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Enforced hamiltonian cycles in generalized dodecahedra

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Abstract

The H-force number of a hamiltonian graph G is the smallest number k with the property that there exists a set $W \subseteq V(G)$ with $|W| = k$ such that each cycle passing through all vertices of W is a hamiltonian cycle. In this paper, we determine the H-force numbers of generalized dodecahedra.

Keywords: hamiltonian graph, H-force number, generalized dodecahedron
Mathematics Subject Classification : 05C45

Throughout this paper we consider graphs without loops or multiple edges; for terminology not defined here we refer to [3].

Let $G = (V, E)$ be a hamiltonian graph and let W be a nonempty subset of $V(G)$. A cycle in G is a W*-cycle* if it contains all vertices of W. The set W *enforces* a hamiltonian cycle in G (or, W is an *H-force set*) if each W-cycle of G is hamiltonian. The *H-force number* of G, denoted $h(G)$, is the cardinality of the smallest H-force set in G.

The H-force number of a graph was introduced in [4] as a possible tool which unifies several concepts in theory of hamiltonian graphs and allows to develop a kind of hierarchic partition in this graph family. Note that there are several different approaches which develop such a hierarchy like pancyclicity or hamiltonian-connectedness, the other way how to study the quality of a hamiltonian graph is to study the existence of hamiltonian cycles passing through particular edges, see [5], [8]. One possible approach how to classify hamiltonian graphs concerns the notion of k-hamiltonicity:

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given an n-vertex graph G and an integer k, $1 \leq k \leq n-3$, G is *k-hamiltonian* if, for all sets $U \subseteq V$, $0 \leq |U| \leq k$, the graph $G-U$ (obtained from G by deleting all vertices of U) is hamiltonian. In particular, a graph is 1-hamiltonian if it is hamiltonian and the graph that results from deletion of any vertex is also hamiltonian. There are several sufficient conditions for graphs to be 1-hamiltonian, in many cases similar to the classical conditions for hamiltonicity (see [1], [2] or [7]). Note that if a graph is k-hamiltonian for $k \geq 1$, then its H-force number is equal to its order, and vice versa; thus, it is interesting to explore graphs with H-force number being less than their orders. The graphs with small H-force number were studied in [4], which provided the complete characterization of graphs with H-force number 2 (or 3 in the case of 3-connected graphs, and 4 for 3-connected planar graphs, respectively). In general, determining the H-force number of a hamiltonian graph is a difficult problem, even for special graphs. The aim of this paper is to determine the H-force numbers of generalized dodecahedra, i.e. the 3-connected planar cubic graphs consisting of two k-gonal faces separated by the strip of $2k$ pentagons.

Given an integer k, the graph G_k is constructed in the following way: take three cycles $C_O = v_1v_2\dots v_kv_1$, $C_M = v_{k+1}v_{k+2}v_{k+3}\dots v_{3k}v_{k+1}$, and $C_I = v_{3k+1}v_{3k+2}\dots\ v_{4k}v_{3k+1}$ drawn in the plane such that C_M lies in the interior of C_O and C_I lies in the interior of C_M (we refer to C_O, C_M, C_I as the outer, middle and inner cycle of G_k, the above described labelling of vertices will be called *primary* in the sequel). Next, for each $i = 1,\dots,k$, add new edges v_iv_{k+2i-1}, $v_{k+2i}v_{3k+i}$. This can be done is such a way that the resulting graph G_k is plane; it contains two k-gons separated by two layers of $2k$ pentagons in total, is 3-connected and cubic, and has $4k$ vertices and $6k$ edges.

Theorem 1. *Let G_k be a generalized dodecahedron, then*

(i) $h(G_3) = 9$;

(ii) $h(G_5) = 15$;

(iii) $h(G_k) = \frac{11k}{3}$ *if* $k \equiv 0$ (*mod* 3), $k \geq 6$;

(iv) $h(G_k) = 4k-2$ *if* $k \equiv 1$ (*mod* 3);

(v) $h(G_k) = \frac{11k-7}{3}$ *if* $k \equiv 2$ (*mod* 3), $k \geq 8$.

Three edges $a, b, c \in E(G_k)$ are *concurrent* if $a \in E(C_O)$, $b \in E(C_M)$, $c \in E(C_I)$ and they belong to two adjacent pentagons with b being their common edge. This term will be also used for any two edges of a concurrent triple.

From the geometrical point of view, when constructing G_k, we can arrange cycles C_M, C_O and C_I in such a way that their drawings are regular polygons, their circumscribed circles are concentric and each half line originating from the common centre of circles intersects either no vertex (and in this case it intersects three edges, one of each polygon, that accord to a concurrent triple in G_k) or exactly two vertices of the polygons (according to adjacent vertices of G_k), see Fig. 1.

Note that, in G_k, there exists an automorphism that maps any vertex $v_i \in V(C_I) \cup V(C_O)$ to arbitrary vertex $v_j \in V(C_I) \cup V(C_O)$ as well as an automorphism that maps any $v_\ell \in V(C_M)$ to any $v_m \in V(C_M)$.

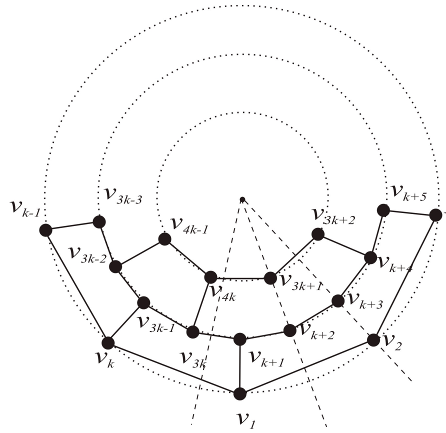

Fig. 1

Let $v_rv_{r+1}, v_sv_{s+1}, v_tv_{t+1} \in E(G_k)$ be three concurrent edges (for the case $v_sv_t \in E(G_k)$ see Fig. 2A, analogously the case $v_rv_s \in E(G_k)$). If we replace the path v_r, v_{r+1} by the path $v_r, v_{r_1^*}, v_{r+1}$, the path v_s, v_{s+1} by the path $v_s, v_{s_1^*}, v_{s_2^*}, v_{s+1}$, the path v_t, v_{t+1} by the path $v_t, v_{t_1^*}, v_{t+1}$, and add two new edges $v_{r_1^*}v_{s_1^*}$, $v_{s_2^*}v_{t_1^*}$ then we obtain the graph G_{k+1}. We say that we *enlarge* the graph G_k to G_{k+1} on the concurrent triple $v_rv_{r+1}, v_sv_{s+1}, v_tv_{t+1}$. Repeating this operation, G_k can be enlarged to G_{k+2}, G_{k+3}, etc. (Note that $V(G_k) \subseteq V(G_{k+1}) \subseteq V(G_{k+2}) \subseteq V(G_{k+3}) \subseteq \dots$)

In the sequel, a cycle C of a graph G *misses exactly the vertices* of a set $S \subset V(G)$, $|S| \leq |V(G)| - 3$, if $V(C) = V(G) \setminus S$. Note, that if C is nonhamiltonian cycle of G then any H-force set of G contains a vertex of $G - C$.

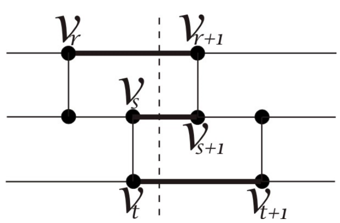

Fig. 2A

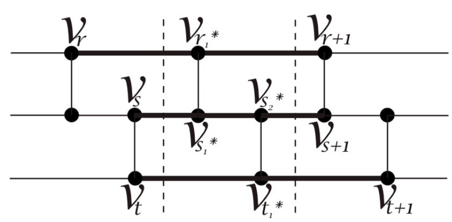

Fig. 2B

Lemma 2. *Let a, b, c be three concurrent edges of G_k and let G_{k+1} (G_{k+3}) be enlarged from G_k on this triple. If C is a cycle of G_k containing all three edges a, b, c (any two of them), then in G_{k+1} (G_{k+3}) there is a cycle C^* missing exactly the same vertices as C in G_k; moreover, C^* contains three (two) concurrent edges as well.*

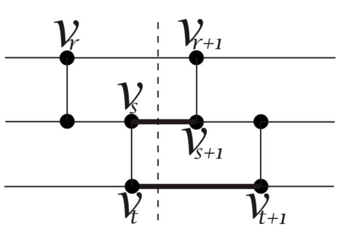

Fig. 3A

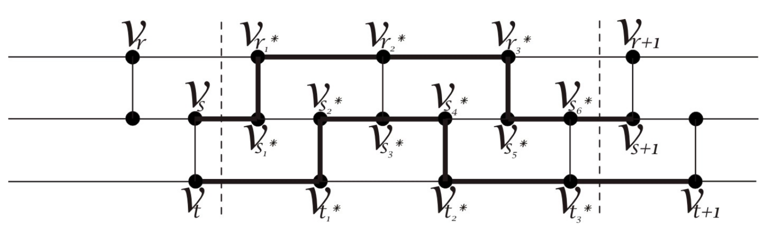

Fig. 3B

Proof. Let C be a cycle of G_k, let $S \subset V(G_k)$ be the set of vertices missed by C, let $v_r v_{r+1} \in E(C_I)$, $v_s v_{s+1} \in E(C_M)$, $v_t v_{t+1} \in E(C_O)$ be concurrent edges, and let G_{k+1} and G_{k+3} be enlarged from G_k on the mentioned triple of edges (see Fig. 2A, 2B and 3A, 3B, respectively).

1. If C contains all three concurrent edges $v_r v_{r+1}$, $v_s v_{s+1}$, $v_t v_{t+1}$, then replace in C the path v_r, v_{r+1} by the path $v_r, v_{r_1^*}, v_{r+1}$, the path v_s, v_{s+1} by the path $v_s, v_{s_1^*}, v_{s_2^*}, v_{s+1}$, and the path v_t, v_{t+1} by the path $v_t, v_{t_1^*}, v_{t+1}$ to create the cycle C^* in G_{k+1} containing all four new vertices and thus missing exactly the vertices of S (Fig. 2B). Moreover C^* contains three concurrent edges of G_{k+1} (for example $v_r v_{r_1^*}$, $v_s v_{s_1^*}$, $v_t v_{t_1^*}$).
2. (a) If C contains exactly two concurrent edges $v_s v_{s+1}$, $v_t v_{t+1}$ (similarly for $v_r v_{r+1}, v_s v_{s+1} \in E(C)$) then replace in C the path v_s, v_{s+1} by the path $v_s, v_{s_1^*}, v_{r_1^*}, v_{r_2^*}, v_{r_3^*}, v_{s_5^*}, v_{s_6^*}, v_{s+1}$ and the path v_t, v_{t+1} by the path $v_t, v_{t_1^*}, v_{s_2^*}, v_{s_3^*}, v_{s_4^*}, v_{t_2^*}, v_{t_3^*}, v_{t+1}$ to create the cycle C^* in G_{k+3} containing all 12 new vertices and thus missing exactly the vertices of S (Fig. 3B). Moreover C^* contains two concurrent edges of G_{k+3} (for example $v_s v_{s_1^*}$, $v_t v_{t_1^*}$).
   (b) If C contains exactly two concurrent edges $v_r v_{r+1}$, $v_t v_{t+1}$ then replace in C the path v_r, v_{r+1} by the path $v_r, v_{r_1^*}, v_{s_1^*}, v_{s_2^*}, v_{s_3^*}, v_{r_2^*}, v_{r_3^*}, v_{r+1}$ and the path v_t, v_{t+1} by the path $v_t, v_{t_1^*}, v_{t_2^*}, v_{s_4^*}, v_{s_5^*}, v_{s_6^*}, v_{t_3^*}, v_{t+1}$ to create the cycle C^* in G_{k+3} containing all 12 new vertices and thus missing exactly the vertices of S. Moreover C^* contains two concurrent edges of G_{k+3} (for example $v_r v_{r_1^*}$, $v_t v_{t_1^*}$).

Now, the vertices in G_{k+1} and G_{k+3} can be relabelled to obtain primary labelling. □

In the next, $d_H(x, y)$ denotes the distance of x, y with respect to the graph H.

Lemma 3. *Let $k \equiv 0$ (mod 3), $k \geq 6$, and let $v_i, v_j \in V(C_M) \cap V(G_k)$ such that $2 \neq d_{C_M}(v_i, v_j) \equiv \pm 2$ (mod 6). Then there exists a cycle in G_k that misses exactly the vertices v_i, v_j.*

Proof. Because of symmetry of G_k and the condition $2 \neq d_{C_M}(v_i, v_j) \equiv \pm 2$ (mod 6) we can assume that $i = k + 1$ and $k + 5 \leq j \leq 2k$. For $j - i = j - k - 1 \equiv \pm 2$ (mod 6) we prove that there exists a cycle in G_k that misses exactly two vertices v_i, v_j. Note that any such cycle contains the following two pairs of concurrent edges: $v_1 v_2$, $v_{k+2} v_{k+3}$ and $v_k v_1$, $v_{3k-1} v_{3k}$.
For $k = 6$ is $i = 7$, $j = 11$ and a desired cycle in G_6 is shown on Fig. 4.
For $k \geq 9$ we consider a cycle C' in G_{k-3} that misses exactly two vertices $v_{k-2}, v_{j-3} \in V(G_{k-3}) \cap V(C_M)$ with the distance (on the cycle C_M in G_{k-3}) $j - 3 - (k - 2) = j - k - 1 \equiv \pm 2$ (mod 6) for $j \leq 2k - 2$ or distance $2k - 6 - (j - k - 1) \equiv \mp 2$ (mod 6) for $2k - 1 \leq j \leq 2k$. The cycle C' contains concurrent edges $v_{k-3} v_1$, $v_{3k-10} v_{3k-9}$ and by previous lemma we obtain a desired cycle in G_k. □

Later we use this lemma in the following way: If W is an H-force set in G_k, $k \equiv 0$ (mod 3), that does not contain a vertex $v_i \in V(C_M)$, then every vertex $v_j \in V(C_M)$ with $2 \neq d_{C_M}(v_i, v_j) \equiv \pm 2$ (mod 6) belongs to W.

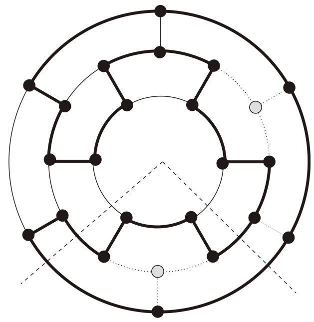

Fig. 4

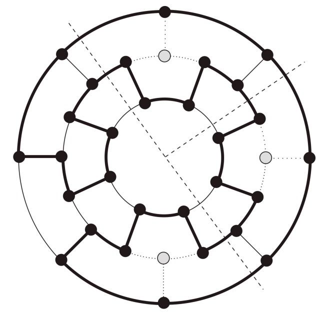

Fig. 5

Lemma 4. *Let $k \equiv 2$ (mod 3), $k \geq 8$, and let $v_i, v_j, v_\ell \in V(C_M) \cap V(G_k)$ such that these vertices split C_M into three paths of lengths at least 4 and congruent to 4, 4 and 2 modulo 6. Then there exists a cycle in G_k that misses exactly the vertices v_i, v_j, v_ℓ.*

Proof. Because of symmetry of G_k we can assume that $k+1 = i < j < \ell$. The vertices $v_i, v_j, v_\ell \in V(C_M)$ split the cycle C_M into three paths of lengths $p_i := j - i \equiv 4 \pmod 6$, $p_j := \ell - j \equiv 4 \pmod 6$, and $p_\ell := 3k + 1 - \ell \equiv 2 \pmod 6$, $p_\ell \neq 2$.
For $k = 8$ is $i = 7$, $j = 11$, $\ell = 15$, and a desired cycle in G_8 is shown on Fig. 5.
For $k \geq 11$ one of p_i, p_j, p_ℓ must be at least 10.

1. If $p_i \geq 10$ then we consider a cycle C' in G_{k-3} that misses exactly three vertices $v_{k-2} = v_{i-3}, v_{j-9}, v_{\ell-9}$ splitting the middle cycle of G_{k-3} into three paths of length $p_i - 6, p_j, p_\ell$ (congruent to 4, 4, and 2 modulo 6). The cycle C' must contain concurrent edges v_1v_2 and $v_{k-1}v_k$. Through enlargement the graph G_{k-3} to G_k on the triple given by mentioned two edges we obtain, by Lemma 2, a desired cycle in G_k.
2. If $p_i = 4$ and $p_j \geq 10$ then we consider a cycle C' in G_{k-3} that misses exactly three vertices $v_{i-3}, v_{j-3}, v_{\ell-9}$ splitting the middle cycle of G_{k-3} into three paths of length $p_i, p_j - 6, p_\ell$ (congruent to 4, 4, and 2 modulo 6). The cycle C' must contain following two concurrent edges: $v_{j-2}v_{j-1}$ and the corresponding edge from the outer cycle of G_{k-3}. By Lemma 2 we obtain a desired cycle in G_k.
3. If $p_i = p_j = 4$ and $p_\ell \geq 10$ then we consider a cycle C' in G_{k-3} that misses exactly three vertices $v_{i-3}, v_{j-3}, v_{\ell-3}$ splitting the middle cycle of G_{k-3} into three paths of length $p_i, p_j, p_\ell - 6$ (congruent to 4, 4, and 2 modulo 6). The cycle C' must contain following two concurrent edges: $v_{\ell-2}v_{\ell-1}$ and the corresponding edge from the outer cycle of G_{k-3}. By Lemma 2 we obtain a desired cycle in G_k.

□

In a similar way, one can prove

Lemma 5. *Let $k \equiv 2$ (mod 3) and let $R \subseteq V(G_k)$ be*

1. *the set of two vertices $v_i \in V(C_M)$ and $v_j \in V(C_I)$ where v_i is adjacent to a vertex on C_O and $2 \neq d(v_i, v_j) \not\equiv 1$ (mod 3) (see Fig. 6), or*

2. *the set of three vertices* $v_i, v_j, v_\ell \in V(C_O)$ *such that these vertices split* C_O *into three paths of lengths congruent to* 1, 1 *and* 0 *modulo* 3 *(see Fig. 7), or*
3. *the set of three vertices* $v_i, v_j \in V(C_O)$ *and* $v_\ell \in V(C_M)$ *where both* v_j *and* v_ℓ *are adjacent to* v_i *(see Fig.* 8*).*

Then there exists a cycle in G_k *that misses exactly the vertices of* R*.*

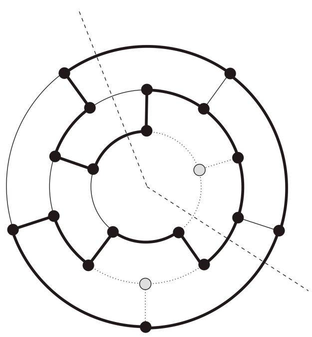

Fig. 6

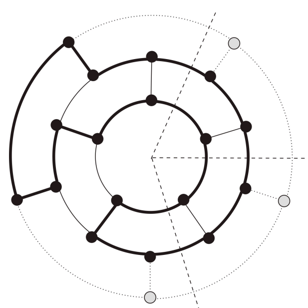

Fig. 7

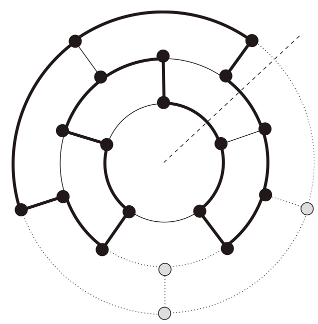

Fig. 8

Lemma 6. *Let* $k \equiv 0$ $(mod\ 3)$ *and let* $R \subseteq V(G_k)$ *be*

1. *the set of one vertex of* C_O *(see Fig.* 9*), or*
2. *the set of two adjacent vertices of* C_M *(see Fig.* 10*).*

Then there exists a cycle in G_k *that misses exactly the vertices of* R*.*

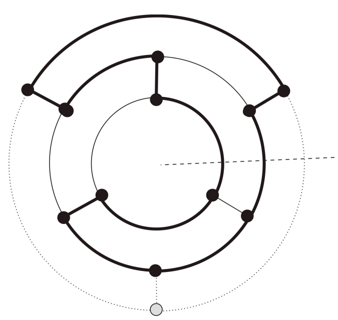

Fig. 9

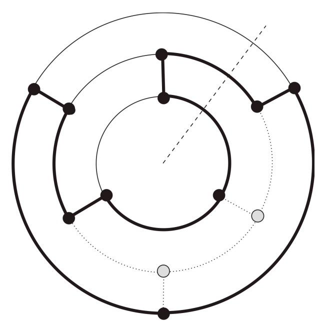

Fig. 10

Lemma 7. *Let* $k \equiv 1$ $(mod\ 3)$ *and let* $R \subseteq V(G_k)$ *be*

1. *a cycle that misses exactly one vertex of* C_M *(see Fig.* 11*), and*
2. *a cycle that misses exactly two vertices* $v_i \in V(C_O)$ *and* $v_j \in V(C_I)$ *with* $d(v_i, v_j) = 4$ *(see Fig.* 12*), and*
3. *a cycle that misses exactly two vertices* $v_i, v_j \in V(C_O)$ *with* $d_{C_O}(v_i, v_j) \not\equiv 2$ $(mod\ 3)$ *(see Fig.* 13*).*

Then there exists a cycle in G_k *that misses exactly the vertices of* R*.*

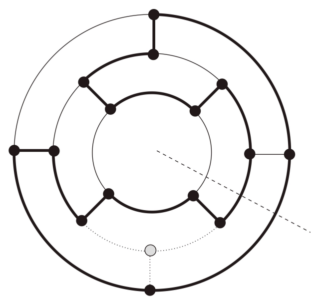

Fig. 11

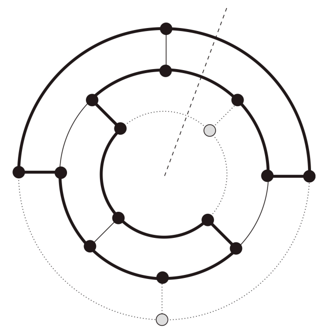

Fig. 12

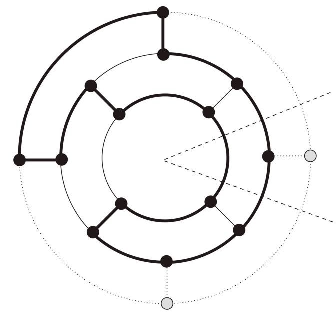

Fig. 13

Lemma 8. *Let $k \geq 3$ and let $R \subseteq V(G_k)$ be*

1. *the set of two vertices $v_i \in V(C_M)$ and $v_j \in V(C_O)$ where v_i is adjacent to a vertex of C_O and $d(v_i, v_j) \geq 3$ (see Fig. 14), or*
2. *the set of two vertices $v_i \in V(C_O)$ and $v_j \in V(C_I)$ with $d(v_i, v_j) \neq 4$ (see Fig. 15A, 15B), or*
3. *the set of two vertices $v_i, v_j \in V(C_M)$ with $1 \neq d_{C_M}(v_i, v_j) \equiv 1 \pmod 2$ (see Fig. 16), or*
4. *the set of three vertices $v_i, v_j \in V(C_M)$ and $v_\ell \in V(C_O)$ where both v_j and v_ℓ are adjacent to v_i (see Fig. 17).*

Then there exists a cycle in G_k that misses exactly the vertices of R.

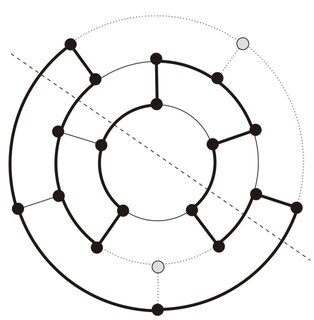

Fig. 14

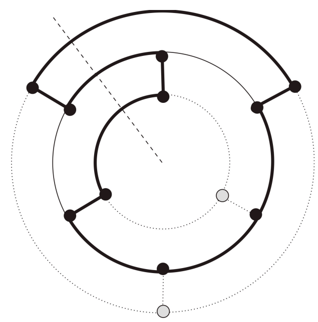

Fig. 15A

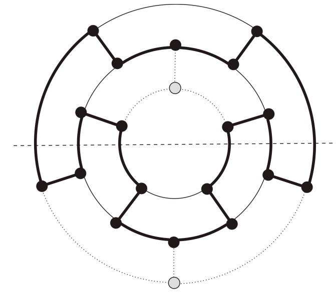

Fig. 15B

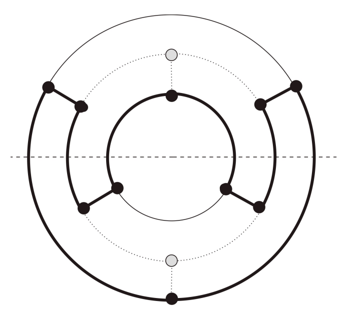

Fig. 16

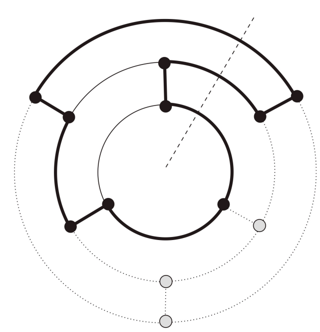

Fig. 17

In the following, we will use the necessary condition for hamiltonicity of plane graphs:

Theorem 9 (Grinberg [6]). *Let G be a plane hamiltonian graph and C be a hamiltonian cycle in G. Let g_i be the number of i-gonal faces in the exterior of C and f_i be the number of i-gonal faces in the interior of C. Then $\sum_{i\geq 3}(i-2)(f_i-g_i)=0$.*

Proof of Theorem 1.

For $1 \leq j \leq 2k$, denote the edges of G_k not belonging to the cycles C_O, C_M, or C_I, as follows

$$a_j = \begin{cases} v_{k+j}v_{\frac{j+1}{2}}, & \text{if } j \text{ odd,} \\ v_{k+j}v_{3k+\frac{j}{2}}, & \text{if } j \text{ even.} \end{cases}$$

Case (ii) The result for dodecahedron ($k = 5$) was proved in [4]. A smallest H-force set is $V(G_5) - \{v_6, v_8, v_{10}, v_{12}, v_{14}\}$.

Case (v) Let $k \equiv 2 \pmod 3$, $k \geq 8$.
Let $S = \{v_{k+1}, v_{3k+1}, v_{3k+3}, v_{3k+6}, v_{3k+9}, \ldots, v_{3k+k-2}, v_{4k}\}$. First, we show that the set $Z = V(G_k) \setminus S$ is an H-force set, that is, for any nonempty subset T of S, there is no cycle that misses exactly the vertices of T (i.e. there is no nonhamiltonian Z-cycle in G_k). In the second step we prove that Z is a smallest H-force set, which will mean $h(G_k) = |Z| = \frac{11k-7}{3}$.

1. Let $T = \{v_j\}$, $v_j \in V(C_M)$. We prove that in the graph G_k, there is no cycle that misses exactly one vertex $v_j \in V(C_M)$. Suppose there exists such cycle. Then the graph $G_k - v_j$ is hamiltonian and, in this graph, $f_i + g_i \neq 0$ holds only for $i \in \{5, 9, k\}$; moreover,

   $$f_5 + g_5 = 2k-3, \qquad f_9 + g_9 = 1, \qquad f_k + g_k = 2.$$

   Then $0 = \sum(i-2)(f_i - g_i) = 3(f_5 - g_5) + (k-2)(f_k - g_k) + 7(f_9 - g_9) \equiv \pm 7 \pmod 3$, a contradiction.
2. Let $T = \{v_j\}$, $v_j \in V(C_O) \cup V(C_I)$. We prove that in the graph G_k, there is no cycle that misses exactly one vertex of C_O (similarly for C_I). Suppose that $G_k - v_j$ is hamiltonian. In this graph we have $f_i + g_i \neq 0$ only for $i \in \{5, k, k+4\}$, and furthermore,

   $$f_5 + g_5 = 2k-2, \qquad f_k + g_k = 1, \qquad f_{k+4} + g_{k+4} = 1.$$

   Hence $0 = \sum(i-2)(f_i - g_i) = 3(f_5 - g_5) + (k-2)(f_k - g_k) + (k+2)(f_{k+4} - g_{k+4}) \equiv \pm(k+2) \equiv \pm 1 \pmod 3$, a contradiction.
3. Let $v_{k+1} \in T$. We prove that no cycle in the graph G_k misses the vertex v_{k+1} and some other vertex from S. Suppose to the contrary that there is a Z-cycle C of G_k with $v_{k+1} \notin V(C)$.
   (a) Let $v_{3k+1}v_{4k} \in E(C)$ and let $a_j \notin E(C)$, for all j with $3 \leq j \leq 2k-1$, $j \equiv 0 \pmod 3$. All edges of C are in this case uniquely determined, but ultimately we obtain two disjoined cycles, a contradiction.

(b) Let $v_{3k+1}v_{4k} \in E(C)$ and let j be the smallest integer with $3 \leq j \leq 2k-1$, $j \equiv 0 \pmod 3$ and $a_j \in E(C)$. The structure of layers of 5-gons of G_k yields that all edges of C are uniquely determined. For j odd, C contains the path $v_{3k-1}, v_{3k}, v_{4k}, v_{3k+1}, v_{k+2}, v_{k+3}, v_{k+4}, v_{3k+2}, v_{3k+3}, v_{3k+4}, v_{k+8}, \ldots, v_{3k+\frac{j-1}{2}}, v_{k+j-1}, v_{k+j}, v_{\frac{j+1}{2}}, v_{\frac{j-1}{2}}, \ldots, v_4, v_{k+7}, v_{k+6}, v_{k+5}, v_3, v_2, v_1, v_k$. Subsequently C does not contain any other a_j with $3 \leq j \leq 2k-1$, $j \equiv 0 \pmod 3$ and does not contain the edges $v_{k+j}v_{k+j+1}$ (since C contains the path $v_{k+j-1}, v_{k+j}, v_{\frac{j+1}{2}}$) and $v_{\frac{j+1}{2}}v_{\frac{j+3}{2}}$ as well (since C contains the path $v_{k+j}, v_{\frac{j+1}{2}}, v_{\frac{j-1}{2}}$). Therefore C contains the edges $v_{k+j+3}v_{k+j+2}$ (since $a_{j+1} = v_{k+j+3}v_{3k+\frac{j+3}{2}} \notin E(C)$), $v_{k+j+1}v_{k+j+2}$ (since $v_{k+j}v_{k+j+1} \notin E(C)$), and $v_{k+j+2}v_{\frac{j+3}{2}}$ (since $v_{\frac{j+1}{2}}v_{\frac{j+3}{2}} \notin E(C)$), a contradiction (analogously for j even).

(c) Let $v_{3k+1}v_{4k} \notin E(C)$. Similarly as in the cases (a) and (b), there is no supposed Z-cycle in G_k.

4. Let $v_{3k+1}, v_{4k} \in T$ and $v_{k+1} \notin T$ (i.e. $v_{k+1} \in V(C)$). Analogously to the previous case 3, the edges of C are uniquely determined and they induce two disjoined cycles, if $a_j \notin E(C)$, for all j with $3 \leq j \leq 2k-1$, $j \equiv 0 \pmod 3$, a contradiction. Otherwise, for the smallest integer j with $3 \leq j \leq 2k-1$, $j \equiv 0 \pmod 3$ such that $a_j \in E(C)$, the cycle C does not contain the vertex $v_{\frac{j+3}{2}} \in Z$, if $j \equiv 3 \pmod 6$, or the vertex $v_{3k+\frac{j}{2}+1} \in Z$, if $j \equiv 0 \pmod 6$, a contradiction.
5. Let $v_{3k+1} \in T$ and $v_{4k}, v_{k+1} \notin T$ (i.e. $v_{4k}, v_{k+1} \in V(C)$). Analogously to the previous cases 3 and 4, if $a_j \notin E(C)$, for all j with $3 \leq j \leq 2k-1$, $j \equiv 0 \pmod 3$, then necessarily $v_1v_2, v_1v_k, v_1v_{k+1} \in E(C)$, a contradiction.
Otherwise, for the smallest integer j with $3 \leq j \leq 2k-1$, $j \equiv 0 \pmod 3$ and $a_j \in E(C)$, the cycle C does not contain vertex v_{k+j+3} or vertex v_{3k-1} belonging to Z, a contradiction. Especially for $j = 3$, let t with $t \equiv 0 \pmod 3$, $j < t < 2k$, be the smallest integer with $a_t \notin E(C)$. Then the cycle C does not contain vertex $v_{\frac{t}{2}}$ (if t even) or vertex $v_{3k+\frac{t-1}{2}}$ or vertex v_{k+t+3} (if t odd), a contradiction. If there is no such t (i.e. if all edges a_t with $t \equiv 0 \pmod 3$, $j < t < 2k$, belong to E(C)) then $W = V(G) \setminus \{v_{3k+1}\}$ and there is no cycle in G_k missing exactly one vertex, a contradiction.

Thus, we just proved that Z is an H-force set.
Now, let W be a smallest H-force set in G_k. We will show that $|W| \geq |Z|$.

1. Let $(V(C_O) \cup V(C_I)) \subseteq W$. Without loss of generality, let $v_{k+1} \notin W$ (otherwise, because of symmetry of G_k, all vertices of C_M belong to W, thus $W = V(G)$). As there exists (by Lemma 8 (3)) a cycle that misses exactly two vertices $v_{k+1}, v_j \in V(C_M)$ with $1 \neq d_{C_M}(v_{k+1}, v_j) \equiv 1 \pmod 2$, all these vertices v_j belong necessarily to W.
   (a) Suppose $v_{k+2} \notin W$. Possibly except of v_{k+3} and v_{3k}, the set W contains all vertices of $V(C_M)$ (again by Lemma 8 (3)), thus $|W| \geq 4k-4 \geq |Z|$.
   (b) Suppose $v_{k+5} \notin W$. By Lemma 8 (3), for $v_i = v_5$, the set W contains the vertices v_{k+2} and v_{3k}. As there exists (by Lemma 4) a cycle that misses exactly three vertices $v_i, v_j, v_\ell \in V(C_M)$ splitting C_M into three paths of lengths at least 4 and congruent to 4, 4 and 2 modulo 6, the set W contains all vertices $v_{k+m} \in V(C_M)$ where $m \not\equiv 5 \pmod 6$, $1 \leq m \leq 2k$, or $m \notin \{k+1, k+3, k+7, 3k-1\}$. By repeated use of

Lemma 4 (for $v_i = v_{3k-1}$, $v_j = v_3$, $v_\ell = v_7$), the set W contains at least one of these three vertices as well. Thus $|W| \geq \frac{11k-7}{3} = |Z|$.

(c) Let $v_{k+2}, v_{3k} \in W$ and let $v_j \notin W$ where $2 \neq d_{C_M}(v_{k+1}, v_j) \equiv 2 \pmod 6$. Then by Lemma 4, $v_{k+m} \in W$ for $m \equiv 5 \pmod 6$, $5 \leq m \leq j-k-4$ and for $m \equiv 1 \pmod 6$, $j-k+4 \leq m \leq 2k-3$. For each mentioned m, the vertices v_{k+m-2} and v_{k+m+2} lie on C_M at distance 4, thus, one vertex of each pair v_{k+m-2}, v_{k+m+2} must belong to W (otherwise we have two vertices from C_M at distance 4 and not belonging to W - already considered in (b)). Ultimately we have $|W| \geq \frac{11k-4}{3} \geq |Z|$.

(d) Let $v_{k+2}, v_{3k}, v_{k+5}, v_{3k-3} \in W$ and let $v_j \in W$ for $2 \neq d_{C_M}(v_{k+1}, v_j) \equiv 2 \pmod 6$. The vertices v_{k+m-2} and v_{k+m+2} for $m \equiv 2 \pmod 6$, $9 \leq m \leq 2k-7$ lie on C_M at distance 4, thus, similarly as in the previous case, one vertex of each pair v_{k+m-2}, v_{k+m+2} must belong to W. From the same reason, one of the vertices v_{k+3}, v_{3k-1} belongs to W as well. Ultimately we have $|W| \geq \frac{11k-1}{3} \geq |Z|$.

2. Let $(V(C_M) \cup V(C_I)) \subseteq W$. Without loss of generality, let $v_1 \notin W$ (otherwise, because of symmetry of G_k, all vertices of C_O belong to W, thus $W = V(G)$).
   (a) Suppose $v_j \notin W$ where $d_{C_O}(v_1, v_j) \equiv 1 \pmod 3$. As there exists (by Lemma 4) a cycle that misses exactly three vertices $v_i, v_j, v_\ell \in V(C_M)$ splitting C_O into three paths of lengths congruent to 1, 1 and 0 modulo 3, the set W contains all vertices $v_m \in V(C_O)$ where $m \not\equiv 0 \pmod 3$, $1 < m < j$, and all vertices $v_m \in V(C_O)$ where $m \not\equiv 1 \pmod 3$, $j < m < k$. Thus $|W| \geq \frac{11k-4}{3} \geq |Z|$.
   (b) Suppose that all $v_j \in V(C_O)$ with $d_{C_O}(v_1, v_j) \equiv 1 \pmod 3$ belong to W. The remaining vertices of C_O are pairwise adjacent, thus, at least half of them belongs to W (otherwise we have two adjacent vertices from C_O not belonging to W - already considered in (a)) and we obtain $|W| \geq \frac{11k-1}{3} \geq |Z|$.
3. Let $V(C_M) \subseteq W$ and let vertices $v_i \in V(C_O)$ and $v_j \in V(C_I)$ do not belong to W. Then by Lemma 8 (2) we have $|W| \geq 4k-4 \geq |Z|$.
4. Let $v_{k+1} \notin W$. Then by Lemma 8 (1), $v_3, \dots, v_{k-1} \in W$. Furthermore, by Lemma 8 (3), the set W contains also the vertices v_{k+m} for m even and $4 \leq m \leq 2k-2$. Finally, by Lemma 5 (1), W contains the vertices v_{3k+m} for $m \not\equiv 0 \pmod 3$ and $2 \leq m \leq k-1$ as well.
   (a) Suppose $v_1 \notin W$. Then using Lemma 5 (3) and Lemma 8 (1),(2),(4) we obtain $|W| \geq 4k-4$.
   (b) Suppose $v_1 \in W$ and $v_2 \notin W$. Then using Lemma 8 (1),(2) we get $|W| \geq 4k-8$.
   (c) Suppose $v_{k+2} \notin W$. Then using Lemma 8 (1),(3) we obtain $|W| \geq 4k-8$.

   Due to symmetry of G_k, the vertices $v_k \in V(C_O)$ and $v_{3k} \in V(C_M)$ belong to W as well, i.e. $V(C_O) \subseteq W$.

   (d) Suppose W does not contain a vertex $v_j \in V(C_I)$. Then by Lemma 5 (1), all vertices $v_\ell \in C_M$ with $2 \neq d(v_j, v_\ell) \not\equiv 1 \pmod 3$ belong to W. The remaining vertices (i.e. $\{v_{3k+m} : m \equiv 0 \pmod 3, 2 \leq m \leq k-1\} \cup \{v_{3k+1}, v_{4k}\} \setminus \{v_j\} \subseteq V(C_I)$ and $\{v_{k+m} : m \equiv 2j-6k-1+6t \pmod{2k}, 0 \leq t \leq \frac{2k+2}{6}\} \setminus \{v_{k+1}\} \subseteq V(C_M)$) form pairs in the distance 5. By Lemma 5 (1), one vertex of each pair must belong to W, thus $|W| \geq \frac{11k-7}{3} \geq |Z|$.

We showed, that the smallest H-force set of G_k contains at least $\frac{11k-7}{3}$ vertices.

Case (i) For $k = 3$ it is easy to check that $h(G_3) = 9$, the set $V(G_3) \setminus \{v_4, v_6, v_8\}$ is a smallest H-force set in G_3.

Case (iii) Let $k \equiv 0 \pmod 3$, $k \geq 6$. Let W be an arbitrary H-force set in G_k.
In G_k, there exists a cycle that misses exactly one vertex of C_O (by Lemma 6 (1)). Hence, all vertices of C_I and C_O belong to W.

Without loss of generality, let $v_{k+1} \notin W$. We already know, that there exist cycles in G_k missing exactly two vertices

(a) v_{k+1} and $v_j \in V(C_M)$ with $d_{C_M}(v_{k+1}, v_j) \equiv 1 \pmod 2$ (by Lemma 8 (3) and 6 (2)), or
(b) v_{k+1} and $v_j \in V(C_M)$ with $2 \neq d_{C_M}(v_{k+1}, v_j) \equiv \pm 2 \pmod 6$ (by Lemma 3).

Hence, for $S = \{v_{k+1}\} \cup \{v_j : d_{C_M}(v_{k+1}, v_j) \equiv 0 \pmod 6\}$ we just found out that $V(G_k) \setminus S \subseteq W$. Now we prove that $V(G_k) \setminus S$ is an H-force set.

In G_k, there is no cycle that misses exactly one vertex v_{k+1}. Assume to the contrary that the graph $G_k - v_{k+1}$ is hamiltonian. We have $f_i + g_i \neq 0$ only for $i \in \{5, 9, k\}$, moreover

$$f_5 + g_5 = 2k - 3, \qquad f_9 + g_9 = 1, \qquad f_k + g_k = 2.$$

If $f_k - g_k = 0$, then $0 = \sum (i-2)(f_i - g_i) = 3(f_5 - g_5) + (k-2)(f_k - g_k) + 7(f_9 - g_9) \equiv \pm 7 \pmod 3$, a contradiction.
Otherwise $f_k - g_k = \pm 2$ and $0 = \sum (i-2)(f_i - g_i) = 3(f_5 - g_5) \pm 2(k-2) \pm 7$. Thus $\pm 2(k-2) \pm 7 \equiv 0 \pmod 3$ and the equality from the Grinberg's theorem is fulfilled only if both k-gons with 9-gon appear in the same region determined by a hamiltonian cycle of $G_k - v_{k+1}$. On the other hand, edges v_1v_2, v_kv_1 belong to any hamiltonian cycle and obviously separate one k-gon and the 9-gon, a contradiction.

Similarly as in the case (v), there is no cycle in G_k missing vertex v_{k+1} and some other vertices from S.

Case (iv) Let $k \equiv 1 \pmod 3$, $k \geq 4$. Let W be an arbitrary H-force set in G_k.
In G_k, there exists a cycle that misses exactly one vertex of C_M (by Lemma 7 (1)). Hence, all vertices of C_M belong to W.

Without loss of generality, let $v_1 \notin W$. In G_k there exist cycles missing exactly two vertices

(a) v_1 and $v_j \in V(C_I)$ (by Lemma 7 (2) and 8 (2)), or
(b) v_1 and $v_j \in V(C_O)$ with $d_{C_O}(v_1, v_j) \not\equiv 2 \pmod 3$ (by Lemma 7 (3)).

Hence, for $S = \{v_1\} \cup \{v_j : d_{C_O}(v_1, v_j) \equiv 2 \pmod 3\}$ we just found out that $V(G_k) \setminus S \subseteq W$.

If there is some $v_j \in S \setminus \{v_1\}$ that does not belong to W, then, according to previous observations, all other vertices of $S \setminus \{v_1\}$ belong necessarily to W because their distance from v_j is congruent to 0 or 1 modulo 3.

In G_k, there is no cycle that misses exactly one vertex $v_1 \in V(C_O)$. Assume to the contrary that the graph $G_k - v_1$ is hamiltonian. We have $f_i + g_i \neq 0$ only for $i \in \{5, k, k+4\}$, moreover

$$f_5 + g_5 = 2k - 2, \qquad f_k + g_k = 1, \qquad f_{k+4} + g_{k+4} = 1.$$

then $0 = \sum(i-2)(f_i - g_i) = 3(f_5 - g_5) + (k-2)(f_k - g_k) + (k+2)(f_{k+4} - g_{k+4}) \equiv \pm(k-2) \equiv \mp 1 \pmod 3$, a contradiction.

Moreover, there is no cycle that misses exactly the vertices v_1 and v_3, because v_2 is a pendant vertex in $G_k - v_1 - v_3$.

Hence, the set $V(G_k) \setminus \{v_1, v_3\}$ is a smallest H-force set. $\square$

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