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On the spectrum of a class of distance-transitive graphs

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Abstract

Let $\Gamma = Cay(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic additive group $\mathbb{Z}_n$ $(n \geq 4)$, where $S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq [\frac{n}{2}] - 1$. In this paper, we will show that $\chi(\Gamma) = \omega(\Gamma) = k+1$ if and only if $k+1|n$. Also, we will show that if n is an even integer and $k = \frac{n}{2} - 1$ then $Aut(\Gamma) \cong \mathbb{Z}_2 wr_I Sym(k+1)$ where $I = \{1, \ldots, k+1\}$ and in this case, we show that Γ is an integral graph.

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1. Introduction

In this paper, a graph $\Gamma = (V, E)$ always means a simple connected graph with n vertices (without loops, multiple edges and isolated vertices), where $V = V(\Gamma)$ is the vertex set and $E = E(\Gamma)$ is the edge set. The size of the largest clique in the graph Γ is denoted by $\omega(\Gamma)$ and the size of the largest independent sets of vertices by $\alpha(\Gamma)$. A graph Γ is called a vertex-transitive graph if for any $x, y \in V$ there is some π in $Aut(\Gamma)$, the automorphism group of Γ, such that $\pi(x) = y$. Let Γ be a graph, the complement $\overline{\Gamma}$ of Γ is the graph whose vertex set is $V(\Gamma)$ and whose edges are the pairs of nonadjacent vertices of Γ. It is well known that for any graph Γ, $Aut(\Gamma) = Aut(\overline{\Gamma})$

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[8]. If Γ is a connected graph and $\partial(u,v)$ denotes the distance in Γ between the vertices u and v, then for any automorphism π in $Aut(\Gamma)$ we have $\partial(u,v) = \partial(\pi(u), \pi(v))$.

Let k be a positive integer, a k-colouring of a graph Γ is a mapping $f\colon V(\Gamma) \longrightarrow \{1,\dots,k\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices x and y in Γ, and if such a mapping exists we say that Γ is k-colorable. The chromatic number $\chi(\Gamma)$ of Γ is the minimum number k such that Γ is k-colorable. Let Γ be a graph and $\mathcal{I}(\Gamma)$ denote the set of all independent sets of the graph Γ. A fractional colouring of a graph Γ is a weight function $\mu\colon \mathcal{I}(\Gamma) \longrightarrow [0,1]$ such that for any vertex x of Γ, $\sum_{x\in I\in\mathcal{I}(\Gamma)} \mu(I) \geq 1$, and if such a weight function exists we say that Γ is fractional colouring. The fractional chromatic number of a graph Γ is denoted by $\chi_f(\Gamma)$ and defined in [9, Page 134]. Also a fractional clique of a graph Γ is denoted by $\psi_f(\Gamma)$ and defined in [9, Page 134].

Let $\Upsilon = \{\gamma_1,\dots,\gamma_{k+1}\}$ be a set and K be a group then we write $Fun(\Upsilon, K)$ to denote the set of all functions from Υ into K, we can turn $Fun(\Upsilon, K)$ into a group by defining a product:

$$(fg)(\gamma) = f(\gamma)g(\gamma) \quad \text{for all} \quad f, g \in Fun(\Upsilon, K) \quad \text{and} \quad \gamma \in \Upsilon,$$

where the product on the right is in K. Since Υ is finite, the group $Fun(\Upsilon, K)$ is isomorphic to K^{k+1} (a direct product of $k+1$ copies of K) via the isomorphism $f \to (f(\gamma_1),\dots,f(\gamma_{k+1}))$. Let H and K be groups and suppose H acts on the nonempty set Υ. Then the wreath product of K by H with respect to this action is defined to be the semidirect product $Fun(\Upsilon, K) \rtimes H$ where H acts on the group $Fun(\Upsilon, K)$ via

$$f^x(\gamma) = f(\gamma^{x^{-1}}) \quad \text{for all} \quad f \in Fun(\Upsilon, K), \gamma \in \Upsilon \quad \text{and} \quad x \in H.$$

We denote this group by $Kwr_\Upsilon H$. Consider the wreath product $G = Kwr_\Upsilon H$. If K acts on a set Δ then we can define an action of G on $\Delta \times \Upsilon$ by

$$(\delta, \gamma)^{(f,h)} = (\delta^{f(\gamma)}, \gamma^h) \quad \text{for all} \quad (\delta, \gamma) \in \Delta \times \Upsilon,$$

where $(f, h) \in Fun(\Upsilon, K) \rtimes H = Kwr_\Upsilon H$ [6].

Eigenvalues of an undirected graph Γ are the eigenvalues of an arbitrary adjacency matrix of Γ. Harary and Schwenk [10] defined Γ to be integral, if all of its eigenvalues are integers. For a survey of integral graphs see [3]. In [2] the number of integral graphs on n vertices is estimated. Known characterizations of integral graphs are restricted to certain graph classes, see [1].

Let G be a finite group and S a subset of G that is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = Cay(G, S)$ is a graph whose vertex-set and edge-set are defined as follows:

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x,y\} \mid x^{-1}y \in S\}.$$

It is well known that every Cayley graph is vertex-transitive.

For any graph Γ, $\omega(\Gamma) \leq \chi(\Gamma)$ [8]. Also it is well known that for bipartite graphs $\omega(\Gamma) = \chi(\Gamma) = 2$. Let Γ be the $Cay(\mathbb{Z}_n, S_k)$ where $\mathbb{Z}_n$ $(n \geq 4)$, is the cyclic additive group with identity $\{0\}$, and for any $k \in \mathbb{N}$, $1 \leq k \leq [\frac{n}{2}] - 1$, $S_1 = \{1, n-1\}, \dots, S_k = S_{k-1} \cup \{k, n-k\}$ are inverse-closed subsets of $\mathbb{Z}_n - \{0\}$. In this paper we will show that $\chi(\Gamma) = \omega(\Gamma) = k+1$ if and only if $k+1|n$, also we show that if n is an even integer and $k = \frac{n}{2} - 1$ then $Aut(\Gamma) \cong \mathbb{Z}_2 wr_I Sym(k+1)$, where $I = \{1,\dots,k+1\}$.

2. Definitions and Preliminaries

Proposition 2.1. *[11] For any graph Γ we have*

$$\omega(\Gamma) \le \omega_f(\Gamma) \le \chi_f(\Gamma) \le \chi(\Gamma).$$

Proposition 2.2. *[8] If Γ is vertex transitive graph, then we have*

$$\omega_f(\Gamma) = \frac{|V(\Gamma)|}{\alpha(\Gamma)}$$

Definition 1. *[4] Let Γ be a graph with automorphism group $Aut(\Gamma)$. We say that Γ is symmetric if, for all vertices u, v, x, y of Γ such that u and v are adjacent, also, x and y are adjacent, there is an automorphism π such that $\pi(u) = x$ and $\pi(v) = y$. We say that Γ is distance-transitive if, for all vertices u, v, x, y of Γ such that $\partial(u, v) = \partial(x, y)$, there is an automorphism π such that $\pi(u) = x$ and $\pi(v) = y$.*

Remark 2.1. [4] Let Γ be a graph. It is clear that we have a hierarchy of conditions:
distance-transitive $\Rightarrow$ symmetric $\Rightarrow$ vertex-transitive

Definition 2. *[4], [5] For any vertex v of a connected graph Γ we define*

$$\Gamma_r(v) = \{u \in V(\Gamma) \mid \partial(u, v) = r\},$$

where r is a non-negative integer not exceeding d, the diameter of Γ. It is clear that $\Gamma_0(v) = \{v\}$, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(v), \ldots, \Gamma_d(v)$, for each v in $V(\Gamma)$. The graph Γ is called distance-regular with diameter d and intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$, if it is regular of valency k and for any two vertices u and v in Γ at distance r we have $|\Gamma_{r+1}(v) \cap \Gamma_1(u)| = b_r$, and $|\Gamma_{r-1}(v) \cap \Gamma_1(u)| = c_r$, $(0 \le r \le d)$. The numbers c_r, b_r and a_r, where

$$a_r = k - b_r - c_r \quad (0 \le r \le d),$$

is the number of neighbours of u in $\Gamma_r(v)$ for $\partial(u, v) = r$, are called the intersection numbers of Γ. Clearly $b_0 = k$, $b_d = c_0 = 0$ and $c_1 = 1$.

Remark 2.2. [4] It is clear that if Γ is distance-transitive graph then Γ is distance-regular.

Lemma 2.1. *[4] A connected graph Γ with diameter d and automorphism group $G = Aut(\Gamma)$ is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer G_v is transitive on the set $\Gamma_r(v)$, for each $r \in \{0, 1, \ldots, d\}$, and $v \in V(\Gamma)$.*

Theorem 2.1. *[5] Let Γ be a distance-regular graph which the valency of each vertex as k, with diameter d, adjacency matrix A and intersection array,*

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}.$$

Then the tridiagonal $(d + 1) \times (d + 1)$ matrix

$$\jmath(\Gamma) = \begin{bmatrix} a_0 & b_0 & 0 & 0 & \dots & & & & \\ c_1 & a_1 & b_1 & 0 & \dots & & & & \\ 0 & c_2 & a_2 & b_2 & & & & & \\ & & & \dots & & & & & \\ & & & & & & & & \\ & & & & c_{d-2} & a_{d-2} & b_{d-2} & 0 \\ & & & \dots & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & \dots & 0 & 0 & c_d & a_d \end{bmatrix},$$

determines all the eigenvalues of Γ.

Theorem 2.2. *[7] Let* Γ *be a graph such that contains* $k+1$ *components* $\Gamma_1, \dots, \Gamma_{k+1}$. *If for any* $i \in I = \{1, \dots, k+1\}$, $\Gamma_i \cong \Gamma_1$ *then* $Aut(\Gamma) \cong Aut(\Gamma_1) wr_I Sym(k+1)$.

3. Main Results

Proposition 3.1. *Let* $\Gamma = Cay(\mathbb{Z}_n, S_k)$ *be the Cayley graph on the cyclic group* $\mathbb{Z}_n$ $(n \geq 4)$, *where* $S_1 = \{1, n-1\}, \dots, S_k = S_{k-1} \cup \{k, n-k\}$ *are the inverse-closed subsets of* $\mathbb{Z}_n - \{0\}$ *for any* $k \in \mathbb{N}$, $1 \leq k \leq [\frac{n}{2}] - 1$. *Then* $\chi(\Gamma) = \omega(\Gamma) = k+1$ *if and only if* $k+1|n$.

Proof. By definition of S_i, $1 \leq i \leq k$ clearly $\mid S_i \mid = 2i$, hence $\mid S_k \mid = 2k$. Let $\Gamma = Cay(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ and S_k be the set of inverse-closed subset of $\mathbb{Z}_n - \{0\}$ which is defined as before. By definition of Γ clearly $\omega(\Gamma) = k+1$. So, if $\chi(\Gamma) = \omega(\Gamma) = k+1$ then by Proposition 2.1, $\chi_f(\Gamma) = \omega_f(\Gamma) = k+1$. Also we know that Γ is a vertex transitive graph, so by Proposition 2.2, $k+1 = \omega_f(\Gamma) = \frac{|V(\Gamma)|}{\alpha(\Gamma)}$ therefore $k+1|n$. Conversely, if $k+1|n$ then $\chi(\Gamma) = k+1$, because Γ is a vertex transitive graph and the size of every clique in the graph Γ is $k+1$, therefore $\chi(\Gamma) = \omega(\Gamma) = k+1$. □

Example 1. *Suppose* $\Gamma_1 = Cay(\mathbb{Z}_{12}, S_2)$ *and* $\Gamma_2 = Cay(\mathbb{Z}_{12}, S_3)$ *are two Cayley graphs, then* $\chi(\Gamma_1) = \omega(\Gamma_1) = 3$ *and* $\chi(\Gamma_2) = \omega(\Gamma_2) = 4$.

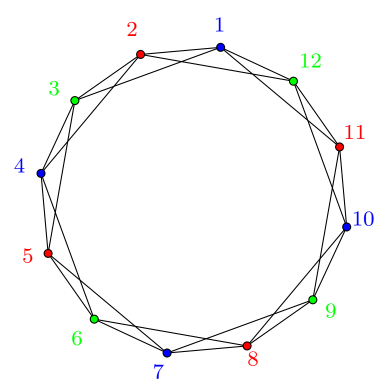

Figure 1: $\chi(\Gamma_1) = \omega(\Gamma_1) = 3$

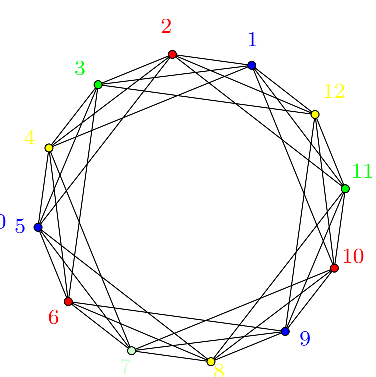

Figure 2: $\chi(\Gamma_2) = \omega(\Gamma_2) = 4$

Proposition 3.2. *Let $\Gamma = Cay(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ $(n \geq 4)$, where $S_1 = \{1, n-1\}$, ..., $S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq [\frac{n}{2}] - 1$. If n is an even integer and $k = \frac{n}{2} - 1$ then $Aut(\Gamma) \cong \mathbb{Z}_2 wr_I Sym(k+1)$, where $I = \{1, \ldots, k+1\}$.*

Proof. Let $V(\Gamma) = \{1, \ldots, n\}$ be the vertex set of Γ. By assumptions and Proposition 2.2, the size of the largest independent set of vertices in the Γ is 2, because Γ is a vertex transitive graph and the size of every clique in the graph Γ is $k+1$. Thus, the size of the every independent set of vertices in the Γ is 2. Therefore for any $x \in V(\Gamma)$, there is exactly one $y \in V(\Gamma)$ such that $x^{-1}y = k+1$. Hence, if $x^{-1}y = k+1$ then two vertices x and y adjacent in the complement $\overline{\Gamma}$ of Γ, so $\overline{\Gamma}$ contains $k+1$ components $\Gamma_1, \ldots, \Gamma_{k+1}$ such that for any $i \in I = \{1, \ldots, k+1\}$, $\Gamma_i \cong \Gamma_1 \cong K_2$, where K_2 is the complete graph of 2 vertices. Therefore $\overline{\Gamma} \cong (k+1)K_2$, hence by Theorem 2.2, $Aut(\overline{\Gamma}) \cong Aut(K_2) wr_I Sym(k+1) = \mathbb{Z}_2 wr_I Sym(k+1)$, so $Aut(\Gamma) \cong \mathbb{Z}_2 wr_I Sym(k+1)$. □

Example 2. *Let $\Gamma = Cay(\mathbb{Z}_{12}, S_5)$ be the Cayley graph on the cyclic group $\mathbb{Z}_{12}$, then $\chi(\Gamma) = \omega(\Gamma) = 6$, and $Aut(\Gamma) = \mathbb{Z}_2 wr_I Sym(6)$, where $I = \{1, \ldots, 6\}$.*

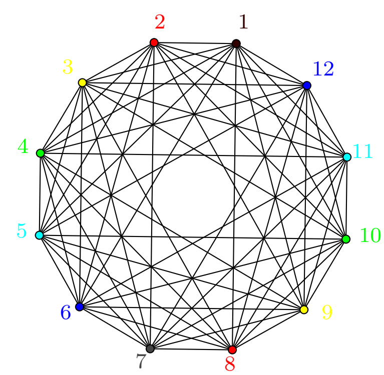

Figure 3: $\chi(\Gamma) = \omega(\Gamma) = 6$

Proposition 3.3. *Let $\Gamma = Cay(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ $(n \geq 4)$, where $S_1 = \{1, n-1\}$, ..., $S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq [\frac{n}{2}] - 1$. If n is an even integer and $k = \frac{n}{2} - 1$ then Γ is a distance-transitive graph.*

Proof. By Lemma 2.1, it is sufficient to show that vertex-stabilizer G_v is transitive on the set $\Gamma_r(v)$ for every $r \in \{0, 1, 2\}$ and $v \in V(\Gamma)$, because of Γ is a vertex-transitive graph. We know $V(\Gamma) = \{1, 2, \ldots, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+1, \ldots, n\}$ is the vertex set of Γ. Let $G = Aut(\Gamma)$. Consider the vertex $v = 1$ in the $V(\Gamma)$, then $\Gamma_0(v) = \{1\}$, $\Gamma_1(v) = \{2, \ldots, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+2, \ldots, n\}$ and $\Gamma_2(v) = \{\frac{n}{2}+1\}$. Let $\rho = (2, 3, \ldots, \frac{n}{2}, \frac{n}{2}+2, \ldots, n)$ be the cyclic permutation of the vertex set of Γ. It is an easy task to show that ρ is an automorphism of Γ. We can show that $H = \langle (2, 3, \ldots, \frac{n}{2}, \frac{n}{2}+2, \ldots, n) \rangle$ acts transitively on the set $\Gamma_r(v)$ for each $r \in \{0, 1, 2\}$, because H is a cyclic group. Note that if $1 \neq v \in V(\Gamma)$ then, we can show that vertex-stabilizer G_v is transitive on the set $\Gamma_r(v)$ for each $r \in \{0, 1, 2\}$, because Γ is a vertex-transitive graph. □

Proposition 3.4. *Let* $\Gamma = Cay(\mathbb{Z}_n, S_k)$ *be the Cayley graph on the cyclic group* $\mathbb{Z}_n$ $(n \geq 4)$, *where* $S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\}$ *are the inverse-closed subsets of* $\mathbb{Z}_n - \{0\}$ *for any* $k \in \mathbb{N}$, $1 \leq k \leq [\frac{n}{2}] - 1$. *If* n *is an even integer and* $k = \frac{n}{2} - 1$ *then* Γ *is an integral graph.*

Proof. By Remark 2.2, it is clear that Γ is distance-regular, because Γ is a distance-transitive graph. Let $V(\Gamma) = \{1, 2, \ldots, n\}$ be the vertex set of Γ. Consider the vertex $v = 1$ in the $V(\Gamma)$, then $\Gamma_0(v) = \{1\}$, $\Gamma_1(v) = \{2, \ldots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 2, \ldots, n\}$ and $\Gamma_2(v) = \{\frac{n}{2} + 1\}$. Let be u in the $V(\Gamma)$ such that $\partial(u, v) = 0$ then $u = v = 1$ and $|\Gamma_1(v) \cap \Gamma_1(u)| = 2k$, hence $b_0 = 2k$ and by Definition 2, $a_0 = 2k - b_0 = 0$. Also, if u in the $V(\Gamma)$ and $\partial(u, v) = 1$ then two vertices u, v are adjacent in Γ, so $|\Gamma_0(v) \cap \Gamma_1(u)| = 1$ and $|\Gamma_2(v) \cap \Gamma_1(u)| = 1$, hence $c_1 = 1$, $b_1 = 1$ and $a_1 = 2k - b_1 - c_1 = 2k - 2$. Finally, if u in the $V(\Gamma)$ and $\partial(u, v) = 2$ then two vertices u, v are not adjacent in Γ, so $|\Gamma_1(v) \cap \Gamma_1(u)| = 2k$, hence $c_2 = 2k$ and $a_2 = 2k - c_2 = 0$. So the intersection array of Γ is $\{2k, 1; 1, 2k\}$. Therefore by Theorem 2.1, the tridiagonal $(3) \times (3)$ matrix,

$$\begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 2k & 0 \\ 1 & 2k-2 & 1 \\ 0 & 2k & 0 \end{bmatrix},$$

determines all the eigenvalues of Γ. It is clear that all the eigenvalues of Γ are $2k, 0, -2$ and their multiplicities are $1, k+1, k$, respectively. So Γ is an integral graph. □

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