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On an edge partition and root graphs of some classes of line graphs

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Abstract

The Gallai and the anti-Gallai graphs of a graph G are complementary pairs of spanning subgraphs of the line graph of G. In this paper we find some structural relations between these graph classes by finding a partition of the edge set of the line graph of a graph G into the edge sets of the Gallai and anti-Gallai graphs of G. Based on this, an optimal algorithm to find the root graph of a line graph is obtained. Moreover, root graphs of diameter-maximal, distance-hereditary, Ptolemaic and chordal graphs are also discussed.

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1. Introduction

The line graph $L(G)$ of a graph G has as its vertices the edges of G, and any two vertices are adjacent in $L(G)$ if the corresponding edges are incident in G. The Gallai graph $Gal(G)$ [10, 15] of a graph G has as its vertices the edges of G, and any two vertices are adjacent in $Gal(G)$ if the corresponding edges are incident in G, but do not span a triangle in G. The anti-Gallai graph $antiGal(G)$[13] of a graph G has as its vertices the edges of G, and any two vertices of G are adjacent in $antiGal(G)$ if the corresponding edges are incident in G and lie on a triangle in G.

In [13] it is shown that the four color theorem can be equivalently stated in terms of anti-Gallai graphs. The problems of determining the clique number and the chromatic number of $Gal(G)$ are

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NP-Complete[13]. In [3] it is shown that there are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs and anti-Gallai graphs. In [2] it is shown that the complexity of recognizing anti-Gallai graphs is NP-complete.

A graph H is forbidden in a graph family $\mathcal{G}$, if H is not an induced subgraph of any $G \in \mathcal{G}$. For any finite graph H, there exist a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be H-free [3]. However, both Gallai graphs and anti-Gallai graphs cannot be characterized using forbidden subgraphs [13].

The Gallai and the anti-Gallai graphs are spanning subgraphs of line graphs. In fact, they are complement to each other in $L(G)$. Therefore a natural question arises: is it possible to identify the edges of $Gal(G)$ and $antiGal(G)$ from $L(G)$? A positive answer to this is given in this paper by introducing an algorithm to partition the edge set of a line graph into the edges of Gallai and anti-Gallai graphs, using the adjacency properties of common neighbors of the edges of a line graph in a hanging [8].

A graph G is a root graph of the line graph H if $L(G) \cong H$. The root graph of a line graph is unique, except for the triangle and $K_{1,3}$ [16]. In this paper, using the edge-partition, an algorithm is obtained to find the root graph of a line graph. Also, the root graphs of diameter-maximal, distance-hereditary, Ptolemaic and chordal graphs are obtained.

Let $H = (V, E)$ be a graph with vertex set $V = V(H)$ and edge set $E = E(H)$. Let $N(v)$ denote the set of all vertices adjacent to v and $N_M(v) = M \cap N(v)$, where $M \subseteq V$. The edge joining u and v is denoted by uv. The common neighbors of uv is $N(u) \cap N(v)$ and $N(uv) = N(u) \cup N(v)$. The subgraph induced by $\{v_1, v_2, ..., v_k\} \subseteq V$ is denoted by $< v_1, v_2, ..., v_k >$. A clique is a complete subgraph of a graph. An edge clique cover of H is a family of cliques $\mathcal{E} = \{q_1, q_2, ..., q_k\}$ such that each edge of H is in at least one of $E(q_1), E(q_2), ...E(q_k)$.

A path on n vertices P_n is the graph with vertex set $\{v_1, v_2, ..., v_n\}$ and $v_i v_{i+1}$ for $i = 1, 2, ..., n-1$ are the only edges. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in H. The diameter of H, denoted by $d(H)$, is the maximum length of a shortest path in H.

The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

All graphs mentioned in this paper are simple and connected, unless otherwise specified. Also, all other basic concepts and notations not mentioned in this paper are from [4].

2. Adjacency properties of edges of $L(G)$

The hanging [8] of a graph $H = (V, E)$, with $|V| = n$ and $|E| = m$, by a vertex z is the function $h_z(x)$ that assigns to each vertex x of H the value $d(z, x)$. The i-th level of H in a hanging h_z is defined as $L_i = \{x \in H : h_z(x) = i\}$. A hanging can be obtained using a breadth first search(BFS) [1], which has a time complexity of $O(m + n)$.

For a vertex v in L_i, a supporter of v is a vertex in L_{i-1}, which is adjacent to v. A vertex in L_i is an ending vertex if it has no neighbors in L_{i+1}. An arbitrary supporter of v is denoted by $S(v)$. It is clear that any vertex v in the level L_i for $i \geq 1$ has at least one supporter.

We use the following, well known, forbidden subgraph characterization of a line graph.

Theorem 2.1. *[6] A graph H is a line graph if and only if the nine graphs in Fig 1 are forbidden subgraphs for H.*

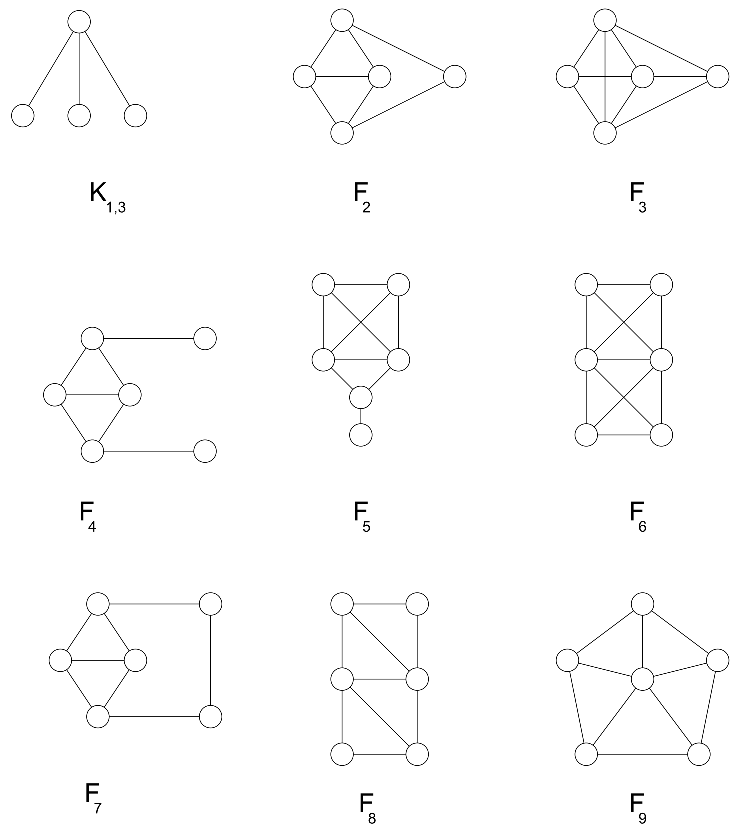


Figure 1. Forbidden Subgraphs of line graph

Theorem 2.2. *Consider a hanging of a line graph H by an arbitrary vertex in H and let uv denote the edge joining u and v in the same level L_i. Then, the following statements hold*

1. *All common neighbors of uv in L_{i-1} are adjacent to each other.*
2. *All common neighbors of uv in L_{i+1} are adjacent to each other.*
3. *If uv has no common neighbor in L_{i-1}, then all the common neighbors of uv in L_i which are adjacent to all other neighbors of uv are adjacent to each other.*
4. *There is at most one common neighbor of uv in L_i, which is adjacent to all the neighbors of uv but not adjacent to the common neighbors of uv in L_{i-1} and L_i.*

Proof.

1. Let x and x' be two (distinct) common neighbors of an edge uv in L_{i-1}, then $i \geq 2$. Assume that x and x' are not adjacent. Now, if x and x' have a common neighbor w in L_{i-2}, then

$<w, x, x', u, v> \cong F_2$ in Fig 1 which contradicts the fact that H is a line graph. So, let w and w' be any two vertices in L_{i-2} adjacent to x and x' respectively. Then $<w, w', x, x', u, v> \cong F_7$ or F_4 according as, w and w' are adjacent or not.

2. Let w and x be two common neighbors of an edge uv in L_{i+1}. Assume that x and w are not adjacent. Now, if z is a supporter of u in L_{i-1}, then $< z, u, w, x > \cong K_{1,3}$, which is a contradiction.

3. Let uv has no common neighbor in the level L_{i-1} and hence $i \geq 2$. Let x and w be two common neighbors of uv in L_i which are adjacent to all the neighbors of uv. Assume that x and w are not adjacent. Now u and v cannot have a common supporter. So let z_1 and z_2 be two supporters of u and v respectively. Since z_1 and z_2 are neighbors of uv, both x and w are adjacent to them. Now, the vertices z_1, x, w and $S(z_1)$ induce a $K_{1,3}$ which is a contradiction.

4. Assume that x and w are two nonadjacent common neighbors of uv in L_i which are not adjacent to the common neighbors of uv but adjacent to all the other neighbors of uv in L_{i-1} and L_i. So, it is clear that $i \geq 2$. Let z be a common neighbor of uv in L_{i-1}. Now u must have at least one neighbor in L_{i-1} other than the common neighbors of uv in L_{i-1}, for otherwise, the vertices u, x, w and z induce a $K_{1,3}$ which is a contradiction. Similar is the case for the vertex v. So let z_1 and z_2 be two neighbors (but not common neighbors) of u and v in L_{i-1} respectively. But, we have, $< S(z_1), z_1, x, w > \cong K_{1,3}$, which is also a contradiction.

□

Remark 2.1. In fact the above theorem is applicable to a larger class of graphs than line graphs as only some of the forbidden sub graphs of line graphs are used in the proof.

3. Anti-Gallai triangles in $L(G)$

Let uvw be a triangle in $L(G)$ and let $\bar{u}, \bar{v}$ and $\bar{w}$ be the edges in G representing the vertices u, v and w respectively in $L(G)$. If the edges $\bar{u}, \bar{v}$ and $\bar{w}$ induce a triangle in G then the triangle uvw in $L(G)$ is referred to as an anti-Gallai triangle. All the triangles in $antiGal(G)$ need not be an anti-Gallai triangle and the number of anti-Gallai triangles in $L(G)$ is equal to the number of triangles in G. Since each edge of an anti-Gallai graph belongs to some anti-Gallai triangle, the set of all anti-Gallai triangles in $L(G)$ induces $antiGal(G)$.

Remark 3.1. When a triangle uvw in $L(G)$ is not an anti-Gallai triangle, the edges $\bar{u}, \bar{v}$ and $\bar{w}$ in G have a vertex in common.

Lemma 3.1. *Consider a line graph $H \ncong K_3$. If a triangle uvw in H is an anti-Gallai triangle, then for all $x \in V(H) \setminus \{u, v, w\}$, one of the following holds.*

a) $<u, v, w, x> \cong K_4 - e$

b) $<u, v, w, x>$ *is disconnected.*

Proof. Let G be the graph such that $L(G) \cong H$ and assume that the triangle uvw is an anti-Gallai triangle in H. Then the edges $\bar{u}, \bar{v}$ and $\bar{w}$ in G induce a triangle in G. Now corresponding to any vertex x in H, there is an edge $\bar{x}$ in G. If $\bar{x}$ is adjacent to the triangle $\bar{u}\bar{v}\bar{w}$, then $\bar{x}$ is adjacent to exactly two of the edges of $\bar{u}\bar{v}\bar{w}$ and hence $<u, v, w, x> \cong K_4 - e$ in H. If $\bar{x}$ is not adjacent to the triangle $\bar{u}\bar{v}\bar{w}$, then $<u, v, w, x>$ is disconnected. □

Lemma 3.2. *If a triangle uvw is not an anti-Gallai triangle in a line graph $H \cong L(G)$, then there is at most one common neighbor z for an edge of uvw in H such that $<u, v, w, z> \cong K_4 - e$.*

Proof. Let $\bar{u}, \bar{v}$ and $\bar{w}$ be the edges in G, representing the vertices u, v and w respectively in H. Let z be such that $<u, v, w, z> \cong K_4 - e$ in $L(G)$ and let it be a common neighbor of uv. Then the edge $\bar{z}$ in G is adjacent to both the edges $\bar{u}$ and $\bar{v}$ and not adjacent to $\bar{w}$. clearly $\bar{u}, \bar{v}$ and $\bar{z}$ induce a triangle in G and hence uvz is an anti-Gallai triangle in $L(G)$. Now assume that z' is a vertex different from z such that it is a common neighbor of uv and $<u, v, w, z'> \cong K_4 - e$. Then the vertices z and z' cannot be adjacent, otherwise $<u, v, z, z'> \cong K_4$ and by Lemma 3.1 it will contradict the fact that u, v, z is an anti-Gallai triangle. But, we have, $<u, w, z, z'> \cong K_{1,3}$ and hence H cannot be a line graph by Theorem 2.1. □

Theorem 3.1. *Consider a line graph $H \ncong K_3, K_4 - e, C_4 \vee K_1$ and $C_4 \vee 2K_1$. A triangle uvw in H is an anti-Gallai triangle if and only if $<u, v, w, x> \cong K_4 - e$ or disconnected for all $x \in V(H) \setminus \{u, v, w\}$.*

Proof. Let G be the graph such that $L(G) \cong H$. The necessary part of the theorem follows from Lemma 3.1.

Conversely, assume that uvw is a triangle in H such that $<u, v, w, x> \cong K_4 - e$ or disconnected for all $x \in V(H)$ and that uvw is not an anti-Gallai triangle. Then the edges $\bar{u}, \bar{v}$ and $\bar{w}$ induce a $K_{1,3}$ in G. Note that any vertex which induces a $K_4 - e$ with the triangle uvw is adjacent to exactly two vertices among u, v and w. Now, since H is connected and not a K_3, there is a vertex x adjacent to the triangle uvw. Assume that x is adjacent to u and w. Then in G, $\bar{u}, \bar{v}$ and $\bar{x}$ induce a triangle so that uwx is an anti-Gallai triangle. Since $H \ncong K_4 - e$ and also connected, there is a vertex y adjacent to at least one of the vertices u, v, w and x. If there is no vertex adjacent to the triangle uvw, then it must be adjacent to x alone, which is a contradiction to the fact that uwx is anti-Gallai triangle. So let y be adjacent to uvw. By Lemma 3.2 y cannot be adjacent to u and w. So let y be adjacent to v and w. Now we have vwy is also an anti-Gallai triangle. But, since $H \ncong C_4 \vee K_1$ and connected, using the same arguments as before, we have a vertex z adjacent to the triangle uvw again. The only possibility then is that z is adjacent to the vertices u and v. Now we show that there are no more vertices possible in H. If not, let p be a vertex in H different from u, v, w, x, y and z. But, by Lemma 3.2, the vertex p cannot be adjacent to uvw. Now if p is adjacent to x, it must be adjacent to u or w as uwx is an anti-Gallai triangle, which again is not possible. Similarly, p cannot be adjacent to y and z. Hence no such vertex p can be adjacent to any of the vertices u, v, w, x, y and z. So such a vertex does not exist in H, as H is a connected graph. Now we have $H \cong <u, v, w, x, y, z> \cong C_4 \vee 2K_1$, which is a contradiction. □

We observe that it is possible to suitably re-label the edges in the root graph of $C_4 \vee K_1$ so that no triangles in $C_4 \vee K_1$ can be claimed to be an anti-Gallai triangle, see Figure 2. It can be seen

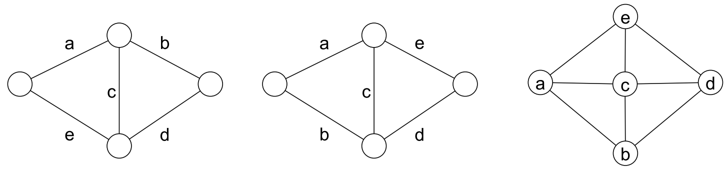


Figure 2. Two possible labellings of $K_4 - e$ and its line graph $C_4 \vee K_1$

that $K_4 - e$ and $C_4 \vee 2K_1$ also have this property. Theorem 3.1 shows that these three graphs are the only exceptions (the graph K_3 is excluded as it is a trivial case with 3 vertices). Hence, the graphs $K_4 - e$, $C_4 \vee K_1$ and $C_4 \vee 2K_1$ are excluded in the following discussions.

Definition 1. *A triangle in a hanging of a line graph is an $L\triangle$ ($M\triangle, R\triangle$) if it is an anti-Gallai triangle and it is induced by two vertices in one level and one vertex from the lower (same, higher) level of the ordering.*

We can see that any anti-Gallai triangle is either an $L\triangle$, $M\triangle$ or $R\triangle$ in a hanging of $L(G)$

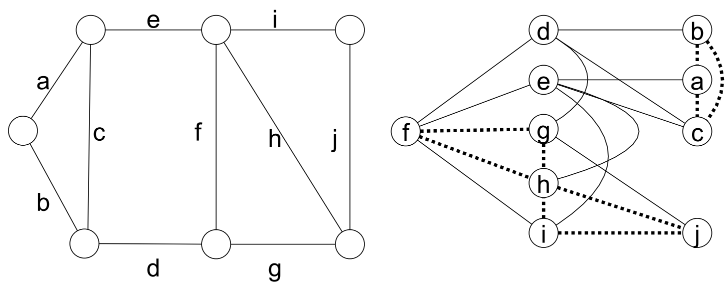


Figure 3. A graph and the hanging of its line graph by vertex f. The dotted lines show an $L\triangle$ $fgh, R\triangle$ hij and an $M\triangle$ abc

Theorem 3.2. *Let uv be an edge in any level of a hanging of $H \cong L(G)$ by an arbitrary vertex in H, then*

1. *uv cannot be an edge of an $L\triangle$ in any level L_i for $i > 1$.*
2. *uv cannot be an edge of an $M\triangle$ in L_1.*
3. *If uv is an edge in an $M\triangle$ then uv cannot be an edge of an $L\triangle$.*
4. *If uv is an edge in an $M\triangle$ then uv cannot be an edge of an $R\triangle$.*
5. *If uv is an edge in an $L\triangle$ then uv cannot be an edge of an $R\triangle$.*
6. *uv can be an edge of at most one $L\triangle$ or $R\triangle$ or $M\triangle$.*

Proof.

1. Let uv be an edge in an L_i for $i > 1$ and let it belong to an $L\triangle$ uvx, where $x \in L_{i-1}$. Let w be the vertex in L_{i-2} which is adjacent to x. Then $< w, x, u, v >$ induces a subgraph which is neither a $K_4 - e$ nor disconnected, which is a contradiction.

2. Let uvx be an $M\triangle$ in L_1 and z be the vertex, from where the hanging of H being considered. Then $d(z) \geq 3$ and $< z, x, u, v >$ induce a K_4 and hence uvx cannot be an anti-Gallai triangle, which is a contradiction.

3. Let uv be an edge in $L\Delta$ then uv is in L_1 by (1) and hence uv cannot be an edge of an $M\Delta$ by (2).

   From (3) and Theorem 3.1, it follows that anti-Gallai triangles of a graph cannot share an edge in a line graph. Hence the proof of (4) to (6) follows.

□

Now, Lemma 3.3 follows.

Lemma 3.3. *Exactly one triangle of a $K_4 - e$ in a line graph is an anti-Gallai triangle.*

From Theorems 2.2 and 3.1, we have the following propositions.

Proposition 3.1. *The edge uv is in an $L\triangle$, with both its ends in the same level of a hanging of a line graph if and only if it satisfies the following conditions*

1. *Each vertex in L_1 is either adjacent to u or v but not to both.*
2. *Each neighbor of uv in L_2 is a common neighbor of uv.*

Proposition 3.2. *The edge uv is in an $M\triangle$ in a hanging of a line graph if and only if it satisfies the following conditions*

1. *The edge uv has a common neighbor x in L_i which is not adjacent to the other common neighbors of uv in L_{i-1} and L_i.*
2. *Either u or v is adjacent to each neighbor of x.*
3. *Each non neighbor of x is either a common neighbor of uv or not a neighbor of uv.*

Proposition 3.3. *The edge uv is in an $R\triangle$ with both its ends in the i^{th} level of a hanging of a line graph if and only if it satisfies the following conditions*

1. *The edge uv has exactly one common neighbor x in L_{i+1}.*
2. *The vertex x is an ending vertex.*
3. *Either u or v is adjacent to each neighbor of x.*
4. *Each non neighbor of x in $L_{i-1} \cup L_i$ is either a common neighbor of uv or not a neighbor of uv.*

4. Partitioning the edges of a line graph

We now provide an algorithm to partition the edge set of a line graph into edge sets of its Gallai and anti-Gallai graphs. The following three tests checks whether an edge $uv \in L_i$ belongs to an $L\triangle, M\triangle$ or $R\triangle$.

Algorithm 1. $L\triangle$ *test*

1. If $i \neq 1$ go to step 7.
2. Find $N(u)$ and $N(v)$.
3. If $N_{L_i}(u) \cup N_{L_i}(v) \neq L_i$ then go to step 7.
4. If $N_{L_i}(u) \cap N_{L_i}(v) \neq \emptyset$ then go to step 7.
5. If $N_{L_{i+1}}(u) \neq N_{L_{i+1}}(v)$ then go to step 7.
6. Triangle uvz is an $L\triangle$.
7. The edge uv is not in $L\triangle$.

Algorithm 2. $M\triangle$ *test*

1. If $i = 1$ go to step 9.
2. Find the set C of common neighbors w_j of uv in L_i. If $C = \emptyset$, go to step 9.
3. Find the set B of common neighbors x_j of uv in L_{i-1} and L_{i+1}.
4. For each $x_j \in B$, delete the members of the set $N_C(x_j)$ from C. If $C = \emptyset$ go to step 9.
5. For each w_j, if $|N_C[w_j]| > 1$, delete the members of the set $N_C[w_j]$. If $|C| \neq 1$ go to step 9.
6. Find the set $N(uv)$ in H.
7. If $|N_C(y_j)| = 1$, for each $y_j \in N(uv) \setminus (B \cup C)$, go to step 8. Else go to step 9.
8. Triangle uvx is an $M\triangle$.
9. The edge uv is not in $M\triangle$.

Algorithm 3. $R\triangle$ *test*

1. Find the set C_R of common neighbors of uv in L_{i+1}.
2. If $|C_R| \neq 1$ go to step 7. Else choose the common neighbor of uv in L_{i+1} as x.
3. If the vertex x is not an ending vertex, go to step 7.

4. Either u or v is adjacent to each neighbor of x. Else go to step 7.

5. Each non neighbor of x is either a common neighbor of uv or not a neighbor of uv. Else go to step 7.

6. Triangle uvx is an $R\triangle$.

7. The edge uv is not in $R\triangle$.

Given a line graph $H \cong L(G)$, obtain a hanging h_z by an arbitrary vertex z. Consider all the edges starting from a vertex u in L_1. For each edge of the form uv for some $v \in L_1$, apply tests 1, 2 and 3 one by one. Choose another edge whenever an anti-Gallai triangle is found or when all the tests fail. When all the edges in a level are considered, go to the next level and repeat the procedure. This algorithm ends when all the edges in the last level of the hanging are considered and uses a time complexity of $O(m)$

We now observe that in a line graph $L(G)$, any edge that is in the edge set of $antiGal(G)$ belongs to some anti-Gallai triangle. Hence the set of all the edges of the anti-Gallai triangles gives the edge set of $antiGal(G)$ and the remaining edges of the $L(G)$ corresponds to the edge set of $Gal(G)$.

5. An algorithm to find the root graph of a line graph

An optimal algorithm to recognize a line graph and out put its root graph can be seen in [14], the time complexity of which is $O(n) + m$. Using the above edge partition, an algorithm, which uses a time complexity of $O(m) + O(n)$, is provided to find the root graph of a line graph H. The same algorithm can be used as a recognition algorithm for line graphs. For this, applying the above three tests for the edges in an arbitrary graph, we call a triangle type I if it belongs to the category of anti-Gallai triangles and type II otherwise.

Algorithm 4. *Root graph of a line graph*

Consider a connected graph $H = (V, E)$ with $|V| = n$, $|E| = m$ and its hanging h_z, by an arbitrary vertex z.

Let $M = \{z, u\}$, where u is a neighbor of z. Let G be a path on three vertices with $V(G) = \{\{z\}, \{z, u\}, \{u\}\}$ and $E(G) = \{(\{z\}, \{z, u\}), (\{z, u\}, \{u\})\}$. Here the labels of vertices of G are represented as sets which can be re-labeled, in the steps of the following algorithm, using set operations.

1. Choose a vertex v from $V(H) \setminus M$ with $N_M(v) \neq \emptyset$.

2. If v induces a clique in $N_M(v)$ and does not induce a type I triangle go to step 3. Else go to step 4.

3. Make $V(G) = V(G) \cup \{v\}$, and join {v} with a vertex $C \in V(G)$, where $C = N_M(v)$, and make $M = M \cup \{v\}$ and $C = C \cup \{v\}$. If no such vertex C exists, go to step 4.

4. Find two vertices A and B in $V(G)$ such that $A \cup B = N_M(v)$ and make $M = M \cup \{v\}$, $A = A \cup \{v\}$ and $B = B \cup \{v\}$. Go to step 1.

The algorithm ends whenever $M = V(H)$ or there does not exist C or A and B as required. Here the graph G represents the root graph of the line graph H and in the latter case it can be concluded that the graph H is not a line graph of any graph.

The correctness of the algorithm can be verified with the help of the following theorem due to Krausz [12].

Theorem 5.1. *A graph H is a line graph if and only if it has an edge clique cover $\mathcal{E}$ such that both the following conditions hold:*

1. *Every vertex of H is in exactly two members of $\mathcal{E}$.*
2. *Every edge of H is in exactly one member of $\mathcal{E}$.*

Since the vertex labels of G are represented as sets, a vertex in $<M>$ is an element of some vertex label(set), of G. Here the elements of each vertex label in $V(G)$ induce a clique in $<M>$ of H, since x, y are in a vertex label of G if and only if x and y are adjacent in $<M>$ of H. Now from the construction of G, each vertex of $<M>$ is an element of exactly two vertex labels of G and also any adjacent vertices in $<M>$ belong to a vertex label of G. Now $V(G)$ gives an edge clique cover of $<M>$ which satisfies the two conditions given in Krausz's theorem. Hence the algorithm obtains a graph G with $L(G) \cong H$ if and only if $M = V(H)$.

We now provide the difference between our algorithm and the algorithm in [14].

Given a graph H, the algorithm in [14] assumes that H is a line graph and defines a graph G such that H is necessarily the line graph of G. A comparison of $L(G)$ and H is then made to check whether the given graph is actually a line graph. The algorithm starts with two adjacent basic nodes, labeled 1-2 and 2-3, and labels the vertices in H, on the go, depending on their adjacency. The algorithm proceeds to determine all connections in G corresponding to a clique, containing the basic nodes in H, simultaneously finding an anti-Gallai triangle {1-2, 2-3, 1-3}, if it exists. In each step, the cliques sharing the vertices, which are already worked out, are considered and the algorithm finally outputs a labeled graph G.

In our algorithm, the types of triangles are found using the first three algorithms, the time complexity of which is calculated as follows. We can see that a hanging of the graph H can be obtained in $O(m + n)$ steps. In each of the algorithms 1, 2 and 3 only a subset of $E(H)$ are considered (as edges between the levels are not included) and the algorithm 4, which assumes that algorithms 1, 2 and 3 are already done, finishes in $O(n)$ steps. Hence using these algorithms the root graph of a line graph can be obtained in $O(m) + O(n)$ steps. It can be noted, as a consequence of Theorem 3.1, that irrespective of the starting set M of nodes, any pre-labeled line graph H with more than four vertices gives a uniquely labeled root graph G.

6. Root graphs of diameter-maximal line graphs

A graph G is diameter-maximal [7], if for any edge $e \in E(\overline{G})$, $d(G + e) < d(G)$.

Theorem 6.1. *[7] A connected graph G is diameter-maximal if and only if*

1. *G has a unique pair of vertices u and v such that $d(u, v) = d(G)$.*
2. *The set of nodes at distance k from u induce a complete sub graph.*
3. *Every node at distance k from u is adjacent to every node at distance $k + 1$ from u.*

Lemma 6.1. *Let G be a diameter-maximal line graph and u, v be two vertices of G with $d(u, v) = d(G)$. Let $L^* = (|L_0|, |L_1|, \ldots, |L_d|)$ be the sequence generated from the hanging h_u. Then, $|L_i| \leq 2$ for $i = 0, 1, ..., d$.*

Proof. Clearly $|L_0| = |L_d| = 1$ in L^*. If possible, let u, v and w be three vertices in L_i for some i for $0 < i < d$. By Theorem 6.1, $< u, v, w >\cong K_3$ and there exist vertices x in L_{i-1} and y in L_{i+1} such that u, v and w are adjacent to both x and y. But, then, $< x, u, v, w, y >\cong F_3$ which is a contradiction. □

A sequence S is forbidden in L^* if the consecutive terms of S do not appear consecutively in L^*.

Theorem 6.2. *For every $d \geq 3$, there exists three diameter-maximal line graphs with diameter d.*

Proof. First, we show that the sequence $(a_1, a_2, 2, a_3, a_4)$, where $a_i \in \{1, 2\}$, is forbidden in L^*. For, assuming the contrary, let $|L_i| = 2$ for some i, $2 \leq i \leq d - 2$, and $L_i = \{v_1, v_2\}$. Let v_3, v_4, v_5 and v_6 be arbitrary vertices in L_j, for $j = i - 2, i - 1, i + 1$ and $i + 2$ respectively. But $< v_1, \ldots, v_6 >\cong F_4$ which is a contradiction.

Applying the same argument, we see that the sequences $(a_1, a_2, 2, 2)$, $(2, 2, a_1, a_2)$ and $(2, 2, 2)$ are also forbidden in L^*, so that the integer 2 appears at most twice in L^* and hence either (i) $|L_1| = |L_{d-1}| = 2$, (ii) $|L_1| = 2$ or (iii) all the entries of L^* are 1. Note that the case when L^* has $|L_{d-1}| = 2$ is not considered, as it is similar to (ii). Hence there are only three possible sequences of L^* when $d \geq 3$. As the three sequences are different and the pair (u, v) in Theorem 6.1 is unique, there exist exactly three diameter-maximal line graphs. □

Corollary 6.1. *The root graphs of diameter-maximal line graphs with diameter d are of the form G in Table 1.*

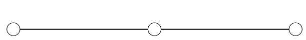
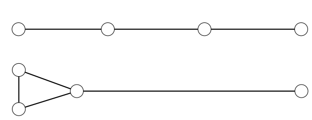
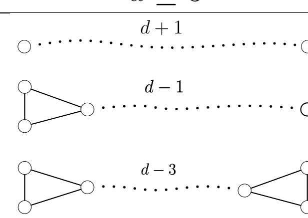
Diameter of $L(G)$	$d = 1$	$d = 2$	$d \geq 3$
G			

Table 1. Graph G, for Corollary 6.1

7. Root graphs of DHL graphs

A graph G is distance-hereditary if for any connected induced subgraph H, $d_H(u,v) = d_G(u,v)$, for any $u, v \in V(H)$. A detailed study can be seen in [5]. A graph G is chordal if every cycle of length at least four in G has an edge(chord) joining two non-adjacent vertices of the cycle [4]. A graph is Ptolemaic if it is both distance-hereditary and chordal [11].

In this section, the family of root graphs of distance-hereditary line (DHL) graphs is obtained. The root graphs of chordal and Ptolemaic graphs are also discussed.

Theorem 7.1. *[5] Let G be a connected graph. Then G is distance-hereditary if and only if the graphs of Fig 4 and the cycles C_n with $n \geq 5$ are forbidden subgraphs of G.*

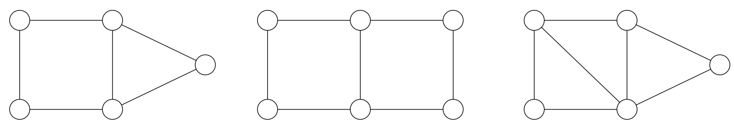


Figure 4. The graphs for Theorem 7.1: house, domino and gem graphs

Theorem 7.2. *[11] Let G be a graph. The following conditions are equivalent*

1. *G is a Ptolemaic graph*
2. *G is distance-hereditary and chordal*
3. *G is chordal and does not contain an induced gem*

A vertex v is simplicial if $N(v)$ is a clique. The ordering $\{v_1, \ldots, v_n\}$ of the vertices of H is a perfect elimination ordering if, for all $i \in \{1, \ldots n\}$, the vertex v_i is simplicial in $H_i =< v_i, \ldots, v_n >$.

Theorem 7.3. *[9]Let G be a graph. The following statements are equivalent:*

1. *G is a chordal graph.*
2. *G has a perfect elimination ordering. Moreover, any simplicial vertex can start a perfect elimination ordering.*

Theorem 7.4. *In a DHL graph if a vertex is adjacent to at least one vertex in a C_4 then it must be adjacent to all the vertices of that C_4 and to no other vertices in the graph.*

Proof. Let H be a DHL graph which contains a C_4 and let a vertex u be adjacent to at least one vertex of the C_4. If u is adjacent to exactly one vertex of C_4 then a $K_{1,3}$ is formed in H, which is a contradiction. Let u be adjacent to exactly two vertices of C_4 . Then either a house, when u is adjacent to two adjacent vertices of C_4, or a $K_{1,3}$, when u adjacent to two non-adjacent vertices of

C_4 is formed, which is also a contradiction. Since an F_2 is obtained when u is adjacent to three vertices of a C_4, u must be adjacent to all the four vertices of the C_4.

Next we show that two adjacent vertices can not be made adjacent to a C_4 in H. For, otherwise each of the two vertices must be adjacent to all the vertices of C_4 and hence induces $C_4 \vee K_2$. But a copy of F_3 is induced in $C_4 \vee K_2$, which is a contradiction. If only one vertex of two adjacent vertices is adjacent to C_4, a $K_{1,3}$ is induced in H which is also a contradiction.

□

Corollary 7.1. *A DHL graph contains at most one C_4.*

Corollary 7.2. *The root graphs of DHL graphs which contain a C_4 are K_4, $K_4 - e$ and C_4.*

Proof. The proof is complete as we see from Corollary 7.1 that the only DHL graphs which contain a C_4 are $C_4 \vee 2K_1$, $C_4 \vee K_1$ and itself. □

As there are only three DHL graphs containing a C_4, we restrict our discussion in the following sections to DHL graphs not containing C_4's.

If H is a DHL graph containing no anti-Gallai triangle then its root graph contains no triangles. Also, a DHL graph is C_n-free, $n \geq 5$. Now, together with Corollary 7.2, we have the following result.

Theorem 7.5. *Let $H \ncong C_4$ be a DHL graph not containing an anti-Gallai triangle, then H is a line graph of a tree.*

Lemma 7.1. *An anti-Gallai triangle in a DHL graph has a vertex of degree two.*

Proof. Let uvx be an anti-Gallai triangle in a DHL graph $H \ncong K_3$. Then uvx is in some $K_4 - e$ in H. Let uvy be a triangle such that $u, x, y, w \cong K_4 - e$. We now show that degree of the vertex x is two. Consider h_x, we just need to show that L_1 contains no vertices other than u and v. For, let w be a vertex in L_1. Then wx is an edge and, by Theorem 3.1, either u or v is adjacent to w. Then y cannot be adjacent to w as $N(w) \cap \{u, v, x, y\}$ together with w induce $C_4 \vee K_1$. But, $<u, v, w, x, y>$ is a gem, a contradiction. □

By lemma 7.1, it now follows that each triangle in the root graph of a DHL graph is attached to the graph by sharing at the most one vertex. Let $\mathcal{T}$ be the family of trees. Let $\mathcal{T}_{\triangle}$ be the family of graphs obtained by attaching some triangles to some vertices in a tree T, for each $T \in \mathcal{T}$.

Theorem 7.6. *A graph G is a root graph of a C_4-free DHL graph if and only if $G \in \mathcal{T}_{\triangle}$.*

Proof. The proof is by induction on the number of edges in a $T \in \mathcal{T}_{\triangle}$. It can be verified that the root graphs of distance-hereditary graphs of size ≤ 3 are in $\mathcal{T}_{\triangle}$ and hence the theorem is true for all $m \leq 3$.

Let $T \in \mathcal{T}_{\triangle}$ has m edges and T is a root graph of a DHL graph. Let T' be a graph in $\mathcal{T}_{\triangle}$ with $E(T') = E(T') \cup \{e\}$. Since T' must be connected, there can be two cases: either (i) the edge e is added as a pendent edge to T or (ii) the edge e is formed by joining two vertices in T.

Let l_e be the vertex in $L(T')$ corresponding to the edge e in T'. In case(i), since e is a pendant edge in T', l_e is simplicial in $L(T')$. We can now show that $L(T')$ is gem-free. If possible let a gem is there in $L(T')$. Since $L(T)$ is distance-hereditary and C_4-free, it is chordal. By Theorem 7.2 $L(T)$ is gem-free, l_e must be a vertex in the induced gem. But, $N(l_e)$ is complete so that l_e is one of the degree two vertices in the gem. Now l_e is in a $K_4 - e$. By Lemma 7.1, one of the two triangles in the $K_4 - e$ must be an anti-Gallai triangle. But the triangle containing l_e cannot be so, as e is a pendant edge in T'. But the other triangle has no vertex of degree 2 in the induced gem. This is a contradiction, by Lemma 7.1, to the assumption that $L(T')$ contains a gem.

In case(ii), as T is connected, adding an edge e joining two vertices of T makes a cycle in T'. But $T \in \mathcal{T}_\triangle$ is C_n-free,$n \geq 4$, and contains no $K_4 - e$. Hence e joins two pendant vertices of T, forming a triangle and has end vertices of degree two. Therefore in $L(T')$, the corresponding vertex l_e is in an anti-Gallai triangle and has degree two. It now follows that l_e is simplicial. If $L(T')$ contains a gem, l_e must be one of the degree two vertices in the induced gem. But in this case the anti-Gallai triangle containing l_e do not satisfy Theorem 3.1 with the other vertex of degree two in the induced gem, which is again a contradiction.

In both the cases we have a one-vertex extension $L(T')$ of a gem-free chordal graph $L(T)$ and hence $L(T')$ is a DHL graph. □

Corollary 7.3. *A graph $L(G)$ is Ptolemaic if and only if $G \in \mathcal{T}_\triangle$*

Corollary 7.4. *Let $\mathcal{T}_\triangle^c$ be the family of graphs obtained by attaching some triangles to some vertices in a tree T and identifying each edge of T by an edge of at most one triangle, for each $T \in \mathcal{T}$. Then $L(G)$ is a chordal graph if and only if $G \in \mathcal{T}_\triangle^c$*

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References

[1] A.V. Aho, J.E. Hopcroft and J.D. Ullman, *The design and analysis of computer algorithms*, Addison Wesley, (1976).

[2] P. Anand, et al., On the hardness of recognizing triangular line graphs, *Discrete Math.* **312** (2012), 2627–2638.

[3] L.S. Aparna, S.B. Rao and A. Vijayakumar, Gallai and anti-Gallai graphs of a graph, *Math. Bohem.* **132** (2007), 43–54.

[4] R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, second edition, Heidelberg: Springer (2012).

[5] H.J. Bandelt and H.M. Mulder, Distance-hereditary graphs, *J. Combin. Theory Ser.* **B 41** (1986), 182–208.

[6] L.W. Beineke, On derived graphs and digraphs, *Beitrage zur Graphentheorie*, Leipzig (1968), 17–23.

[7] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA (1990).

[8] A. D'Atri and M. Moscarini, Distance hereditary graphs, Steiner trees, and connected domination, *SIAM J. Comput.* **17** (1988), 521–538.

[9] D.R. Fulkerson and O.A. Gross, Incidence matrices and interval graphs, *Pacific J. Math.* **15** (1965), 835–855.

[10] T. Gallai, Transitiv Orientierbare Graphen, *Acta math. Acad. Sci. Hung.* **18** (1967), 25–66.

[11] E. Howorka, A characterization of distance-hereditary graphs, *Quart. J. Math. Oxford. Ser.* **28** (1977), 417–420.

[12] J. Krausz, Démonstration nouvelle d'une théorème de Whitney sur les réseaux, *Mat. Fiz. Lapok* **50** (1943), 75–85.

[13] V.B. Le, Gallai graphs and anti-Gallai graphs, *Discrete Math.* **159** (1996), 179–189.

[14] P.G.H. Lehot, An optimal algorithm to detect a line graph and output its root graph, *J. ACM* **21** (1974), 569–575.

[15] L. Sun, Two classes of perfect graphs, *J. Combin. Theory Ser. B* **53** (1991), 273–292.

[16] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), 150–168.