{1, b, a\}$. Thus, $deg_{\overline{\Gamma}_G} b = 1$. Furthermore, $o(a) = o(b) = 3$. □

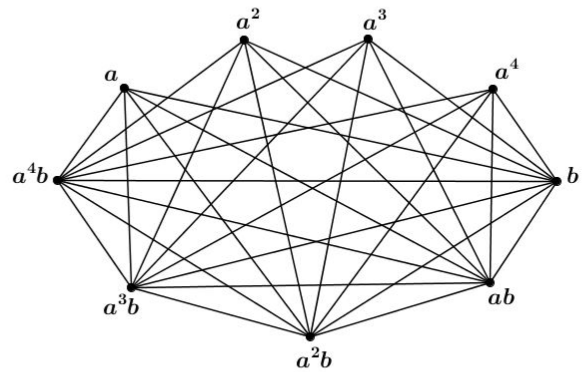

Figure 4. The non-commuting graph of D_{10}.

Lemma 2.4. *Let G be a non-abelian group. Then $\overline{\Gamma}_G$ does not have $C_n, (n > 3)$ as component.*

Proof. Let C_n be a component of $\overline{\Gamma}_G$ and $\{a, b\} \subset V(C_n)$ such that $b \in N_{\overline{\Gamma}_G}(a)$. Then $N_{\overline{\Gamma}_G}[a] = \{a, b, f\}$ and $N_{\overline{\Gamma}_G}[b] = \{a, b, c\}$. So $C_G(a) = \{a, b, f\} \cup Z(G)$ and $C_G(b) = \{a, b, c\} \cup Z(G)$. Since $|C_G(a) \cap C_G(b)|$ divides $|C_G(a)|$, then $2 + |Z(G)|$ divides $3 + |Z(G)|$, that is a contradiction. □

Definition 2.1. *Let G be a group. Write S and T for the set of elements of G of order two and three, respectively. Then G is a acceptable when neither S nor T is empty and $G = S^* \cup T$, where $S^* = S \cup \{e\}$.*

Theorem 2.5. *[3] If G is acceptable, then either $S^* \leq G$ or $T^* \leq G$.*

3. Main results

In this section, we prove our main results.

Theorem 3.1. *Let G be a non-abelian group. Then, the following holds.*

i) $\delta(\Gamma_G) = 3$ *if and only if G is isomorphic to S_3.*

ii) $\delta(\Gamma_G) = 4$ *if and only if G is isomorphic to D_8 or Q_8.*

iii) $\delta(\Gamma_G) = 5$ *if and only if G is isomorphic to D_{10}.*

Proof. We prove as follows:

i) Let $\delta(\Gamma_G) = 3$ and $deg_{\Gamma_G}(v) = 3$, for some $v \in V(\Gamma_G)$. Since $deg_{\Gamma_G}(v) \geqslant \frac{|G|}{2}$, then $|G| \leqslant 6$. The only non-abelian group of order less than 7 is S_3.
Conversely, Suppose that G is isomorphic to S_3. By considering the figure of the non-commuting graph associated to symmetric group S_3 (See Figure 1), we obtain $\delta(\Gamma_G) = 3$.

ii) Let $\delta(\Gamma_G) = 4$. Then $|G| \leqslant 8$ and so $G \in \{S_3, D_8, Q_8\}$. By (i), $G \ncong S_3$. So $G \cong D_8$ or Q_8. Conversely, Suppose that G is isomorphic to D_8 or Q_8. By considering the figure of the non-commuting graph associated to D_8 and Q_8 (See Figures 2 and 3), we obtain $\delta(\Gamma_G) = 4$.

iii) Let $\delta(\Gamma_G) = 5$. Then $|G| \leqslant 10$. By (i) and (ii), $|G| = 10$. Since G is not an abelian group, then G is isomorphic to D_{10}. Conversely, Suppose that G is isomorphic to D_{10}. By considering the figure of the non-commuting graph associated to the dihedral group D_{10} (See Figure 4), we obtain $\delta(\Gamma_G) = 5$.

□

Corollary 3.1. *All of* 3*-regular and* 5*-regular graphs cannot be non-commuting graphs.*

Theorem 3.2. *Let* Γ *be a* $(n-2)$*-regular graph of order* n*. Then* Γ *is the non-commuting graph associated to a non-abelian group* G *if and only if* $n = 6$ *and* G *is isomorphic to* D_8 *or* Q_8*.*

Proof. Let Γ be a $(n-2)$-regular graph of order n and G be a group such that $\Gamma_G = \Gamma$. Then n is even and $\overline{\Gamma}_G$ is a disjoint union of $\frac{n}{2}$ edges. If a and b are adjacent in $\overline{\Gamma}_G$, then $C_G(a) = Z(G) \cup \{a, b\}$. It is clear that $|Z(G)| \leqslant 2$.

If $|Z(G)| = 1$, then $C_G(a) = \{1, a, a^{-1} = b\}$ and $o(a) = 3$. Since $\overline{\Gamma}_G$ is a disjoint union of some edges, then for each $x \in G$ we have $o(x) = 3$. Thus, there is a positive integer s such that $|G| = 3^s$. Hence $|Z(G)| > 1$, which is a contradiction.

If $|Z(G)| = 2$ and $Z(G) = \{1, x\}$, then for each $a \in G \setminus Z(G)$, $C_G(a) = \{1, x, a, b\}$ and so we have $a^2 = 1$ or $a^2 = x$. Therefore for each $a \in G \setminus Z(G)$ we have $a^2 \in Z(G)$. Hence $\frac{G}{Z(G)}$ is an elementary abelian 2-group. So $G' \leq Z(G)$, which implies $|G'| = 1$ or 2. Since G is not an abelian group, then $G' \neq \{1\}$. Thus $G' = Z(G)$. Also we have $cl(a) = \{g^{-1}ag : g \in G\} \subseteq aG'$ and so $|cl(a)| \leqslant 2$. Since $|cl(a)| = \frac{|G|}{|C_G(a)|}$, then $|G| \leqslant 8$. Hence $G \cong S_3$, D_8 or Q_8. Since $Z(S_3) = 1$, then $G \cong D_8$ or $G \cong Q_8$.

Conversely, suppose that $G \cong D_8$ or $G \cong Q_8$. Then by considering the figures of the non-commuting graphs of these two groups (See Figures 2 and 3), we obtain Γ_G is a 4-regular graph of order 6. □

Theorem 3.3. *Every* $(n-3)$*-regular graph of order* n *is not the non-commuting graph.*

Proof. Let Γ be a graph of order n and $(n-3)$-regular. Also, suppose that G is a group and $\Gamma_G = \Gamma$. Thus, $\overline{\Gamma}_G$ is a 2-regular graph. It means that $\overline{\Gamma}_G$ is a disjoint union of cycles.

By Lemma 2.4, $\overline{\Gamma}_G$ is a disjoint union of triangles. Hence for every $a \in G \setminus Z(G)$, $C_G(a) = Z(G) \cup \{a, b, c\}$ such that $bc = cb$. Thus, $|Z(G)| \leqslant 3$.

Case 1. If $|Z(G)| = 1$, then $|C_G(a)| = 4$. So there is a positive integer s such that $|G| = 2^s$. Thus, $|Z(G)| > 1$, which is a contradiction.

Case 2. If $|Z(G)| = 2$, then $|C_G(a)| = 5$. Since $|Z(G)|$ divides $|C_G(a)|$, then $2 \mid 5$, which is a contradiction.

Case 3. If $|Z(G)| = 3$, then $|C_G(a)| = 6$. Thus, $C_G(a) \cong Z_6$ and so for each $x \notin Z(G)$, $o(x) \in \{2, 6\}$. So for each $a \in G \setminus Z(G)$, $a^2 \in Z(G)$ which implies $G/Z(G)$ is an elementary abelian 2-group. Therefore, $G' \leq Z(G)$. Since G is not an abelian group, then $G' = Z(G)$. We know that $cl(a) \subseteq aG'$. So $|cl(a)| \leqslant 3$ Hence $6 < |G| \leqslant 18$. Since $|Z(G)| = 3$ and $|Z(G)|$ divides $|G|$, then $|G| \in \{9, 12, 15, 18\}$. Furthermore, since G is not an abelian group, then we have $|G| \in \{12, 18\}$.

If $|G| = 12$, then $G \cong A_4, D_{12}$ or $\langle a\,,\, b \ \mid a^6 = 1\ , a^3 = b^2\ , b^{-1}ab = a^{-1}\rangle$. Since G has only 2 elements of order three, then $G \ncong A_4$. Also since $|Z(D_{12})| = 2$, then $G \ncong D_{12}$. Hence $G \cong \langle a\,,\, b \ \mid a^6 = 1\ , a^3 = b^2\ , b^{-1}ab = a^{-1}\rangle$. So G has an element of order 4, which is a contradiction.

If $|G| = 18$, then $\overline{\Gamma}_G \cong 5K_3$. So G has exactly 5 elements of order 2. Let n_p be the number of Sylow $p-$subgroup of G. By Sylow Theorem, $n_2(G) \in \{1, 3, 9\}$. Hence G has 1, 3 or 9 elements of order 2, which is a contradiction. □

In [10], Ore proved that if Γ is a graph with no isolated vertices, then $\gamma(\Gamma) \leq \frac{n}{2}$. In Theorems 3.4 and 3.5, we show that if G is a non-abelian group with $|Z(G)| = t$, then $\gamma(\Gamma_G)$ and $\gamma(\overline{\Gamma}_G)$ are less than $\frac{n-t}{2}$.

Theorem 3.4. *Let G be a non-abelian group of order n and $|Z(G)| = t$. Then $\gamma(\Gamma_G) < \frac{n-t}{2}$.*

Proof. We know $|V(\Gamma_G)| = n-t$. Since Γ_G is a connected graph, it contains no isolated vertex. By Theorem 2.1, $\gamma(\Gamma_G) \leqslant \frac{n-t}{2}$. Now, we show $\gamma(\Gamma_G) < \frac{n-t}{2}$. Assume to the contrary, $\gamma(\Gamma_G) = \frac{n-t}{2}$. By Theorem 2.2, each component of the graph Γ_G is the cycle C_4 or the Corona product K_1 and a connected graph H, that is HoK_1. By Lemma 2.1, is a contradiction. Hence $\gamma(\Gamma_G) < \frac{n-t}{2}$. □

Theorem 3.5. *Let G be a non-abelian group of odd order n and $|Z(G)| = t$. Then $\gamma(\overline{\Gamma}_G) < \frac{n-t}{2}$.*

Proof. Since n is odd, then t is odd. By Lemma 2.2, the graph $\overline{\Gamma}_G$ contains no isolated vertex. By Theorem 2.1, $\gamma(\overline{\Gamma}_G) \leqslant \frac{n-t}{2}$. Now, suppose that $\gamma(\overline{\Gamma}_G) = \frac{n-t}{2}$. Then, by Theorem 2.2, $\overline{\Gamma}_G$ has connected components of kind of C_4 or HoK_1. By Lemma 2.4, all components of $\overline{\Gamma}_G$ are the corona product K_1 and a connected graph H. Let a be a vertex of degree 1 in $\overline{\Gamma}_G$ and b is adjacent to a. Then $C_G(a) = \{a, b\} \cup Z(G)$ and so $|Z(G)| \leqslant 2$. Since $|G|$ is odd, then $|Z(G)| = 1$. By Lemma 2.3, $|G| = 3^s$. Hence $|Z(G)| > 1$, which is a contradiction. □

In the following theorem, we characterize all groups G of order n with $|Z(G)| = t$, in which $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) \in \{n-t+1, n-t, n-t-1, n-t-2\}$.

Theorem 3.6. *Let G be a non-abelian group of order n and $|Z(G)| = t$. Then the following holds.*

i) $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) < n - t + 1$.

ii) $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t$ *if and only if* $G \cong S_3$.

iii) $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t - 1$ *if and only if* $G \cong D_8$ *or* $G \cong Q_8$.

iv) $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t - 2$ *if and only if* $G \cong D_{10}$.

Proof. Since $|G| = n$ and $|Z(G)| = t$, then $|V(\Gamma_G)| = n - t$. By Theorem 3.4, $\gamma(\Gamma_G) < \frac{n-t}{2}$.

i) By Theorem 2.4, $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) \leqslant n - t + 1$. Let $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t + 1$. Then $\gamma(\overline{\Gamma}_G) > n - t + 1 - \frac{n-t}{2} = \frac{n-t+2}{2}$. By Theorem 2.1, $\overline{\Gamma}_G$ contains at least one isolated vertex. So $t = 1$ and $\gamma(\Gamma_G) = 1$. Thus $\gamma(\overline{\Gamma}_G) = n - 1$. Therefore $\Gamma_G = K_{n-1}$. By Theorem 2.3, it is impossible. Hence $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) < n - t + 1$.

ii) Let $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t$. Then $\gamma(\overline{\Gamma}_G) > \frac{n-t}{2}$. By Theorem 2.1, $\overline{\Gamma}_G$ contains at least one isolated vertex. So $t = 1$, $\gamma(\Gamma_G) = 1$ and $\gamma(\overline{\Gamma}_G) = n - 2$. Thus, $\overline{\Gamma}_G$ is a disjoint union of P_2 and isolated vertices $\{x_1, x_2, \dots, x_{n-3}\}$. All isolated vertices x_i are of order 2. Let a and b be the vertices of the path P_2. By Lemma 2.3, $b = a^{-1}$ and $o(a) = o(b) = 3$. We claim that if x_1 and x_2 are two elements of order 2, then $o(x_1x_2) = 3$. To see this, suppose $o(x_1x_2) = 2$. Then $x_1x_2x_1x_2 = 1$ and so $x_1x_2 = x_2x_1$. Thus x_1 is adjacent to x_2 in $\overline{\Gamma}_G$, which is a contradiction. It is easy to see that $o(x_ib) = 2$, for $i = 1, 2, \dots, n-3$.
If $T = \langle x_1, b \mid x_1^2 = b^3 = 1, (x_1b)^2 = 1\rangle$, then $T \cong S_3$. We prove $G \cong T$. If $x_2 \in G \setminus T$ and $x_1x_2 = b$, then $x_2 = x_1b$. Since $x_1b \in T$, $x_2 \in T$. Also if $x_1x_2 = b^{-1}$, then $x_2 = x_1b^{-1} \in T$. However it is a contradiction. Hence $G \cong T \cong S_3$. Conversely, if $G \cong S_3$, then by Figure 1, $\gamma(\Gamma_G) = 1$ and $\gamma(\overline{\Gamma}_G) = 4$ and the proof is complete.

iii) Let $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = n - t - 1$. Then $\gamma(\overline{\Gamma}_G) > \frac{n-t-2}{2}$. We consider two cases:

Case 1. Let $\gamma(\overline{\Gamma}_G) = \frac{n-t}{2}$. Then $\gamma(\Gamma_G) = \frac{n-t}{2} - 1$. If $\overline{\Gamma}_G$ has an isolated vertex, then $t = 1$ and so n is odd. By Theorem 3.5, $\gamma(\overline{\Gamma}_G) < \frac{n-1}{2}$, which is false. If $\overline{\Gamma}_G$ does not have an isolated vertex, then by Theorem 2.2 and Lemma 2.4, $\overline{\Gamma}_G$ has a vertex of degree one. So $t = 1$ or $t = 2$. If $t = 1$, then n is odd. By Theorem 3.5, $\gamma(\overline{\Gamma}_G) < \frac{n-1}{2}$, which is false.
If $t = 2$, then $\gamma(\overline{\Gamma}_G) = \frac{n-2}{2}$ and $\gamma(\Gamma_G) = \frac{n-4}{2}$. By Theorem 2.4, $\frac{n-2}{2}.\frac{n-4}{2} \leqslant n - 2$. Since $n \neq 2$, then $n \leqslant 8$. So G is isomorphic to S_3, D_8 or Q_8. But $Z(S_3) = 1$. Hence G is isomorphic to D_8 or Q_8.

Case 2. If $\gamma(\overline{\Gamma}_G) > \frac{n-t}{2}$, then $\overline{\Gamma}_G$ contains at least one isolated vertex. So $t = 1$ and so $\gamma(\Gamma_G) = 1$. Therefore $\gamma(\overline{\Gamma}_G) = n - 3$. By Lemmas 2.3, 2.4 and $\gamma(\overline{\Gamma}_G) = n - 3$, we have the following subcases.

Subcase 1. Let $\overline{\Gamma}_G$ be a union of the isolated vertices $\{x_1, x_2, \dots, x_{n-4}\}$ and K_3 with vertices a, b, c. Then $C_G(a) = C_G(b) = C_G(c) = \{1, a, b, c\}$. So orders of a, b and c are 2 or 4. If $o(a) = o(b) = o(c) = 2$, then order of each element of G is 2 and so G is an abelian group, which is a contradiction. If $o(a) = 2$ and $o(b) = 4$, then $a = b^2$ and for each i $(i = 1, 2, ..., n-4)$, $x_ibx_i = b$ or $x_ibx_i = b^{-1}$. If $x_ibx_i = b$, then $bx_i = x_ib$. So b is adjacent to x_i, which is a contradiction. If $x_ibx_i = b^{-1}$, then $\langle b, x_i\rangle \cong D_8$. We claim that $G \cong \langle b, x_1\rangle$. Suppose that $x \in G \setminus \langle b, x_1\rangle$. Since $xbx = b^{-1}$ and $x_1bx_1 = b^{-1}$, then $xbx = x_1bx_1$ and so $x_1x \in C_G(b)$. We know that $C_G(b) = \langle b\rangle$. So $x \in \{x_1b, x_1b^2, x_1b^3\}$, Hence $x \in \langle b, x_1\rangle$. Therefore $G \cong D_8$. Since in this case $Z(G) = \{1\}$, then we have $Z(D_8) = \{1\}$, which is a contradiction.

Subcase 2. Let $\overline{\Gamma}_G$ be a union of the isolated vertices $\{x_1, x_2, \dots, x_{n-5}\}$ and two edges with vertices $a \sim b$ and $c \sim d$. By Lemma 2.3, $o(a) = o(b) = o(c) = o(d) = 3$. Since $\{1, a, b, c, d\}$ is not a subgroup of G, by Theorem 2.5, $\{1, x_1, x_2, \dots, x_{n-5}\}$ is a subgroup of G. So $o(x_1x_2) = 2$. Hence $x_1x_2 = x_2x_1$ and so x_1 is adjacent to x_2 in $\overline{\Gamma}_G$, which is a contradiction. Conversely, by using the figures 2 and 3 we can obtain the proof.

iv) Let $\gamma(\Gamma_G)+\gamma(\overline{\Gamma}_G) = n-t-2$. By Theorem 3.4, $\gamma(\Gamma_G) < \frac{n-t}{2}$. We have the following cases.

Case 1. If $\gamma(\Gamma_G) = \frac{n-t-2}{2}$, then $\gamma(\overline{\Gamma}_G) = \frac{n-t-2}{2}$. By Theorem 2.4, $\gamma(\Gamma_G).\gamma(\overline{\Gamma}_G) \leqslant n-t$. Hence, $1 \leqslant n-t \leqslant 7$. Since $n-t$ is even and $t \mid n$, then $n-t \in \{4, 6\}$. If $n-t = 4$, then $(n, t) = (6, 2)$ and so $G \cong S_3$. Which is a contradiction to the fact that $Z(S_3) = 1$. If $n-t = 6$, then $(n, t) \in \{(10, 4), (8, 2)\}$. Since $t \mid n$, we have $(n, t) = (8, 2)$. So $G \cong D_8$ or $G \cong Q_8$. By Figures 2 and 3, we have a contradiction.

Case 2. Let $\gamma(\Gamma_G) = \frac{n-t-4}{2}$ and $\gamma(\overline{\Gamma}_G) = \frac{n-t}{2}$. If $\overline{\Gamma}_G$ has an isolated vertex, then $t = 1$ and so n is odd. By Theorem 3.5, $\gamma(\overline{\Gamma}_G) < \frac{n-1}{2}$, which is false.
If $\overline{\Gamma}_G$ does not have an isolated vertex, then by Theorem 2.2 and Lemma 2.4, $\overline{\Gamma}_G$ has a vertex of degree 1. So $t = 1$ or $t = 2$. If $t = 1$, then n is odd, which is false. If $t = 2$, then $\gamma(\overline{\Gamma}_G) = \frac{n-2}{2}$ and $\gamma(\Gamma_G) = \frac{n-6}{2}$. Thus $n \leqslant 10$. By Figures 2, 3 and 4, we have a contradiction.

Case 3. Let $\gamma(\Gamma_G) < \frac{n-t-4}{2}$. Then $\gamma(\overline{\Gamma}_G) > \frac{n-t}{2}$ and so $\overline{\Gamma}_G$ has at least one isolated vertex. Thus $t = 1$, $\gamma(\Gamma_G) = 1$ and $\gamma(\overline{\Gamma}_G) = n-4$. Let $u \in V(\overline{\Gamma}_G)$. If $deg_{\overline{\Gamma}_G}(u) > 3$, then $\gamma(\overline{\Gamma}_G) < n-4$, which is not true. Hence for every $u \in V(\overline{\Gamma}_G)$, $deg_{\overline{\Gamma}_G}(u) \leqslant 3$ and $o(u) \leqslant 5$. We have the following subcases.

Subcase 1. Let $deg_{\overline{\Gamma}_G}(u) \leqslant 1$, where $u \in V(\overline{\Gamma}_G)$. Then $\overline{\Gamma}_G$ is isomorphic to union of 3 copies of P_2 and $n-7$ isolated vertices. It is clear that isolated vertices are of order 2. By Lemma 2.3, G is an abelian acceptable group. By Theorem 2.5, $T \cup \{1\} \leq G$ or $S \cup \{1\} \leq G$. Since 3 does not divide $\mid T \cup \{1\} \mid$, then $T \cup \{1\}$ is not a subgroup of G. Hence $S^* = S \cup \{1\}$ is a subgroup of G. Since $|S^*| \leqslant \frac{|G|}{2}$, then $|G| \leqslant 12$. We know that there is no group of order less than 12 with exactly 6 elements of order 3, which implies that $|G| = 12$. Hence $G \cong A_4, D_{12}$ or $L = \langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$. Since G has exactly 6 elements of order 3, we have $G \notin \{A_4, D_{12}\}$. Also in L we have $o(b) = 4$. So $G \ncong L$.

Subcase 2. For each $u \in V(\overline{\Gamma}_G)$, $deg_{\overline{\Gamma}_G}(u) \leqslant 2$. Let $u, v \in V(\overline{\Gamma}_G)$ and $deg_{\overline{\Gamma}_G}(u) = deg_{\overline{\Gamma}_G}(v) = 2$. If u and v are not adjacent in $\overline{\Gamma}_G$, then by Lemmas 2.3 and 2.4, $\gamma(\overline{\Gamma}_G) < n-4$, which is not true. If u and v are adjacent in $\overline{\Gamma}_G$, then by Lemmas 2.3 and 2.4, u and v are vertices of a K_3. Since $\gamma(\overline{\Gamma}_G) = n-4$, then $\overline{\Gamma}_G$ is isomorphic to union of exactly one copy of K_3, one copy of P_2 and some isolated vertices. Suppose $V(K_3) = \{u_1, u_2, u_3\}$, $V(P_2) = \{v_1, v_2\}$ and isolated vertices are denoted by x_i, where $1 \leqslant i \leqslant n-6$. We have

$C_G(u_i) = \{1, u_1, u_2, u_3\}$. So $o(u_i) \in \{2, 4\}$. By Lemma 2.3, $o(v_1) = o(v_2) = 3$. Also we have $o(x_i) = 2$. So $|G| = 2^{\ell}3$, where ℓ is natural. By Sylow Theorem, there are two subgroup H and K of G such that $|H| = 2^{\ell}$ and $K = \{e, v_1, v_1^{-1} = v_2\}$.
If $h \in H$ and $o(h) = 2$, then $o(hv_1h) = o(v_1) = 3$. So $hv_1h = v_1$ or v_1^{-1}. If $hv_1h = v_1$, then $v_1h = hv_1$ and so $o(v_1h) = 6$, which is false. If $hv_1h = v_1^{-1}$, then $v_1h = hv_1^{-1}$ or $hv_1 = v_1^{-1}h$. It is well known that $|Z(H)| > 1$.

Let $z \in Z(H)$ and $o(z) = 2$. Then for every $h \in H \setminus \{z\}$, we have
$(zh)v_1 = (hz)v_1 = h(zv_1) = h(v_1^{-1}z) = (hv_1^{-1})z = (v_1h)z = v_1(hz) = v_1(zh)$ Hence $zh \in C_G(v_1)$. So $zh \in \{v_1, v_1^{-1}\}$.
If $|H| > 4$, then $|H| \geqslant 8$. Thus there are $h_1, h_2 \in H$, $h_1 \neq h_2$ such that $zh_1 = zh_2$. Hence $h_1 = h_2$, which is a contradiction. Thus $|H| = 4$ and so $|G| = 12$. Since G has exactly two elements of order 3, then $|cl(v_1)| = 1$ or 2. Since $|C_G(v_1)| = 3$ and $[G : C_G(v_1)] = |cl(v_1)|$, we have $4 = 1$ or $4 = 2$, which is not true.

Subcase 3. Let $u \in V(\overline{\Gamma}_G)$, $deg_{\overline{\Gamma}_G}(u) = 3$ and $N_{\overline{\Gamma}_G}(u) = \{x, y, z\}$. Then $C_G(u) = \{1, u, x, y, z\}$ and so $o(u) = 5$. Hence induced subgraph on $N_{\overline{\Gamma}_G}[u]$ is isomorphic to K_4. Since $\gamma(\overline{\Gamma}_G) = n - 4$, then $\overline{\Gamma}_G \cong K_4 \cup (n-5)K_1$. On the other hand if x is a isolated vertex in $\overline{\Gamma}_G$, then $o(x) = 2$. Since $xu \notin \{1, u, u^2, u^3, u^4\}$, we have $o(xu) = 2$. Thus $xux = u^{-1}$ and so $\langle x, u\rangle \cong D_{10}$. Now let $y \in G \setminus \langle x, u\rangle$ and y be an isolated vertex in $\overline{\Gamma}_G$. Then $o(yu) = 2$. Hence $yuy = xux$. This implies that $xy \in C_G(u) = \{1, u, u^2, u^3, u^4\}$. Therefore $y \in \{x, xu, xu^2, xu^3, xu^4\}$ and so $y \in \langle x, u\rangle$, which is a contradiction. Hence $G \cong \langle x, u\rangle \cong D_{10}$.
Conversely, if $G \cong D_{10}$, then $t = 1$ and by Figure 4, $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) = 7$ and the proof is complete.

□

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