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Graceful labeling of triangular extension of complete bipartite graph

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Abstract

For positive integers m, n, $K_{m,n}$ represents the complete bipartite graph. We name the graph $G = K_{m,n} \odot K_2$ as triangular extension of complete bipartite graph $K_{m,n}$, since there is a triangle hanging from every vertex of $K_{m,n}$. In this paper we show that G is graceful when $m = n = 2\ell$, for any integer ℓ.

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1. Introduction

Let $G = (V, E)$ be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$, where $e = uv \in E(G)$ if and only if edge e connects vertex u to vertex v. In this paper $K_{m,n}$ denotes a regular complete bipartite graph. For all other terminology and notations we follow [3]. A function f is called a graceful labeling of a graph G if $f : V(G) \to \{0, 1, 2, \cdots, |E(G)|\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, \cdots, |E(G)|\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. This type of graph labeling, first introduced by Rosa [1] in 1967 as a β-valuation, was used as an efficient tool for decomposing a complete graph into isomorphic subgraphs. Even though Graham and Sloane [8] claimed that most graphs are not graceful, it is still an interesting problem to identify which graphs are graceful. However, as per the rigorous

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survey by Gallian [4], it is obvious that a lot of work has been devoted in seeking the answers to this problem for different family of graphs, and substantial progress in this area has been made in last few decades, but there are still numerous number of families of graphs of important structures for which the answer must be found for future reference. Both Rosa [1] and Golomb [11] proved that complete bipartite graphs are graceful. Also it is known that K_n is graceful if and only if $n \leq 4$ [4]. The corona $G_1 \odot G_2$ of two graphs is the graph obtained by taking one copy of G_1, which has n_1 vertices, and n_1 copies of G_2 , and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 by an edge. Knowing the fact that the family of bipartite graphs $K_{m,n}$ and complete graphs K_p (for $p \leq 4$) are graceful, we are interested to investigate whether the corona of these two family of graceful graphs, $K_{m,n} \odot K_p$ is also graceful. In 2001, Sethuraman and Elumalai [2] have shown that pendant edge extension of a complete bipartite graph, that is, $K_{m,n} \odot K_1$ is graceful for $n \leq m \leq 2n+4$ when n is even, and for $n \leq m \leq 2n-1$ when n is odd. Recently, Bhoumik and Mitra [9] presented a graceful labeling of $K_{m,n} \odot K_1$ for the particular case, when $m = n$. The bounds of m, in terms of n was extended in [10], that is, $m \simeq \mathcal{O}(n^2)$, to be precise $n \leq m \leq n^2+n$. So far, a few authors have worked on the gracefulness of $K_{m,n} \odot K_1$ as mentioned above, But to the best of our knowledge this is the first attempt to prove the gracefulness of $K_{m,n} \odot K_2$. In this paper we investigate whether $G = K_{n,n} \odot K_2$ is graceful, and show that G is graceful, when n is even.

The paper is organized in the following manner. In section 2, we provide the graceful labeling of the few particular graphs from the family of $K_{2m,2m} \odot K_2$ ($m = 2, 3, 8$). In Section 3 we define the function that assigns the vertex labels. Section 4 describes the bijective property of the induced function that defined the edge labels and finally we conclude in Section 5.

2. Graceful Labeling of $K_{n,n} \odot K_2$, where $n = 2, 4, 6, 16$

The generalized pattern of graceful labeling for the family $K_{2m,2m} \odot K_2$ for any integer m, which we discuss the later sections, does not work if $m \in \{1, 2, 3, 8\}$. So in this section we provide the explicit graceful labelings for each of the four particular graphs from the family of $K_{2m,2m} \odot K_2$ (see Figure 1,2,3). Hence these graphs will be omitted in later sections, where we discuss the graceful labelings for the graphs from the family $K_{2m,2m} \odot K_2$ for any integer $m \in \mathbb{Z}^+ \setminus \{1, 2, 3, 8\}$.

3. Vertex Labeling

Now on, through out this paper we view each element of $v \in V(G)$ uniquely as $v = (i, j, k)$, where $i \in \mathbb{Z}_2, j \in \mathbb{Z}_n, k \in \mathbb{Z}_3$, such that $\{(0, j, 0)|j \in \mathbb{Z}_n\}$ and $\{(1, j, 0)|)j \in \mathbb{Z}_n\}$ form the two partite sets of $K_{n,n}$ and $(i, j, 1)$ and $(i, j, 2)$ are the two vertices that are adjacent to $(i, j, 0)$.

Here in terms of the orientation, $k = 0$ denotes the vertices on $K_{2m,2m}$, and we call them stem-vertices for our convenience. Also, $k = 1, 2$ denote the vertices on the triangles hanging from each vertex of $K_{2m,2m}$. These vertices are the parts of the extended triangles of the graph, not included in the stem. We will now present the function f that would assign non-negative integers to the vertices of the graph $K_{2m,2m} \odot K_2$. For the sake of easiness of the reading we introduce the

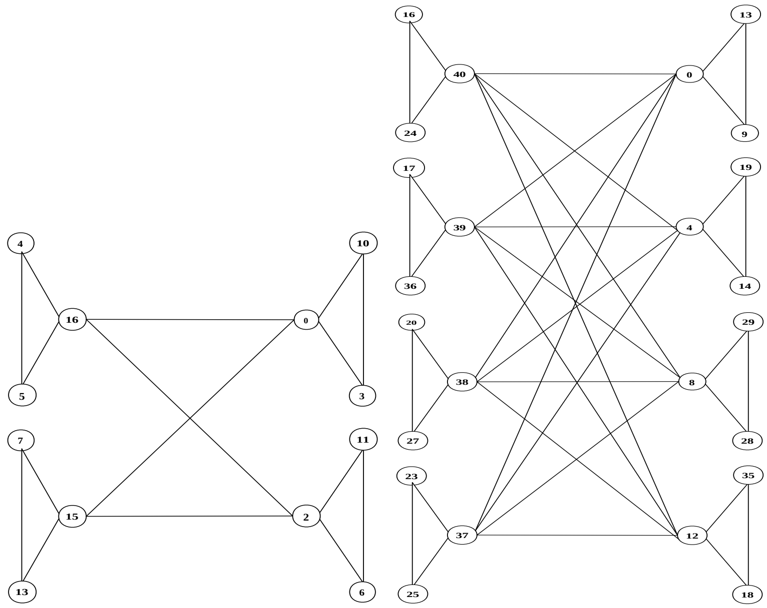

Figure 1. Graceful labeling of $K_{2,2} \odot K_2$ and $K_{4,4} \odot K_2$.

function f in pieces. First piece of the function, $f(i,j,0)$, that assigns labeling to the stem-vertices, that is, the vertices of complete bipartite graph $K_{2m,2m}$ inside $K_{2m,2m} \odot K_2$, is as follows

$$f(i,j,0) = \begin{cases} 4m^2 + 12m - j, & \text{if } i = 0, \\ 2mj, & \text{if } i = 1, \end{cases}$$

for $j \in \{0, 1, 2, \cdots, 2m-1\}$. The following functions assign the labeling to the outer vertices of the triangles of one side.

$$f(0,j,1) = \begin{cases} 4m^2 + j, & \text{if } j = 0, 1, \cdots, m-1, \\ 4m^2 - 2m + j + 1, & \text{if } j = m, m+1, \cdots, 2m-2, \\ 4m^2 + m + 1, & \text{if } j = 2m-1, \end{cases}$$

$$f(0,j,2) = \begin{cases} 4m^2 + 4m - 3j, & \text{if } j = 0, 1, \cdots, \lceil \frac{m}{2} \rceil - 1, \\ 4m^2 + 4m - 3j - 1, & \text{if } j = \lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1, \cdots, m-1, \\ 4m(m+1) + 3j + 2, & \text{if } j = m, m+1, \cdots, \lfloor \frac{3m}{2} \rfloor - 1, \\ 4m(m+1) + 3j + 3, & \text{if } j = \lfloor \frac{3m}{2} \rfloor, \lfloor \frac{3m}{2} \rfloor + 1, \cdots, 2m-2, \\ 4m^2 + 6m + 2, & \text{if } j = 2m-1. \end{cases}$$

Similarly, the following functions $f(1,j,1), f(1,j,2)$ assign the labeling to the outer vertices of

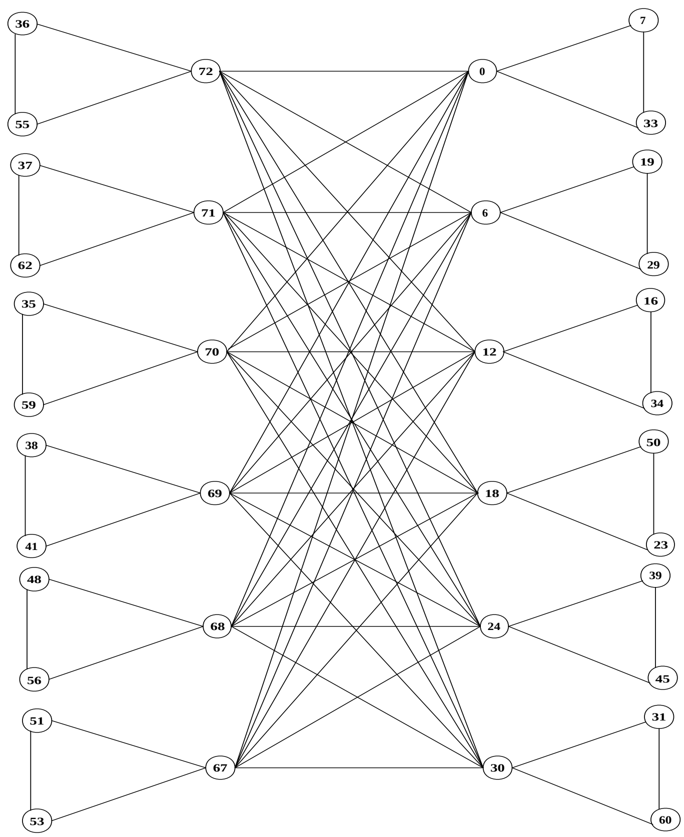

Figure 2. Graceful labeling of $K_{6,6} \odot K_2$.

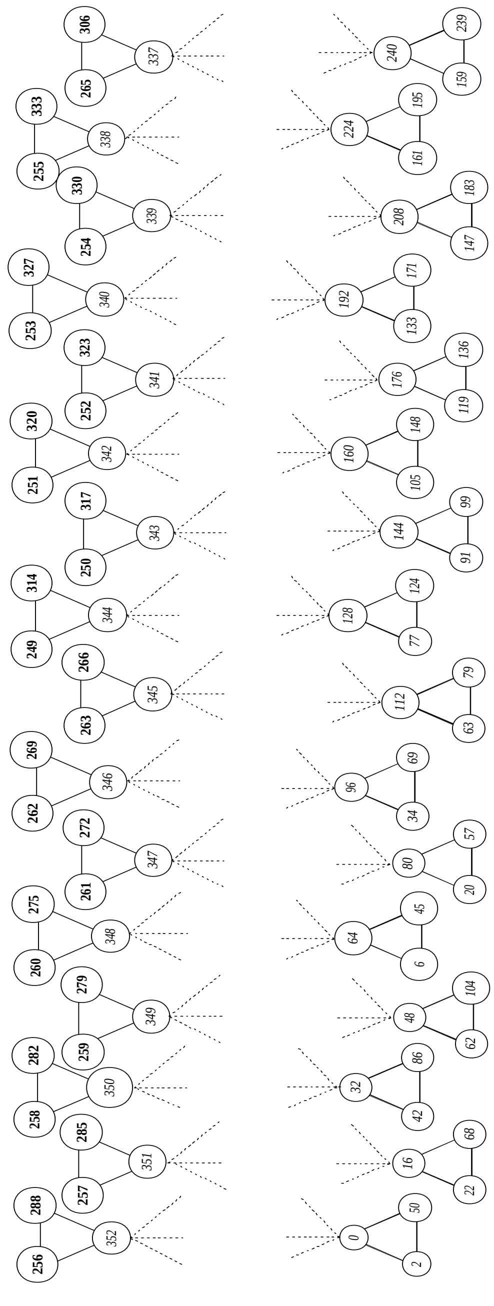

Figure 3. Graceful labeling of $K_{16,16} \odot K_2$

the triangles on the other side.

$$f(1,j,1)=\begin{cases}6m+j(2m+2)+2, & \text{if } j=0,1,\cdots,m-2,\\ 2j(m-1)-4m-3, & \text{if } j=m-1,m,\cdots,2m-2,\\ 4m^2-12m-1, & \text{if } j=2m-1,\end{cases}$$

$$f(1,j,2)=\begin{cases}2j(m+2)+2, & \text{if } j=0,1,\cdots,\lfloor\frac{m}{2}\rfloor-1,\\ 2j(m+2)+3, & \text{if } j=\lfloor\frac{m}{2}\rfloor,\lfloor\frac{m}{2}\rfloor+1,\cdots,m-2,\\ 2m(j-2)-1, & \text{if } j=m-1,\\ 2j(m-2)+4m-4, & \text{if } j=m,m+1,\cdots,\lceil\frac{3m}{2}\rceil-2,\\ 2j(m-2)+4m-5, & \text{if } j=\lceil\frac{3m}{2}\rceil-1,\lceil\frac{3m}{2}\rceil,\cdots,2m-2,\\ 2mj-1, & \text{if } j=2m-1.\end{cases}$$

Figure 4 shows the explicit vertex labels (according to the above mentioned functions) and the induced edge labels of $K_{12,12}\odot K_2$.

Theorem 3.1. *f is injective for any $m\in\mathbb{Z}^+\setminus\{1,2,3,8\}$.*

Proof. To prove that f is injective, first we partition all the vertices into six subsets, and we will prove that there are neither any intra nor inter-overlapping among these sets. The multisets of vertex labels are as follows:

- Left stem : V_L (denoted by $f(0,j,0)$).
- Right stem : V_R (denoted by $f(1,j,0)$).
- Extended triangles on the left side (excluding the labels of the vertices on the bipartite graph) V_{L_1},V_{L_2} (denoted by $f(0,j,1)$, and $f(0,j,2)$).
- Extended triangles on the right side (excluding the labels of the vertices on the bipartite graph) V_{R_1},V_{R_2} (denoted by $f(1,j,1)$, and $f(1,j,2)$).

First, observe that $V_R=\{2mj\mid j=0,1,\cdots,2m-1\}$ contains $2m$ distinct elements, each of which is a multiple of $2m$; which implies the uniqueness of all the elements of V_R. On the other hand, V_L contains all the consecutive elements starting from $4m^2+10m+1$ to $4m^2+12m$. Hence there is no chance of repetition in V_L, since all the elements are consecutive and can be arranged in an order. Therefore, there are no overlapping labels in the vertex sets V_L and V_R. Next we describe how the rest of proof is designed.

The skeleton of our proof is based on the five claims as follows, assuming that there is no overlapping in either V_L or V_R.

- Claim I : $V_L\bigcap\{V_{L_1}\cup V_{L_2}\cup V_R\cup V_{R_1}\cup V_{R_2}\}=\emptyset$.
- Claim II : $V_R\bigcap\{V_{L_1}\cup V_{L_2}\cup V_{R_1}\cup V_{R_2}\}=\emptyset$.

- Claim III : There is no intra overlapping in V_{L_1}, V_{L_2} and $V_{L_1} \cap V_{L_2} = \emptyset$.
- Claim IV : There is no intra overlapping in V_{R_1}, V_{R_2} and $V_{R_1} \cap V_{R_2} = \emptyset$.
- Claim V : $\{V_{L_1} \cup V_{L_2}\} \bigcap \{V_{R_1} \cup V_{R_2}\} = \emptyset$.

Claim I states that any element of the set V_L can not coincide with any other vertex label of the graph, that is, V_L is pairwise disjoint with any other vertex set. Similarly, assuming Claim I to be true, Claim II states the same for the set V_R. Therefore, The first two claims jointly imply that $\{V_L \cup V_R\} \bigcap \{V_{L_1} \cup V_{L_2} \cup V_{R_1} \cup V_{R_2}\} = \emptyset$. Hence, it remains to show that the four vertex sets, namely $V_{L_1}, V_{L_2}, V_{R_1}, V_{R_2}$ are pairwise exclusive and there is no self-overlap in any of these four vertex-sets. Considering two more specific claims in Claim III and Claim IV, it is sufficient to prove Claim V to show the sets $V_{L_1}, V_{L_2}, V_{R_1}, V_{R_2}$ are pairwise disjoint. These five claims once proved, consequently lead us to conclude that all the six multisets are basically sets and theses six sets are mutually exclusive. For the sake of the proof, the maximum and the minimum element of each of the six vertex sets are listed in Table 1.

Table 1. Minimum and Maximum of each of the six sets of vertex-labels.

Vertex Sets	Minimum	Maximum
V_L	$4m^2+10m+1$	$4m^2+12m$
V_{L_1}	$4m^2-m+1$	$4m^2+m+1$
V_{L_2}	$4m^2+m+2$	$4m^2+10m-3$
V_R	0	$4m^2-2m$
V_{R_1}	$6m+2$	$4m^2-12m+1$
V_{R_2}	2	$4m^2-2m-1$

Claim I From Table 1 it is evident that $\min\{V_L\} > \max\{X\}$ for every set $X \in \{V_{L_1}, V_{L_2}, V_R, V_{R_1}, V_{R_2}\}$, therefore, it proves the first claim, that is, $V_L \bigcap \{V_{L_1}, V_{L_2}, V_R, V_{R_1}, V_{R_2}\} = \emptyset$,

Claim II We have already pointed out that the elements of V_R are multiples of $2m$. Claim I assures $V_R \cap V_L = \emptyset$ We now prove this claim through these following four subclaims where we show that V_R is pairwise disjoint with each of the four vertex sets $V_{L_1}, V_{L_2}, V_{R_1}$ and V_{R_2} respectively.

Subclaim 1 $V_R \cap V_{L_1} = \emptyset$

It is easy to observe that,

(i) For, $j \in \{0, 1, \cdots, m-1\}$, $2m \nmid j \implies 2m \nmid (4m^2 + j)$.

(ii) For, $j \in \{m, m+1 \cdots, 2m-2\}$, $2m \nmid (j+1)$
$\implies 2m \nmid (4m^2 - 2m + j + 1)$.

(iii) For, $j = 2m-1$, $2m \nmid j \Longrightarrow 2m \nmid (4m^2 + j)$.

This shows that if $x \in V_{L_1}$ then $2m \nmid x$, which implies $V_R \cap V_{L_1} = \emptyset$.

Subclaim 2 $V_R \cap V_{L_2} = \emptyset$

If possible let us assume that $V_R \cap V_{L_2} \neq \emptyset$. So there must be one element which is contained in both V_R and V_{L_2}. Since, all the elements of V_R is a multiple of $2m$, we must assume that $2m$ divides at least one element of V_{L_2}. We now check for the possibility in these following cases.

(i) First as we know that $2m \nmid 3j$ for any $j \in \{0, 1, \cdots, \lceil \frac{m}{2} \rceil - 1\}$, as a consequence we get $2m \nmid (4m^2 + 4m - 3j)$.

(ii) If possible let $2m \mid (4m^2 + 4m - 3j - 1)$, that is, we must have $2m \mid (3j + 1)$ for some $j \in \{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1, \cdots, m-1\}$. Now, minimum and maximum values of $(3j+1)$, where $j \in \{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1, \cdots, m-1\}$ are $3m/2 + 1$, and $3m-2$ respectively. Thus $2m \mid (3j+1)$ only when $3j+1 = 2m$. But then, the corresponding element of V_R becomes $4m^2 + 2m$, which is greater than the maximum possible element of V_R, as listed in Table 1. Therefore, $2m \nmid (4m^2 + 4m - 3j - 1)$ for any $j \in \{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1, \cdots, m-1\}$.

(iii) If possible let $2m \mid (4m^2 + 4m + 3j + 2)$, that is, we must have $2m \mid (3j+2)$ for any $j \in \{m, m+1, \cdots, \lfloor \frac{3m}{2} \rfloor - 1\}$. Similar to the previous case, it is easy to observe the minimum and maximum values of $(3j+2)$, for any j in that specific interval are $3m+2$, and $9m/2 - 1$ respectively. Hence $2m \mid (3j+2)$ only when $3j + 2 = 4m$. But then, the corresponding element of V_R becomes $4m^2 + 8m$, which is greater than the maximum possible element of V_R, as listed in Table 1. Therefore, $2m \nmid (4m^2 + 4m + 3j + 2)$ for any $j \in \{m, m+1, \cdots, \lfloor \frac{3m}{2} \rfloor - 1\}$, .

(iv) If possible let $2m \mid (4m^2 + 4m + 3j + 3)$, then we must have $2m \mid (3j+3)$ for some $j \in \{\lfloor \frac{3m}{2} \rfloor, \lfloor \frac{3m}{2} \rfloor + 1, \cdots, 2m-2\}$. Once again the maximum and minimum values of $3j+3$ where $j \in \{\lfloor \frac{3m}{2} \rfloor, \lfloor \frac{3m}{2} \rfloor + 1, \cdots, 2m-2\}$, are $9m/2 + 3$ and $6m-3$ respectively. Hence $2m \nmid (3j+3)$, and therefore $2m \nmid (4m^2 + 4m + 3j + 3)$, for any $j \in \{\lfloor \frac{3m}{2} \rfloor, \lfloor \frac{3m}{2} \rfloor + 1, \cdots, 2m-2\}$.

(v) For $j = 2m-1$, clearly $2m \nmid (4m^2 + 6m + 2)$ for any m.

We see that for any set of values of j, we arrive at a contradiction when we assume the possibility that the elements of V_{L_2} is a multiple of $2m$.

Subclaim 3 $V_R \cap V_{R_1} = \emptyset$

In this case also we assume that $V_R \cap V_{R_1} \neq \emptyset$. Then according to the similar logic as before, there is at least one element of V_{R_1}, which is divisible by $2m$. We now proceed as follows.

(i) As we know $2m \nmid (2j+2)$ for any $j \in \{0, 1, \cdots, m-2\}$, as a consequence $2m \nmid (6m + j(2m+2) + 2)$.

(ii) Let us assume that $2m \mid (2j(m-1) - 4m - 3)$, which implies $2m \mid (2j+3)$ for some $j \in \{m-1, m, \cdots, 2m-2\}$. Now, we observe that $\max(2j+3)_{j \in \{0,1,\cdots,m-2\}} = 4m-1$ and $\min(2j+3)_{j \in \{0,1,\cdots,m-2\}} = 2m+1$. Thus, $2m \nmid (2j+3)$ and consequently, $2m \nmid (2j(m-1) - 4m - 3)$ for any $j \in \{m-1, m, \cdots, 2m-2\}$.

(iii) Again, we notice that $2m \nmid (4m^2 - 12m - 1)$ for any m.

Hence, it is clear from the above three subcases that $2m$ does not divide any element V_{R_1}, which implies $V_R \cap V_{R_1} = \emptyset$.

Subclaim 4 $V_R \cap V_{R_2} = \emptyset$

Similar to the previous subclaims we assume that $V_R \cap V_{R_2} \neq \emptyset$ that is, $2m$ divides at least one element of V_{R_2}.

(i) If possible let us assume that $2m \mid (2j(m+2)+2)$; hence $2m \mid (4j+2)$ for some $j \in \{0, 1, \cdots, \lfloor \frac{m}{2} \rfloor - 1\}$. Now, $\max(4j+2)_{j \in \{0,1,\cdots,\lfloor \frac{m}{2} \rfloor - 1\}} = 2m - 1$. This implies there is no j for which $2m \mid (4j+2)$ or $2m \mid (2j(m+2)+2)$.

(ii) Let us assume that $2m \mid (2j(m+2)+3)$, which implies $2m \mid (4j+3)$ for some $j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}$. Now, we observe that $\max(4j+3)_{j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}} = 4m - 5$ and $\min(4j+3)_{j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}} = 2m + 1$. Thus, $2m \nmid (4j+3)$ and consequently, $2m \nmid (2j(m+2)+3)$ for any $j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}$.

(iii) Clearly $2m \nmid (2m^2 - 6m - 1)$ for any m.

(iv) Let us assume that $2m \mid (2j(m-2)+4m-4)$, hence $2m \mid (4j+4)$ for some $j \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}$. Now, we observe that $\max(4j+4)_{j \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}} = 6m - 2$ and $\min(4j+4)_{j \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}} = 4m + 4$. Thus, $2m \nmid (4j+4)$ and consequently, $2m \nmid (2j(m-2)+4m-4)$ for any $j \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}$.

(v) Let us assume that $2m \mid (2j(m-2)+4m-5)$, which implies $2m \mid (4j+5)$ where $j \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}$. Now, we observe that $\max(4j+5)_{j \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}} = 8m - 3$ and $\min(4j+5)_{j \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}} = 6m + 1$. Thus, $2m \nmid (4j+5)$ and consequently, $2m \nmid (2j(m-2)+4m-5)$ for any $j \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}$.

(vi) Clearly $2m \nmid (4m^2 - 2m - 1)$ for any m.

Thus, by thorough analysis it was observed that no multiple of $2m$ was contained in any of the sets $V_{L_1}, V_{L_2}, V_{R_1}, V_{R_2}$ and hence we conclude that V_R is pairwise disjoint with any of the sets $(V_{L_1}, V_{L_2}, V_{R_1}, V_{R_2})$.

This completes the proof of the second claim.

Step III In this step we are going to show that there is no intra-overlapping in V_{L_1}, V_{L_2}; and also $V_{L_1} \cap V_{L_2} = \emptyset$. The vertex labels of all upper vertices of the extended triangles on the left side are as follows:

$$\begin{aligned} V_{L_1} &= \{4m^2, 4m^2+1, 4m^2+2, \cdots, 4m^2+m-1\} \cup \{4m^2-m+1, \\ &\quad 4m^2-m+2, 4m^2-m+3, \cdots, 4m^2-1\} \cup \{4m^2+m+1\}. \end{aligned}$$

Rearranging the entries of the set in increasing order we get

$$V_{L_1} = \{4m^2-m+1, 4m^2-m+2, \cdots, 4m^2+m-1\} \cup \{4m^2+m+1\}.$$

From the explicit expression of the elements of the set V_{L_1} of vertex labels, it is very clear that all the elements can be arranged in increasing order, which implies that they are distinct, so that there is no intra-overlapping in V_{L_1}.

Now, the vertex labels of the lower vertices of the extended triangles on the left side are as follows:

When m is even

$$\begin{aligned} V_{L_2} &= \Big\{4m^2+4m, 4m^2+4m-3, 4m^2+4m-6, \cdots, 4m^2+\frac{5}{2}m+3\Big\} \\ &\bigcup \Big\{4m^2+\frac{5}{2}m-1, 4^2+\frac{5}{2}m-4, \cdots, 4m^2+m+2\Big\} \\ &\bigcup \Big\{4m^2+7m+2, 4m^2+7m+5, \cdots, 4m^2+\frac{17}{2}m-1\Big\} \\ &\bigcup \Big\{4m^2+\frac{17}{2}m+3, 4m^2+\frac{17}{2}m+6, \cdots, 4m^2+10m-3\Big\} \\ &\bigcup \Big\{4m^2+6m+2\Big\}. \end{aligned}$$

Rearranging the entries in increasing order we get the following

$$\begin{aligned} V_{L_2} = &\Big\{4m^2+m+2, 4m^2+m+5, \cdots, 4m^2+\frac{5}{2}m-1; \\ &4m^2+\frac{5}{2}m+3, 4m^2+\frac{5}{2}m+6, \cdots, 4m^2+4m; 4m^2+6m+2; \\ &4m^2+7m+2, 4m^2+7m+5, \cdots, 4m^2+\frac{17}{2}m-1; \\ &4m^2+\frac{17}{2}m+3, 4m^2+\frac{17}{2}m+6, \cdots, 4m^2+10m-3\Big\}. \end{aligned}$$

When m is odd

$$\begin{aligned} V_{L_2} &= \Big\{4m^2+4m, 4m^2+4m-3, 4m^2+4m-6, \cdots, 4m^2+\frac{5}{2}m+\frac{3}{2}\Big\} \\ &\bigcup \Big\{4m^2+\frac{5}{2}m-\frac{5}{2}, 4m^2+\frac{5}{2}m-\frac{11}{2}, \cdots, 4m^2+m+2\Big\} \\ &\bigcup \Big\{4m^2+7m+2, 4m^2+7m+5, \cdots, 4m^2+\frac{17}{2}m-\frac{5}{2}\Big\} \\ &\bigcup \Big\{4m^2+\frac{17}{2}m+\frac{3}{2}, 4m^2+\frac{17}{2}m+\frac{9}{2}, \cdots, 4m^2+10m-3\Big\} \\ &\bigcup \Big\{4m^2+6m+2\Big\}. \end{aligned}$$

Rearranging the entries in increasing order we get the following

$$\begin{aligned} V_{L_2} = & \Big\{4m^2+m+2, 4m^2+m+5, \cdots, 4m^2+\frac{5}{2}m-\frac{5}{2}; 4m^2+\frac{5}{2}m+\frac{3}{2}, \\ & 4m^2+\frac{5}{2}m+\frac{9}{2}, \cdots, 4m^2+4m; 4m^2+6m+2; 4m^2+7m+2, \\ & 4m^2+7m+5, \cdots, 4m^2+\frac{17}{2}m-\frac{5}{2}; 4m^2+\frac{17}{2}m+\frac{3}{2}, \\ & 4m^2+\frac{17}{2}m+\frac{9}{2}, \cdots, 4m^2+10m-3\Big\}. \end{aligned}$$

Similar to V_{L_1}, in this case also, the explicit entries and their arrangement in increasing order imply the non-overlapping phenomena of the vertex labels. Hence there is no overlap in the vertex labels in V_{L_2} for both m even and odd.

Now, from Table 1 we have $\max(V_{L_1}) = 4m^2+m+1$ and $\min(V_{L_2}) = 4m^2+m+2$, which implies $\max(V_{L_1}) < \min(V_{L_2})$. Hence, we must have $V_{L_1} \cap V_{L_2} = \emptyset$.

Step IV We are going to show that there is no intra overlapping in V_{R_1}, V_{R_2} and $V_{R_1} \cap V_{R_2} = \emptyset$ in this step. In this case, instead of considering V_{R_1} and V_{R_2} as two different sets and we consider the set $V_{R_1} \cup V_{R_2}$, and see if it is possible to have any coincidence in that set. For our convenience, we partition $V_{R_1} \cup V_{R_2}$ into two following partitions, even subsets (E_i) and odd subsets (O_i) for $i = 1, 2$ defined as follows: $(O_i) = \{x \in V_{R_i} | \text{x is odd}\}$, $(E_i) = \{x \in V_{R_i} | \text{x is even}\}$. To prove that $V_{R_1} \cap V_{R_2} = \emptyset$, it needs to be shown that $O_1 \cap O_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$ and as well as there is no intra-repetition in E_1,E_2. For our convenience, through out the proof we consider $j_1 - j_2 = p$, where p is any integer.

$$O_1 = \begin{cases} 2m^2-8m-1, & \text{if } j = m-1, \\ 2j(m-1)-4m-3, & \text{if } j \in \{m, m+1, \cdots, 2m-2\}, \\ 4m^2-12m-1, & \text{if } j = 2m-1. \end{cases}$$

$$O_2 = \begin{cases} 2mj+3+4j, & \text{if } j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}, \\ 2j(m-2)+4m-5, & \text{if } j \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}, \\ 4m^2-2m-1, & \text{if } j = 2m-1, \\ 2m^2-6m-1, & \text{if } j = m-1. \end{cases}$$

Case A First we need to see if there is any intra-repetition in O_1 or O_2. For that we consider the following nine subcases. In the first three subcases we deal with the possibility of intra-repetition in O_1, whereas in the remaining six subcases we discuss the possibility of intra-repetition in O_2.

Subcase 1 Let us assume $2m^2 - 8m - 1 = 2j(m-1) - 4m - 3$ for some $j \in \{m, m+1, \cdots, 2m-2\}$. But then we get $j = m - 6 + \frac{10}{m+2}$. Simplifying we get $j = m - 1 \notin \{m, m+1, \cdots, 2m-2\}$.

Subcase 2 If possible let us assume $2j(m-1) - 4m - 3 = 4m^2 - 12m - 1$ for some $j \in \{ m, m+1, \cdots, 2m-2\}$. Simplifying we get $j = 2m - 2 - \frac{1}{m-1}$, which is an integer only when $m = 2$, which contradicts our assumption $m > 3$.

Subcase 3 Now, let us assume that $2m^2 - 8m - 1 = 4m^2 - 12m - 1$, which on simplification gives us $m = 2$; a contradiction.

Subcase 4 If possible let us assume $2mj_1 + 3 + 4j_1 = 2j_2(m-2) + 4m - 5$ for some $j_1 \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1, \cdots, m-2\}$ and $j_2 \in \{\lceil\frac{3m}{2}\rceil - 1, \lceil\frac{3m}{2}\rceil, \cdots, 2m-2\}$. For the sake of computation, now-on we will consider $j_1 - j_2 = p$, wherever required, where p is some integer. Hence after simplification we have $j_1 = \frac{m(2-p)+2p}{4} - 1$ and $j_2 = \frac{m(2-p)-2p}{4} - 1$. As, $j_2 \in \{\lceil\frac{3m}{2}\rceil - 1, \lceil\frac{3m}{2}\rceil, \cdots, 2m-2\}$, the possible values of p are $-3, -4, -5$. On the other hand, as $j_1 \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1, \cdots, m-2\}$, p should be less than -3 (since $m > 3$). Hence we arrive at a contradiction.

Subcase 5 Let us suppose that $2mj+3+4j = 4m^2-2m-1$ for some $j \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1, \cdots, m-2\}$. On simplification we get, $(m+2)j = 2m^2 - m - 2$ which finally gives $j = 2m - 5 + \frac{8}{m+2}$. Now, possible positive values (since $m = 0$ is unacceptable as well) of m which leads j to be a whole number, are $m = 2, 6$.
Now, if $m = 2$ then we have $j = 1 = m-1$, which does not belong to the specified range $\{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1, \cdots, m-2\}$.
Again, if $m = 6$ then we have $j = 8 > m - 2$, which is not included in the specified range of j either. Hence, we arrive at a contradiction once again.

Subcase 6 Next, let us consider that $2mj + 3 + 4j = 2m^2 - 6m - 1$ for some $j \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1, \cdots, m-2\}$. After simplification from the above equation we get $2(m+2)j = 2m^2 - 6m - 4 = m - 5 + \frac{8}{m+2}$. Since j has to be a whole number, feasible positive value of m is $m = 6$ (since $m > 3$). If $m = 6$ then we have $j = 8 > m - 2$, which is not included in the specified range of j.

Subcase 7 In this case we assume that $2j(m-2) + 4m - 5 = 4m^2 - 2m - 1$ for $j \in \{\lceil\frac{3m}{2}\rceil - 1, \lceil\frac{3m}{2}\rceil, \cdots, 2m-2\}$. On simplification we have $j = 2m - 1 \notin \{\lceil\frac{3m}{2}\rceil - 1, \lceil\frac{3m}{2}\rceil, \cdots, 2m-2\}$, which is not acceptable.

Subcase 8 If $2j(m-2)+4m-5 = 2m^2-6m-1$ for some $j \in \{\lceil\frac{3m}{2}\rceil-1, \lceil\frac{3m}{2}\rceil, \cdots, 2m-2\}$. Then we have, after simplifying, $j = m - 3 - \frac{4}{m-2}$. Possible values of $m > 3$ are $4, 6$ for which j remains an integer.
Now, if $m = 4$, we get $j = -1$, which is absurd.
Otherwise, when $m = 6$, we have $j = 2$, which does not belong to the specified range of j.

Subcase 9 The assumption $4m^2 - 2m - 1 = 2m^2 - 6m - 1$ leads us an impossible case $m = -2$.

In each of the above nine subcases we arrive at a contradiction whenever we assume that there is a conflict between two vertex labels, both belonging to either O_1 or O_2.

This leads us to conclude that there is no intra-repetition in O_1 or O_2.

Case B Now we check for the inter-repetition in between O_1 and O_2. Our aim is to prove $O_1 \cap O_2 = \emptyset$. We consider the following cases depending on the values of j, and the piece-wise function of O_1 and O_2.

Subcase 1 Let us assume $2m^2 - 8m - 1 = 2mj + 3 + 4j$ for some $j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}$. But then we get $j = m - 6 + \frac{10}{m+2}$. We get j as a non-negative integer only when m is 8, but this gives $j = 3 \notin \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}$.

Subcase 2 Let us assume $2m^2 - 8m - 1 = 2j(m-2) + 4m - 5$ for some $j \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}$. But then we get $j = m - 4 - \frac{6}{m-2}$. We get j as a non-negative integer only when m is 8, but this gives $j = 3 \notin \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}$.

Subcase 3 If $2m^2 - 8m - 1 = 4m^2 - 2m - 1$, then m is 0 or -3. On the hand $2m^2 - 8m - 1 = 2m^2 - 6m - 1$ only when $m = 0$. Both the cases are unacceptable.

Subcase 4 Let us assume that $2j_1(m-1) - 4m - 3 = 2mj_2 + 3 + 4j_2$, for some $j_1 \in \{m, m+1, \cdots, 2m-2\}$, and $j_2 \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}$. This gives us $j_2 = \frac{m(p-2)-p}{3} - 1$. Note that when $p \le 3$, $j_2 < \lfloor \frac{m}{2} \rfloor$. When $p = 5$, $j_2 = m - 8/3$, not an integer. $p \ge 6$ gives us $j_2 > m-2$. So the only possible choice of p is 4, which implies $j_2 = \frac{2m-7}{3}$, and $j_1 = \frac{2m+5}{3}$. Observing the domain of j_1, it is clear that it is only possible when $m \le 5$. But then j_2 is non-negative only when $m = 5$, which leads to a contradiction since $j_2 = 1 \notin \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}$, for $m > 3$.

Subcase 5 Let us assume that $2j_1(m-1) - 4m - 3 = 2j_2(m-2) + 4m - 5$, for some $j_1 \in \{m, m+1, \cdots, 2m-2\}$, and $j_2 \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}$, which gives us $j_2 = m(4-p) + p - 1$. Observe that if $p \le 2$, $j_2 \ge 2m+1$ which is not feasible. Similarly, when $p \ge 5$, $j_2 \le -m + 4$ which is also absurd. Therefore, we are left with two cases, viz, $p = 3, 4$.
Let $p = 3$, then we must have $j_2 = m+2$ and $j_1 = m+5$. Note that $j_2 = m+2 \le \lceil \frac{3m}{2} \rceil - 1$, only when $m \le 6$. As $j_1 = m+5$, we arrive at a contradiction as $m + 5 \notin \{m, m+1, \cdots, 2m-2\}$ for any $m \le 6$.
Now, $p = 4$ leads us to a contradiction as we get $j_2 = 3 \notin \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}$, for any m.

Subcase 6 Let us assume that $2j(m-1) - 4m - 3 = 4m^2 - 2m - 1$, for some $j \in \{m, m+1, \cdots, 2m-2\}$, which implies $j = (2m^2+m+1)/(m-1) > 2m+3$, a contradiction.

Subcase 7 Let us assume that $2j(m-1) - 4m - 3 = 2m^2 - 6m - 1$, for some $j \in \{m, m+1, \cdots, 2m-2\}$, which implies $j = m + \frac{1}{m-1}$, a contradiction, as $m > 3$.

Subcase 8 Let us assume that $4m^2 - 12m - 1 = 2mj + 3 + 4j$, for some $j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2\}$, which implies $j = 2m - 10 + \frac{18}{m+2}$. As $j \ge \lfloor \frac{m}{2} \rfloor$, we can conclude that $m \ge 5$. To j be a non-negative integer, m can be either 7 or 16. But in both the cases (when $m = 7$, $j = 6$, and when $m = 16$, $j = 23$) $2m - 10 + \frac{18}{m+2} > m - 2$, hence a contradiction.

Subcase 9 Let us assume that $4m^2 - 12m - 1 = 2j(m-2) + 4m - 5$, for some $j \in \{\lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2\}$, which implies $j = 2m - 4 - \frac{6}{m-2}$. To j be an integer m must be $4, 5$, or 8. But as $j = 2m - 4 - \frac{6}{m-2} \geq \lceil \frac{3m}{2} \rceil - 1$, m must be at least 8. Hence we arrive to a contradiction as we assumed that $m \neq 8$.

Subcase 10 $4m^2 - 12m - 1 = 4m^2 - 2m - 1$ implies $m = 0$. On the other hand $4m^2 - 12m - 1 = 2m^2 - 6m - 1$ implies $m = 3$, a contradiction.

Now we consider E_1 and E_2 where

$$E_1 = 2j(m+1) + 6m + 2, \text{ if } j \in \{0, 1, \cdots, m-2\}$$

$$E_2 = \begin{cases} 2j(m+2) + 2, & \text{if } j \in \{0, 1, \cdots, \lceil \frac{m}{2} \rceil - 1\}, \\ 2j(m-2) + 4m - 4, & \text{if } j \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}. \end{cases}$$

Case A First note that in E_1 all the vertices are in ascending order. In E_2,

$$\max_{j \in \{0,1,\cdots,\lceil \frac{m}{2} \rceil - 1\}} 2j(m+2) + 2 < \min_{j \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}} 2j(m-2) + 4m - 4,$$

when $m > 3$. This rejects any chance of intra-repetition within the set E_1 or E_2.

Case B Let us assume that $2j_1(m+1) + 6m + 2 = 2j_2(m+2) + 2$, for some $j_1 \in \{0, 1, \cdots, m-2\}$, and some $j_2 \in \{0, 1, \cdots, \lceil \frac{m}{2} \rceil - 1\}$. On simplification, we get $j_2 = m(p+3) + p$. Note that if $p \leq -3$ or $p \geq 0$, then the value of j_2 is unacceptable. We discuss the cases when $p = -1, -2$. First $p = -1$ gives $j_2 = 2m - 1 \notin \{0, 1, \cdots, m-2\}$ for all m. On the other hand $p = -2$ gives us $j_2 = m - 2 \notin \{0, 1, \cdots, m-2\}$ for any $m \geq 4$. Hence all the cases lead to a contradiction.

Case C Let us assume that $2j_1(m+1) + 6m + 2 = 2j(m-2) + 4m - 4$, for some $j_1 \in \{0, 1, \cdots, m-2\}$, and some $j_2 \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}$. On simplifying we get $j_2 = -\frac{m(p+1)+p}{3} - 1$. Observe that $j_2 \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}$ only if $-5 \leq p \leq -4$.

Now $p = -4$ gives us $j_2 = m + 1/3$, not an integer for any m. Again $p = -5$ gives that $j_2 = (4m+2)/3$, and consequently $j_1 = (4m-13)/3$. Note that $j_1 = (4m-13)/3 \in \{0, 1, \cdots, m-2\}$, only if $m \leq 7$; whereas, $j_2 = (4m+2)/3 \in \{m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2\}$, only if $m \geq 13$. So there is no common m for which the prescribed values of j_1, and j_2 fall in the defined range. Hence both the possibilities lead to contradiction.

Case 2 and Case 3 together imply that $E_1 \cap E_2 = \emptyset$.

Therefore, from the above discussion on the sets O_1, O_2 and E_1, E_2 we conclude that all the entries in V_{R_1} and V_{R_2} are disjoint and $V_{R_1} \cap V_{R_2} = \emptyset$.

Step V From Table 1 we observe that $\max\{V_{R_1}, V_{R_2}\} = 4m^2 - 2m - 1 < \min\{V_{L_1}, V_{L_2}\} = 4m^2 - m + 1$. As a consequence we conclude that $V_{L_i} \bigcap V_{R_j} = \emptyset$, for all $i, j \in \{1, 2\}$.

□

4. Edge Labeling

Theorem 4.1. *The induced function f^* is bijective.*

Proof. Let us recall function f^* and name it as the edge-labeling function on $E(G)$. First, from Section 3 we observe that the vertex-labeling function is so defined that we must have the induced edge labeling function f^* satisfying

$$f^*(\text{stem-edges}) = \{12m + 112m + 2, \cdots, 12m + 4m^2\}$$

This implies that the edge labels are distinctly and exhaustively assigned to all the edges of the bipartite graph $K_{2m,2m}$. So, f^* is one-to-one and onto as far as stem-edges are considered. Now, if we can show that f^* maintains the similar property for assigning the edge-labels to the remaining $12m$ edges of the extended triangles, then it will be sufficient to show that f^* is bijective on the set $E(G)$. Now in the rest of the section we show that remaining numbers $\{1, 2, \cdots, 12m\}$ are uniquely and exhaustively assigned to edges of the triangles extended from the stem ($K_{2m,2m}$). From the above function, we easily achieve the edge differences on the left and right sides are as follows. Once again L and R stand for left and right side of the stem, and we use a_j, b_j, and c_j to distinguish the edge labels of the extended triangles. Please note that x_{jt} and y_{jt} are formerly called $(0, j, t)$ and $(1, j, t)$ respectively.

$$L_{a_j} := f^*(x_{j0}x_{j1}) = \begin{cases} 12m - 2j, & \text{if } j = 0, 1, \cdots, m-1, \\ 14m - 2j - 1, & \text{if } j = m, m+1, \cdots, 2m-2, \\ 9m, & \text{if } j = 2m-1. \end{cases}$$

$$R_{a_j} := f^*(y_{j0}y_{j1}) = \begin{cases} 6m + 2j + 2, & \text{if } j = 0, 1, \cdots, m-2, \\ 4m + 2j + 3, & \text{if } j = m-1, m, \cdots, 2m-2, \\ 10m + 1, & \text{if } j = 2m-1. \end{cases}$$

$$L_{b_j} := f^*(x_{j1}x_{j2}) = \begin{cases} 8m + 2j, & \text{if } j = 0, 1, \cdots, \lceil \frac{m}{2} \rceil - 1, \\ 8m + 2j + 1, & \text{if } j = \lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1, \cdots, m-1, \\ 6m + 2j + 1, & \text{if } j = m, m+1, \cdots, 3\lfloor \frac{m}{2} \rfloor, \text{ when } m \text{ is odd}, \\ 6m + 2j + 1, & \text{if } j = m, m+1, \cdots, 3\lfloor \frac{m}{2} \rfloor - 1, \text{ when } m \text{ is even}, \\ 6m + 2j + 2, & \text{if } j = \lfloor \frac{3m}{2} \rfloor, \lfloor \frac{3m}{2} \rfloor + 1, \cdots, 2m-2, \\ 5m + 1, & \text{if } j = 2m-1. \end{cases}$$

$$R_{b_j} := f^*(y_{j1}y_{j2}) = \begin{cases} 6m - 2j, & \text{if } j = 0, 1, \cdots, \lfloor \frac{m}{2} \rfloor - 1, \\ 6m - 2j - 1, & \text{if } j = \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-1, \\ 8m - 2j - 1, & \text{if } j = m, m+1, \cdots, 3\frac{m-1}{2}; m \text{ is odd}, \\ 8m - 2j - 1, & \text{if } j = m, m+1, \cdots, 3\frac{m}{2} - 2; m \text{ is even}, \\ 8m - 2j - 2, & \text{if } j = \frac{3m-1}{2}, \frac{3m+1}{2}, \cdots, 2m-2; m \text{ is odd}, \\ 8m - 2j - 2, & \text{if } j = \frac{3m-2}{2}, \frac{3m}{2}, \cdots, 2m-2; m \text{ is even}, \\ 10m, & \text{if } j = 2m-1. \end{cases}$$

$$L_{c_j} := f^*(x_{j0}x_{j2}) = \begin{cases} 4m-4j, & \text{if } j = 0,1,\cdots,\lceil \frac{m}{2} \rceil - 1, \\ 4m-4j-1, & \text{if } j = \lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1, \cdots, m-1, \\ 8m-4j-2, & \text{if } j = m, m+1, \cdots, \lfloor \frac{3m}{2} \rfloor - 1, \\ 8m-4j-3, & \text{if } j = \lfloor \frac{3m}{2} \rfloor, \lfloor \frac{3m}{2} \rfloor + 1, \cdots, 2m-2, \\ 4m-1, & \text{if } j = 2m-1. \end{cases}$$

$$R_{c_j} := f^*(y_{j0}y_{j2}) = \begin{cases} 2+4j, & \text{if } j = 0,1,\cdots,\lfloor \frac{m}{2} \rfloor - 1, \\ 3+4j, & \text{if } j = \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, \cdots, m-2, \\ 2m, & \text{if } j = m-1, \\ 4-4m+4j, & \text{if } j = m, m+1, \cdots, \lceil \frac{3m}{2} \rceil - 2, \\ 5-4m+4j, & \text{if } j = \lceil \frac{3m}{2} \rceil - 1, \lceil \frac{3m}{2} \rceil, \cdots, 2m-2, \\ 1, & \text{if } j = 2m-1. \end{cases}$$

4.1. m is even

We can clearly expand the edge labels as,

$$\begin{aligned} L_{a_j} = \{ & 12m, 12m-2, 12m-4, \cdots, 10m+2;\ \ 12m-1, 12m-3, \\ & 12m-5, \cdots, 10m+3; 9m\}. \end{aligned}$$

$$\begin{aligned} L_{b_j} = \{ & 8m, 8m+2, 8m+4, \cdots, 9m-2;\ \ 9m+1, 9m+3, 9m+5, \cdots, \\ & 10m-1;\ \ 8m+1, 8m+3, 8m+5, \cdots, 9m-1;\ \ 9m+2, 9m+4, \\ & 9m+6, \cdots, 10m-2;\ \ 5m+1\}. \end{aligned}$$

$$\begin{aligned} L_{c_j} = \{ & 4m, 4m-4, 4m-8, \cdots, 2m+4;\ \ 2m-1, 2m-5, 2m-9, \cdots, \\ & 3; 4m-2, 4m-6, 4m-10, \cdots, 2m+2;\ \ 2m-3, 2m-7, \\ & 2m-11, \cdots, 5;\ \ 4m-1\}. \end{aligned}$$

$$\begin{aligned} R_{a_j} = \{ & 6m+2, 6m+4, 6m+6 \cdots, 8m-2;\ \ 6m+1, 6m+3, 6m+5, \cdots, \\ & 8m-1; 10m+1\}. \end{aligned}$$

$$\begin{aligned} R_{b_j} = \{ & 6m, 6m-2, 6m-4, \cdots, 5m+2;\ \ 5m-1, 5m-3, 5m-5, \cdots, \\ & 4m+3, 4m+1; 6m-1, 6m-3, 6m-5, \cdots, 5m+3; 5m; \\ & 5m-2, 5m-4, 5m-6, \cdots, 4m+2; 10m\}. \end{aligned}$$

$$\begin{aligned} R_{c_j} = \{ & 2, 6, 10, \cdots, 2m-2;\ \ 2m+3, 2m+7, 2m+11, \cdots, 4m-5; 2m; \\ & 4, 8, 12 \cdots, 2m-4;\ \ 2m+1, 2m+5, 2m+9 \cdots, 4m-3; 1\}. \end{aligned}$$

Rearranging the sets in consecutive manner we obtain the following:

- $L_{a_j} = \{10m+2, 10m+3, 10m+4, \cdots, 12m-1, 12m\} \bigcup \{9m\}$.
- $R_{a_j} = \{6m+1, 6m+2, 6m+3, \cdots, 8m-2, 8m-1\} \bigcup \{10m+1\}$.
- $L_{b_j} = \{8m, 8m+1, 8m+2, \cdots, 9m-2, 9m-1\} \bigcup \{9m+1, 9m+2, 9m+3, \cdots, 10m-2, 10m-1\} \bigcup \{5m+1\}$.
- $R_{b_j} = \{4m+1, 4m+2, 4m+3, \cdots, 5m-2, 5m-2\} \bigcup \{5m\} \bigcup \{5m+2, 5m+3, 5m+4, \cdots, 6m\} \bigcup \{10m\}$.
- $L_{c_j} \cup R_{c_j} = \{1, 2, 3, \cdots, 2m-2, 2m-1\} \bigcup \{2m\} \bigcup \{2m+1, 2m+2, 2m+3, \cdots, 4m-1, 4m\}$.

With keen observation we obtain from the above:

$$L_{a_j} \cup R_{a_j} \cup L_{b_j} \cup R_{b_j} \cup L_{c_j} \cup R_{c_j} = \{1, 2, 3, \cdots, 12m\}$$

and $L_x \cap L_y = \emptyset$, where $x, y \in \{L_{a_j}, L_{b_j}, L_{c_j}, R_{a_j}, R_{b_j}, R_{c_j}\}$ and $x \neq y$. This implies that f^* assigns the edge-labels in an injective as well as exhaustive manner, provided m is even.

4.2. m is odd

We get a similar set to the previous subsection

$$\begin{aligned} L_{a_j} &= \{12m, 12m-2, 12m-4, \cdots, 10m+2; 12m-1, 12m-3, 12m-5, \\ &\cdots, 10m+3; 9m\}. \end{aligned}$$

$$\begin{aligned} L_{b_j} &= \{8m, 8m+2, 8m+4, \cdots, 9m-1; 9m+2, 9m+4, 9m+6, \cdots, \\ &10m-1; 8m+1, 8m+3, 8m+5, \cdots, 9m-2; 9m+1, 9m+3, \\ &9m+5, \cdots, 10m-2; \;\; 5m+1\}. \end{aligned}$$

$$\begin{aligned} L_{c_j} &= \{4m, 4m-4, 4m-8, \cdots, 2m+2; 2m-3, 2m-7, 2m-11, \cdots, 3; \\ &4m-2, 4m-6, 4m-10, \cdots, 2m+4; 2m-1, 2m-5, 2m-9, \\ &\cdots, 5; 4m-1\}. \end{aligned}$$

On the right side all values of $f^*(y_{it} \sim y_{it'})$ for all $t, t' \in \mathbb{Z}_3$, and $t \neq t'$.

$$\begin{aligned} R_{a_j} &= \{6m+2, 6m+4, 6m+6, \cdots, 8m-2; 6m+1, 6m+3, 6m+5, \cdots, \\ &8m-1; 10m+1\}. \end{aligned}$$

$$\begin{aligned} R_{b_j} &= \{6m, 6m-2, 6m-4, \cdots, 5m+3; 5m, 5m-2, 5m-4, \cdots, \\ &4m+3, 4m+1; 6m-1, 6m-3, 6m-5, \cdots, 5m+4; 5m+2; \\ &5m-1, 5m-3, 5m-5, \cdots, 4m+2; \\ &10m\}. \end{aligned}$$

$$\begin{aligned}R_{c_j} &= \{2, 6, 10, \cdots, 2m-4; 2m+1, 2m+5, 2m+9, \cdots, 4m-5; 2m;\\ &\quad 4, 8, 12\cdots, 2m-2;\\ &\quad 2m+3, 2m+7, 2m+11\cdots, 4m-3; 1\}.\end{aligned}$$

Rearranging the sets in consecutive manner we obtain the following:

$$L_{a_j} \cup R_{a_j} \cup L_{b_j} \cup R_{b_j} \cup L_{c_j} \cup R_{c_j} = \{1, 2, 3, \cdots, 12m\}$$

and $L_x \cap L_y = \emptyset$, where $x, y \in \{L_{a_j}, L_{b_j}, L_{c_j}, R_{a_j}, R_{b_j}, R_{c_j}\}$ and $x \neq y$. This implies that f^* assigns the edge-labels in an injective as well as exhaustive manner, provided m is odd.

This completes the proof that f^* is bijective. □

Theorem 4.2. *f does not label $K_{2m+1,2m+1} \odot K_2$ gracefully.*

Proof. If possible let us assume that $G = K_{n,n} \odot K_2$ is graceful when $n = 2m+1$. Now, The vertex labels are defined by the function $f : V(G) \longrightarrow \{0, 1, 2, \cdots, n^2+6n\}$ such that the induced function $f^* : E(G) \longrightarrow \{1, 2, \cdots,$
$n^2+6n\}$ assigns the edge labels. If we proceed in a similar manner, we observe that

$$f(i,j,0) = \begin{cases} n^2+6n-j, & \text{if } i = 0,\\ nj & \text{if } i = 1,\end{cases}$$

for $j \in \{0, 1, 2, \cdots, n-1\}$. Therefore, we get $f(0,j,0) \in \{n^2+6n, n^2+6n-1, n^2+6n-2, \cdots, n^2+5n+1\}$ on the left stem vertices and $f(1,j,0) \in \{0, n, 2n, , \cdots, n^2-n\}$ on the right stem vertices. As a result, the induced graph f^* assigns the labels $6n+1, 6n+2, cdots, n^2+6n$ to the edges of the bipartite graph. It remains to assign the remaining edge labels $1, 2, \cdots, 6n$ to the edges of the extended triangles. Note that, in each triangle, the sum of the three edge labels must be even which implies that the sum of all the edge labels in the triangles must be even. Now, this is possible only when $3n(6n+1)$ would be even, which is not possible if $n = 2m+1$. Hence, we arrive at a contradiction. □

5. Conclusion

In this paper we have investigated that the graph $G = K_{n,n} \odot K_2$ is graceful when n is even. We formally proposed a function for vertex labeling and shown that the function induces a graceful vertex labeling for graph G. We have also proved that the same technique fails to label the graph gracefully $G = K_{n,n} \odot K_2$ when n is odd. Hence our future work would be to investigate whether $G = K_{2m+1,2m+1} \odot K_2$ is graceful, and if so then find that labeling using a suitable method.

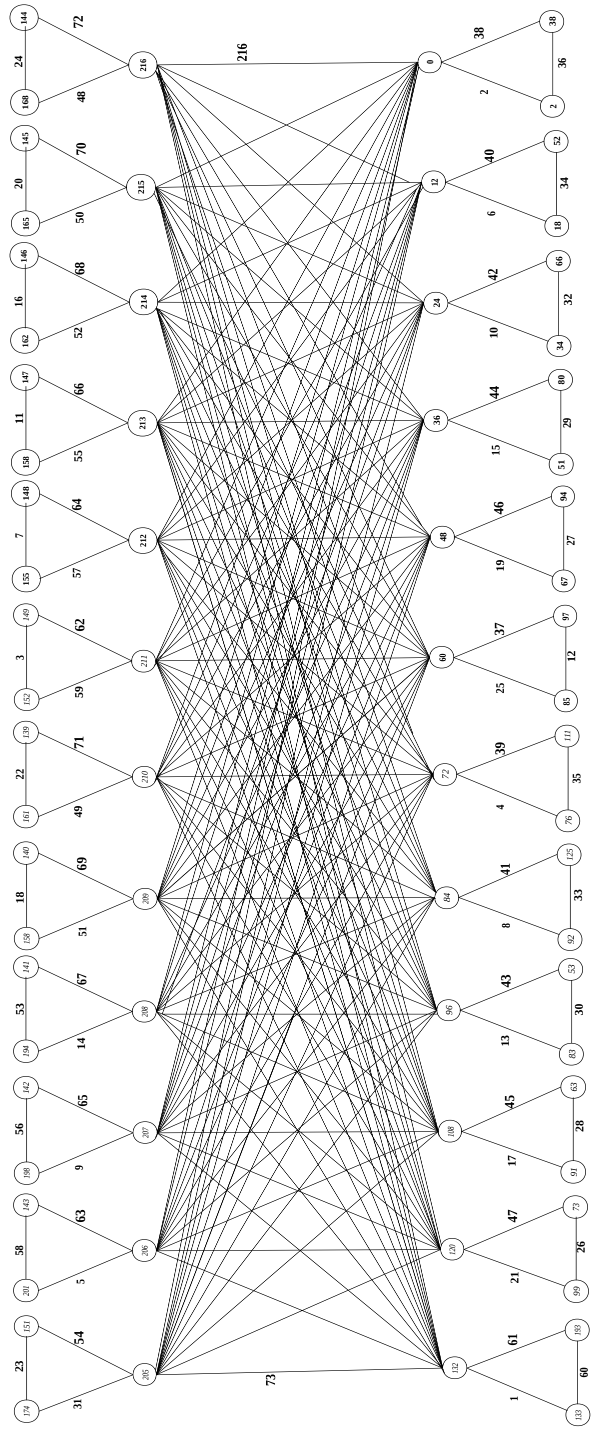

Figure 4. Graceful labeling of $K_{12,12} \odot K_2$.

References

[1] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (International Symposium Rome)* (1966), 349–355.

[2] G. Sethuraman and A. Elumalai, On graceful graphs: pendant edge extensions of a family of complete bipartite and complete tripartite graphs, *Indian J. Pure Appl. Math.* **32** (9) (2001), 1283–1296.

[3] J.A. Bondy, and U.S.R. Murty, *Graph Theory*, Springer, New York, (2008).

[4] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* (2017), #DS6.

[5] K.J. Ma, *Graceful Graphs*, Beijing University Publishing House, Beijing, (1991).

[6] M. Maheo, H. Thuillier, On d-graceful graphs, *Ars Combin.* **13** (1982), 181–192.

[7] P.J. Slater, On k-graceful graphs. In *Proc. of the 13th SE Conf. on Combinatorics*, Graph Theory and Computing (1982), 53–57.

[8] R.L. Graham, N.J. Sloane, On additive bases and harmonious graphs, *SIAM Journal on Algebraic Discrete Methods* **1**(4) (1980), 382–404.

[9] S. Bhoumik and S. Mitra, Graceful labeling of pendant edge extension of complete bipartite graph, *International Journal of Mathematical Analysis* **8** (58) (2014), 2885–2897.

[10] S. Mitra and S. Bhoumik, On graceful labeling of 1-crown for complete bipartite graph, *International Journal of Computational and Applied Mathematics* **10** (1) (2015) 69–75.

[11] S.W. Golomb, How to number a graph, *Graph Theory and Computing*, (1972), 23–37.

[12] W. Li, G. Li, and Q. Yan, Advances in computer science, environment, ecoinformatics, and education, *Communications in Computer and Information Science* **214** (2011), 297–301.

[13] Y.L. Jirimuta, On k-gracefulness of r-crown $I_r(K_{1,n})(n \geq 2, r \geq 2)$ for complete bipartite graph, *Journal of Inner Mongolia University for Nationalities* **2** (2003), 108–110.