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Computation of the eigenvalues of complete signed graphs

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Abstract

A signed graph Σ is the ordered pair (G, σ), where $G = (V, E)$ is a finite simple graph, called the underlying graph, and $\sigma\colon E(G) \longrightarrow \{+1, -1\}$ is a sign function or a signature of Σ. Let (K_n, σ) be a complete signed graph with n vertices. In this paper, we give a complete description of the adjacency, Laplacian and net Laplacian spectrum of a complete signed graph (K_n, σ) whenever its negative edges induce either a complete tripartite graph or a friendship graph. This is an addition to the class of complete signed graphs whose spectra is completely known.

Keywords: complete signed graph; complete tripartite graph; friendship graph; spectrum; Laplacian spectrum; net Laplacian spectrum
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1. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$. The signed graph $\Sigma = (G, \sigma)$ is a graph G together with a function $\sigma\colon E(G) \longrightarrow \{+1, -1\}$ called the signature of G. If $\sigma(e) = 1$ (respectively, $\sigma(e) = -1$) for every edge e, then σ is called the all-positive (respectively, all-negative) signature and $\Sigma = (G, \sigma)$ is called an all-positive (respectively, all-negative) signed graph. The underlying graph G is interpreted as a signed graph where all its edges are positive. Recent work on signed graphs which can be useful in future studies can be seen in [3, 5, 19].

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The adjacency matrix $A(\Sigma) = (a_{ij})$ of Σ is an $n \times n$ matrix such that $a_{ij} = \sigma(ij)$ if i and j are adjacent, and 0 otherwise. The Laplacian matrix of Σ is defined as $L(\Sigma) = D(\Sigma) - A(\Sigma)$, where $D(\Sigma)$ is the diagonal matrix of vertex degrees in Σ. For a vertex v of Σ, the net degree of v is the difference between the number of positive edges and the number of negative edges incident with this vertex. Accordingly, $D^{\pm}(\Sigma)$ is the diagonal matrix of net degrees and the net Laplacian matrix $N(\Sigma)$ is defined as $N(\Sigma) = D^{\pm}(\Sigma) - A(\Sigma)$.

We write $\Phi_M(x)$ and $\mathrm{Spec}(M)$ to denote the characteristic polynomial and the spectrum of a square matrix M, respectively. In particular, if M is the adjacency matrix of a signed graph Σ, this notation is simplified to $\Phi_\Sigma(x)$ and $\mathrm{Spec}(\Sigma)$. Also, we will write $\mathrm{LSpec}(\Sigma)$ and $\mathrm{NSpec}(\Sigma)$ for the Laplacian and net Laplacian spectrum, respectively. The spectrum of any matrix is considered as a multiset, say $\left\{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_t^{m_t}\right\}$, in which an exponent denotes the algebraic multiplicity of the corresponding eigenvalue.

In contrast to the former two matrices, the net Laplacian is less studied. It can be considered as a counterpart to the standard Laplacian matrix of an ordinary graph; moreover, in case of graphs these matrices coincide. Although it appears sporadically in earlier publications, it is known under the given name since 2020 [17]. Observe that 0 belongs to the net Laplacian spectrum of every signed graph; an associated eigenvector is the all-1 vector. Other spectral properties and relationships with the standard Laplacian can be found in [6, 15]. A relevance in control theory has been recognized in [4, 16]. Some spectral properties and relations with other graph matrices are given in [15, 18]. In [6], the net Laplacian spectrum of complete signed graphs is examined and signed graphs with exactly two distinct net Laplacian eigenvalues are characterized.

In recent years, the spectral analysis of signed graphs has attracted considerable attention from researchers. In [1], Akbari et al. determined the spectrum of complete signed graphs and complete bipartite signed graphs whenever negative edges form a matching. More recently, Pirzada et al. [8], determined the spectra of complete bipartite signed graphs whose negative edges induce either a matching, or a complete bipartite graph, or a double star graph. For similar results on signed graphs, we refer the reader to [2, 7, 11, 12, 16, 17]. Some recent work on spectra of signed graphs can be seen in [9, 10, 13, 14].

A complete tripartite graph is a graph whose vertex set is partitioned into three disjoint subsets, with every vertex in one subset adjacent to all vertices in the other two subsets and there are no edges within the same subset. It is denoted by $K_{a,b,c}$. A friendship graph is the graph obtained by joining p triangles at a single common vertex, so that each pair of triangles shares exactly this vertex. It is denoted by F_{2p+1}.

The results in this paper are mainly inspired by the work presented in [1] and [2]. To present our results, we consider Σ as a complete signed graph. In the following sections, we determine the spectrum, Laplacian spectrum, and net Laplacian spectrum of Σ whenever its negative edges induce either a complete tripartite graph or a friendship graph.

Section 2 serves as a preliminary section, providing the necessary notation and an important lemma used throughout the paper. Our main results are presented in Sections 3 and 4: the former addresses the case where the negative edges form a complete tripartite graph, while the latter focuses on the case where they form a friendship graph. Finally, Section 5 presents a brief conclusion.

2. Preliminaries

We write I_p to denote the $p \times p$ identity matrix, and $J_{p\times q}$ (or J_p if $p = q$) for the $p \times q$ all-1 matrix. Also, we denote the all-1 (resp. all-0) column vector of size p by $\mathbf{j}_p$ ($\mathbf{o}_p$).

In what follows, we will frequently deal with a complete signed graph $\Sigma = (K_n, \sigma)$ with n vertices. In this context, if H is a subgraph of K_n, then we write (K_n, H^-) to denote the complete signed graph underlined by K_n whose edge is negative if and only if it belongs to H.

We now consider a known result about the spectrum of a matrix. Let M be an $n \times n$ matrix blocked as

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{pmatrix},$$

where M_{ij} is an $n_i \times n_j$ matrix, for $1 \leq i, j \leq t$, and $n = \sum_{i=1}^{t} n_i$. If b_{ij} denotes the average row sum of M_{ij}, then $Q = (b_{ij})$ is called the *quotient matrix* of M. If, in addition, M_{ij} has a constant row sum, then Q is called the *equitable quotient matrix* of M. The following result is well known in the literature.

Theorem 2.1 ([20, Theorem 2.3])**.** *Let Q be the equitable quotient matrix of M. Then* $\mathrm{Spec}(Q) \subseteq \mathrm{Spec}(M)$.

3. Adjacency, Laplacian and Net Laplacian Spectrum of $(K_n, K^-_{a.b,c})$

We first deal with complete signed graphs Σ whose negative edges induce a complete tripartite graph. We first compute the spectrum.

Theorem 3.1. *Let $\Sigma \cong (K_n, K^-_{a,b,c})$. The spectrum of Σ consists of*
If $n \neq a + b + c$

(i) -1 *with multiplicity* $n - 4$,

(ii) the four roots of $x^4 + (4-n)x^3 + (6-3n)x^2 + C_1x + C_0 = 0$, *where* $C_1 = -4a^2b - 4a^2c - 4b^2a - 8abc - 4c^2a - 4b^2c - 4c^2b + 4abn + 4acn + 4bcn + 4 - 3n$ *and* $C_0 = 1 - n - 4a^2b - 4a^2c - 4b^2a - 8abc - 4c^2a - 4b^2c - 4c^2b + 4abn + 4acn + 4bcn + 16bca^2 + 16acb^2 + 16abc^2 - 16abcn$.

If $n = a + b + c$

(i) -1 *with multiplicity* $n - 3$,

(ii) the three roots of $x^3 - (n-3)x^2 + (3-2n)x - n + 1 + 4abc = 0$.

Proof. **Case 1.** With a suitable labeling of the vertices of Σ, its adjacency matrix is

$$A(\Sigma) = \begin{pmatrix} J_a - I_a & -J_{a\times b} & -J_{a\times c} & J_{a\times(n-a-b-c)} \\ -J_{b\times a} & J_b - I_b & -J_{b\times c} & J_{b\times(n-a-b-c)} \\ -J_{c\times a} & -J_{c\times b} & J_c - I_c & J_{c\times(n-a-b-c)} \\ J_{(n-a-b-c)\times a} & -J_{(n-a-b-c)\times b} & J_{(n-a-b-c)\times c} & J_{n-a-b-c} - I_{n-a-b-c} \end{pmatrix}.$$

Let $\mathbf{x} \in \mathbb{R}^a$, $\mathbf{y} \in \mathbb{R}^b$, $\mathbf{z} \in \mathbb{R}^c$ and $\mathbf{t} \in \mathbb{R}^{n-a-b-c}$ be column vectors orthogonal to $\mathbf{j}_a$, $\mathbf{j}_b$, $\mathbf{j}_c$ and $\mathbf{j}_{n-a-b-c}$ respectively. By setting $\mathbf{w} = (\mathbf{x}^\intercal, \mathbf{y}^\intercal, \mathbf{z}^\intercal, \mathbf{t}^\intercal)^\intercal \in \mathbb{R}^n$, we immediately obtain

$$A(\Sigma)\mathbf{w} = (-\mathbf{x}^\intercal, -\mathbf{y}^\intercal, -\mathbf{z}^\intercal, -\mathbf{t}^\intercal)^\intercal = -1\mathbf{w}.$$

It follows that $\mathbf{w}$ is an eigenvector associated with the eigenvalue -1. Since there are $a-1+b-1+c-1+n-a-b-c-1(=n-4)$ such linearly independent column vectors of size n that are orthogonal to $\mathbf{j}_n$, the algebraic multiplicity of -1 as an eigenvalue of Σ is (at least) $n-4$, which proves (i).

The remaining four eigenvalues are obtained from the 4×4 equitable quotient matrix Q of $A(\Sigma)$ according to Theorem 2.1. The matrix Q is given by

$$Q = \begin{pmatrix} a-1 & -b & -c & n-a-b-c \\ -a & b-1 & -c & n-a-b-c \\ -a & -b & c-1 & n-a-b-c \\ a & b & c & n-a-b-c-1 \end{pmatrix}.$$

The characteristic polynomial of Q is computed as follows. The trace of a matrix Q is given by

$$\text{Tr}(Q) = n-4.$$

We calculate the 2×2 principal minors of Q.

$M_{12} = \det \begin{pmatrix} a-1 & -b \\ -a & b-1 \end{pmatrix} = (a-1)(b-1) - ab = 1-a-b.$

$M_{13} = \det \begin{pmatrix} a-1 & -c \\ -a & c-1 \end{pmatrix} = (a-1)(c-1) - ac = 1-a-c.$

$M_{14} = \det \begin{pmatrix} a-1 & n-a-b-c \\ a & n-a-b-c-1 \end{pmatrix} = b+c+1-n.$

$M_{23} = \det \begin{pmatrix} b-1 & -c \\ -b & c-1 \end{pmatrix} = (b-1)(c-1) + bc = 1-b-c.$

$M_{24} = \det \begin{pmatrix} b-1 & n-a-b-c \\ b & n-a-b-c-1 \end{pmatrix} = a+c+1-n.$

$M_{34} = \det \begin{pmatrix} c-1 & n-a-b-c \\ a & n-a-b-c-1 \end{pmatrix} = b+a+1-n.$

Clearly, the sum of all 2×2 principal minors is $M_2 = 6-3n$. Further, the 3×3 principal minors of Q are given by

$M'_{11} = \det \begin{pmatrix} b-1 & -c & n-a-b-c \\ -b & c-1 & n-a-b-c \\ b & c & n-a-b-c-1 \end{pmatrix} = a(4bc-1) + 4b^2c + 4bc(c-n) + n - 1.$

$M'_{22} = \det \begin{pmatrix} a-1 & -c & n-a-b-c \\ -a & c-1 & n-a-b-c \\ a & c & n-a-b-c-1 \end{pmatrix} = 4c^2c + b(4ac-1) + 4ac(c-n) + n - 1.$

$M'_{33} = \det \begin{pmatrix} a-1 & -b & n-a-b-c \\ -a & b-1 & n-a-b-c \\ a & b & n-a-b-c-1 \end{pmatrix} = 4a^2b + 4ab(b+c-n) - c + n - 1.$

$$M'_{44} = \det \begin{pmatrix} a-1 & -b & -c \\ -a & b-1 & -c \\ -a & -b & c-1 \end{pmatrix} = -4abc + a + b + c - 1.$$

Clearly, the sum of all 3×3 principal minors is $M_3 = 4a^2b+4a^2c+4b^2a+8abc+4c^2a+4b^2c+4c^2b-4abn-4acn-4bcn-4+3n$. Lastly, the determinant of Q is given by $\det(Q) = 1-n-4a^2b-4a^2c-4b^2a-8abc-4c^2a-4b^2c-4c^2b+4abn+4acn+4bcn+16bca^2+16acb^2+16abc^2-16abcn$.

Therefore, the characteristic polynomial of Q can be expressed as

$$\Phi_Q(x) = x^4 - \text{Tr}(Q)x^3 + M_2x^2 - M_3x + det(Q),$$

which after substituting the above values transforms into

$$\Phi_Q(x) = x^4 + (4-n)x^3 + (6-3n)x^2 + C_1x + C_0,$$

where C_0 and C_1 are same as given in the formulation of statement(ii). It is worth noting that some of the eigenvalues obtained from the quotient matrix may coincide with those already found. To avoid repetition, we must ensure that these eigenvalues are distinct from the previously determined ones.

Next, we prove that -1 is not a root of $\Phi_Q(x) = 0$.

$$\Phi_Q(-1) = \det(-I-Q) = (-1)^4\det(I+Q) = \det(I+Q) = \det \begin{pmatrix} a & -b & -c & n-a-b-c \\ -a & b & -c & n-a-b-c \\ -a & -b & c & n-a-b-c \\ a & b & c & n-a-b-c \end{pmatrix}.$$

Subtracting the 4th row from the first three and expanding along the 4th column, gives

$$\Phi_Q(-1) = -16abc(n-a-b-c) \neq 0.$$

This shows that -1 is not a zero of $\Phi_Q(x)$ and thus completing the proof of case 1.

Case 2. With a suitable labelling of the vertices of Σ, its adjacency matrix is

$$A(\Sigma) = \begin{pmatrix} J_a - I_a & -J_{a\times b} & -J_{a\times c} \\ -J_{b\times a} & J_b - I_b & -J_{b\times c} \\ -J_{c\times a} & -J_{c\times b} & J_c - I_c \end{pmatrix}.$$

Part (i) is proved by considering a vector $\mathbf{w} = (\mathbf{x}^\intercal, \mathbf{y}^\intercal, \mathbf{z}^\intercal)^\intercal \in \mathbb{R}^n$, where $\mathbf{x} \perp \mathbf{j}_a$, $\mathbf{y} \perp \mathbf{j}_b$ and $\mathbf{z} \perp \mathbf{j}_c$. The remaining three eigenvalues are obtained from the 3×3 equitable quotient matrix

$$Q = \begin{pmatrix} a-1 & -b & -c \\ -a & b-1 & -c \\ -a & -b & c-1 \end{pmatrix}.$$

Clearly, we have

$$\Phi_Q(x) = x^3 - (n-3)x^2 + (3-2n)x - n + 1 + 4abc$$

and since $\Phi_Q(-1) = 4abc \neq 0$, the proof of the theorem is complete. □

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As the signed graph under consideration is $(n-1)$-regular, each eigenvalue λ of Σ yields a Laplacian eigenvalue $n-1-\lambda$. Hence, the Laplacian spectrum of Σ follows directly from its adjacency spectrum, and the corresponding theorem is omitted. Now, we proceed with the net Laplacian spectrum.

Theorem 3.2. *Let $\Sigma \cong (K_n, K^-_{a.b,c})$, $n \neq a+b+c$. The net Laplacian spectrum of Σ is given by*

$$\Big\{0, (n-2(a+b+c))^2, (n-2b-2c)^{a-1}, (n-2a-2c)^{b-1}, (n-2a-2b)^{c-1}, n^{n-a-b-c}\Big\}.$$

Proof. With a suitable labelling of the vertices of Σ, the diagonal matrix of vertex net degrees is

$$D^{\pm}(\Sigma) = \operatorname{diag}\Big((n-2b-2c-1)I_a, (n-2a-2c-1)I_b, (n-2a-2b-1)I_c, (n-1)I_{n-s}\Big),$$

where $s = a+b+c$. Accordingly, the net Laplacian matrix is

$$N(\Sigma) = \begin{pmatrix} (n-2b-2c)I_a - J_a & J_{a\times b} & J_{a\times c} & -J_{a\times(n-s)} \\ J_{b\times a} & (n-2a-2c)I_b - J_b & J_{b\times c} & -J_{b\times(n-s)} \\ J_{c\times a} & J_{c\times b} & (n-2a-2b)I_c - J_c & -J_{c\times(n-s)} \\ -J_{(n-s)\times a} & -J_{(n-s)\times b} & -J_{(n-s)\times c} & nI_{n-s} - J_{n-s} \end{pmatrix}.$$

Let $\mathbf{x} \in \mathbb{R}^a$ be a vector orthogonal to $\mathbf{j}_a$. By setting $\mathbf{z} = (\mathbf{x}^\intercal, \mathbf{o}_b^\intercal, \mathbf{o}_c^\intercal, \mathbf{o}_{n-s}^\intercal)^\intercal \in \mathbb{R}^n$, we immediately obtain $N(\Sigma)\mathbf{z} = (n-2b-2c)\mathbf{z}$. Therefore, $n-2b-2c$ is the net Laplacian eigenvalue. Since there are $a-1$ linearly independent column vectors of size a that are orthogonal to $\mathbf{j}_a$, the algebraic multiplicity of $n-2b-2c$ as an eigenvalue of $N(\Sigma)$ is (at least) $a-1$.

By considering a vector $\mathbf{z} = (\mathbf{o}_a^\intercal, \mathbf{x}^\intercal, \mathbf{o}_c^\intercal, \mathbf{o}_{n-s}^\intercal)^\intercal \in \mathbb{R}^n$, where $\mathbf{x}$ is one of $b-1$ vectors that belong to $\mathbb{R}^b$ and are orthogonal to $\mathbf{j}_b$, we immediately obtain $N(\Sigma)\mathbf{z} = (n-2a-2c)\mathbf{z}$.

Similarly, for $n-2a-2b$, we consider a vector $\mathbf{z} = (\mathbf{o}_a^\intercal, \mathbf{o}_b^\intercal, \mathbf{x}^\intercal, \mathbf{o}_{n-s}^\intercal)^\intercal \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^c$and $\mathbf{x} \perp \mathbf{j}_c$, with $(c-1)$ possible linearly independent choices for $\mathbf{x}$. Finally, $\mathbf{z} = (\mathbf{o}_a^\intercal, \mathbf{o}_b^\intercal, \mathbf{o}_c^\intercal, \mathbf{x}^\intercal)^\intercal \in \mathbb{R}^n$, where $\mathbf{x} \in \mathbb{R}^{n-s}$and $\mathbf{x} \perp \mathbf{j}_{n-s}$, leads to the analogous conclusion for n.

In order to determine the remaining four net Laplacian eigenvalues, consider a 4×4 equitable quotient matrix

$$Q = \begin{pmatrix} n-2b-2c-a & b & c & s-n \\ a & n-2a-2c-b & c & s-n \\ a & b & n-2a-2b-c & s-n \\ -a & -b & -c & s \end{pmatrix}.$$

Observe that $\mathbf{j}_4$ and $\left(1,1,1,\frac{-s}{n-s}\right)^\intercal$ are eigenvectors corresponding to 0 and n, respectively. Similarly, by considering two linearly independent vectors $\mathbf{v}_1 = (-b,a,0,0)$ and $\mathbf{v}_2 = (-c,0,a,0)$, we immediately obtain $Q\mathbf{v}_i = (n-2(a+b+c))\,\mathbf{v}_i$, $i \in \{1,2\}$. Thus, we conclude that 0 and $n-2(a+b+c)$ are also eigenvalues of $N(\Sigma)$.

Let λ_1 and λ_2 be the remaining two net Laplacian eigenvalues of Σ. We have

$$\operatorname{Tr}(N(\Sigma)) = a(n-2b-2c-a-1)+b(n-2a-2c-b-1)+c(n-2a-2b-c-1)+(n-a-b-c)(n-1),$$

implies

$$\begin{aligned}\lambda_1+\lambda_2 &= a(n-2b-2c-a-1)+b(n-2a-2c-b-1)+c(n-2a-2b-c-1)\\ &\quad+(n-a-b-c)(n-1)-(a-1)(n-2b-2c)-(b-1)(n-2a-2c)\\ &\quad-(c-1)(n-2a-2b)-(n-a-b-c-1)n-(n-2(a+b+c))\\ &= 2(n-a-b-c). \qquad (1)\end{aligned}$$

Also,

$$\begin{aligned}\mathrm{Tr}\big(N(\Sigma)^2\big) &= \mathrm{Tr}\Big(\big(\big(D^{\pm}(\Sigma)-A(\Sigma)\big)\big(D^{\pm}(\Sigma)-A(\Sigma)\big)\Big)\\ &= \mathrm{Tr}\Big(D^{\pm}(\Sigma)^2-D^{\pm}(\Sigma)A(\Sigma)-A(\Sigma)D^{\pm}(\Sigma)+A(\Sigma)^2\Big)\\ &= \mathrm{Tr}(D^{\pm}(\Sigma)^2)-2\mathrm{Tr}(D^{\pm}(\Sigma)A(\Sigma))+\mathrm{Tr}(A(\Sigma)^2)\\ &= \sum_{i=1}^{n} {d_i^{\pm}}^2+n(n-1),\end{aligned}$$

where the later equality is obtained by using the fact $\mathrm{Tr}(D^{\pm}(\Sigma)A(\Sigma))=0$ and $\mathrm{Tr}(A(\Sigma)^2)=\sum_{i=1}^{n} d_i$, where d_i is the degree of the vertex v_i.

Thus, we have

$$\begin{aligned}{\lambda_1}^2+{\lambda_2}^2 &= a(n-2b-2c-a-1)^2+b(n-2a-2c-b-1)^2+c(n-2a-2b-c-1)^2\\ &\quad+(n-a-b-c)(n-1)^2+n(n-1)-(a-1)(n-2b-2c)^2\\ &\quad-(b-1)(n-2a-2c)^2-(c-1)(n-2a-2b)^2-(n-a-b-c-1)n^2\\ &\quad-(n-2(a+b+c))^2\\ &= 2\big((a+b+c)^2+(n-(a+b+c))^2\big). \qquad (2)\end{aligned}$$

We know that $2\lambda_1\lambda_2=(\lambda_1+\lambda_2)^2-{\lambda_1}^2-{\lambda_2}^2$ and hence, in view of Equations (1) and (2), we have

$$\lambda_1\lambda_2=n\big(n-2(a+b+c)\big).$$

Thus, λ_1 and λ_2 satisfy the quadratic equation

$$t^2-2(n-a-b-c)t+n\big(n-2(a+b+c)\big)=0.$$

Solving this equation gives the roots n and $n-2(a+b+c)$, thereby completing the proof of the theorem. □

We illustrate the previous results in an example.

Example 3.3. Consider the complete signed graph $(K_6, K^-_{2,1,1})$ depicted in Figure 1. By plugging $(n,a,b,c)=(6,2,1,1)$ in Theorem 3.1(case 1), the bi-quadratic equation transforms to $x^4-2x^3-12x^2+34x-21=0$ and its roots are $-1\pm2\sqrt{2}$,1,3. Thus, we have

$$\mathrm{Spec}((K_6, K^-_{2,1,1}))=\Big\{(-1)^2,1,-1-2\sqrt{2},-1+2\sqrt{2},3\Big\},$$

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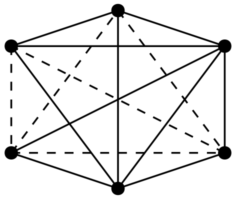

Figure 1. The signed graph $(K_6, K^-_{2,1,1})$. Negative edges are dashed.

$$\mathrm{LSpec}((K_6, K^-_{2,1,1})) = \Big\{2, 6-2\sqrt{2}, 6+2\sqrt{2}, 4, 6^2\Big\}.$$

Similarly, in view of Theorem 3.2,

$$\mathrm{NSpec}((K_6, K^-_{2,1,1})) = \Big\{(-2)^2, 0, 2, 6^2\Big\}.$$

4. Adjacency, Laplacian and net Laplacian spectrum of (K_n, F^-_{2p+1})

Theorem 4.1. *Let* $\Sigma \cong (K_n, F^-_{2p+1})$, $p \notin \{1, \frac{n-2}{2}, \frac{n-1}{2}\}$. *The spectrum of* Σ *is given by*

$$\Big\{(-3)^{p-1}, (-1)^{n-2p-2}, 1^p, \alpha_1, \alpha_2, \alpha_3\Big\}, \textit{where}$$

$\alpha_i, 1 \le i \le 3$ *are the roots of* $x^3 + (5-n)x^2 + (4p-4n+7)x + 8np - 3n - 16p^2 - 4p + 3 = 0.$

Proof. With a suitable labelling of the vertices of Σ, its adjacency matrix is

$$A(\Sigma) = \begin{pmatrix} 0 & -\mathbf{j}_p^\intercal & -\mathbf{j}_p^\intercal & \mathbf{j}_{n-2p-1}^\intercal \\ -\mathbf{j}_p & J_p - I_p & J_p - 2I_p & J_{p\times(n-2p-1)} \\ -\mathbf{j}_p & J_p - 2I_p & J_p - I_p & J_{p\times(n-2p-1)} \\ \mathbf{j}_{n-2p-1} & J_{(n-2p-1)\times p} & J_{(n-2p-1)\times p} & J_{n-2p-1} - I_{n-2p-1} \end{pmatrix}.$$

Let $\mathbf{x} \in \mathbb{R}^p$ be a vector orthogonal to $\mathbf{j}_p$. By setting $\mathbf{z} = (0, \mathbf{x}^\intercal, \mathbf{x}^\intercal, \mathbf{o}_{n-2p-1}^\intercal)^\intercal \in \mathbb{R}^n$, we immediately obtain $A(\Sigma)\mathbf{z} = -3\mathbf{z}$. Therefore, -3 is the eigenvalue with multiplicity (at least) $p-1$.

By setting $\mathbf{z} = (0, \mathbf{x}^\intercal, -\mathbf{x}^\intercal, \mathbf{o}_{n-2p-1}^\intercal)^\intercal \in \mathbb{R}^n$, $x \perp \mathbf{j}_p$, we immediately obtain $A(\Sigma)\mathbf{z} = 1\mathbf{z}$. Therefore, 1 is the eigenvalue with multiplicity (at least) $p-1$.

Next, we consider a vector $\mathbf{z} = (0, \mathbf{o}_p^\intercal, \mathbf{o}_p^\intercal, \mathbf{x}^\intercal)^\intercal \in \mathbb{R}^n$, $x \perp \mathbf{j}_{n-2p-1}$, we immediately obtain $A(\Sigma)\mathbf{z} = -1\mathbf{z}$. Therefore, -1 is the eigenvalue with multiplicity (at least) $n-2p-2$.

In order to determine the remaining four eigenvalues, consider a 4×4 equitable quotient matrix

$$Q = \begin{pmatrix} 0 & -p & -p & n-2p-1 \\ -1 & p-1 & p-2 & n-2p-1 \\ -1 & p-2 & p-1 & n-2p-1 \\ 1 & p & p & n-2p-2 \end{pmatrix}.$$

The characteristic polynomial of Q is

$$\Phi_Q(x) = (x-1)P(x),$$

where $P(x) = x^3 + (5-n)x^2 + (4p-4n+7)x + 8np - 3n - 16p^2 - 4p + 3$.

Now, we will show that the roots of $P(x) = 0$ are distinct from the previously determined ones.
Clearly, $P(1) = 8(2 - n + pn - 2p^2) = 0 \ \Leftrightarrow p = 1$ or $p = \frac{n-2}{2}$,
$P(-1) = 0 \ \Leftrightarrow p = 0$ or $p = \frac{n-1}{2}$ and
$P(-3) = 0 \ \Leftrightarrow p = 0$ or $p = \frac{n-2}{2}$.

Since it is given that $p \notin \{1, \frac{n-2}{2}, \frac{n-1}{2}\}$, it follows that $P(x) \neq 0$ for $x \in \{-3,-1,1\}$. Hence, we conclude that the three roots of $P(x) = 0$ are also eigenvalues of $A(\Sigma)$, distinct from those previously determined. The remaining eigenvalue can be determined using the trace of the matrix, since the sum of all eigenvalues equals zero. It is straightforward to verify that this eigenvalue is 1, and therefore its total multiplicity is p. This completes the proof. □

As before, we omit the theorem related to the Laplacian spectrum of Σ. Finally, we obtain the net Laplacian spectrum of Σ.

Theorem 4.2. *Let $\Sigma \cong (K_n, F^-_{2p+1})$. The net Laplacian spectrum of Σ is given by*

$$\Big\{0, (n-6)^p, (n-2)^{p-1}, n^{n-2p-1}, n-4p-2\Big\}.$$

Proof. The diagonal matrix of vertex net degrees is

$$D^{\pm}(\Sigma) = \mathrm{diag}\Big((n-4p-1)I_1, (n-5)I_p, (n-5)I_p, (n-1)I_{n-2p-1}\Big).$$

Accordingly, the net Laplacian matrix is

$$N(\Sigma) = \begin{pmatrix} n-4p-1 & \mathbf{j}_p^{\intercal} & \mathbf{j}_p^{\intercal} & -\mathbf{j}_{n-2p-1}^{\intercal} \\ \mathbf{j}_p & (n-4)I_p - J_p & 2I_p - J_p & -J_{p\times(n-2p-1)} \\ \mathbf{j}_p & 2I_p - J_p & (n-4)I_p - J_p & -J_{p\times(n-2p-1)} \\ -\mathbf{j}_{n-2p-1} & -J_{(n-2p-1)\times p} & -J_{(n-2p-1)\times p} & nI_{n-2p-1} - J_{n-2p-1} \end{pmatrix}.$$

The proof proceeds similarly to that of the preceding theorem i.e., For the net Laplacian eigenvalues $n-2$, $n-6$ and n, we select the same vectors as those corresponding to the eigenvalues -3, 1, and -1, respectively, in Theorem 4.1.

Next, we consider 4×4 equitable quotient matrix

$$Q = \begin{pmatrix} n-4p-1 & p & p & 2p+1-n \\ 1 & n-p-4 & 2-p & 2p+1-n \\ 1 & 2-p & n-p-4 & 2p+1-n \\ -1 & -p & -p & 2p+1 \end{pmatrix}.$$

The characteristic polynomial of Q is given by

$$\Phi_Q(x) = x(x-n)(x-n+6)(x-n+4p+2).$$

Therefore, 0 and $n-4p-2$ are also net Laplacian eigenvalues of Σ.

Let λ_1 and λ_2 be the remaining two net Laplacian eigenvalues of Σ. It is not hard to see that

$$\lambda_1 + \lambda_2 = 2n-6,\ \lambda_1^2 + \lambda_2^2 = 2(n^2 - 6n + 18) \text{ and } \lambda_1\lambda_2 = n^2 - 6n.$$

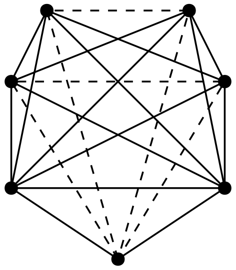

Figure 2. The signed graph (K_7, F_5^-).

Therefore, λ_1 and λ_2 are the roots of the quadratic equation

$$t^2 - (2n-6)t + n^2 - 6n = 0,$$

which upon solving yields $\lambda_1 = n$ and $\lambda_2 = n-6$, completing the proof. □

As before, we provide an example.

Example 4.3. Consider the complete signed graph (K_7, F_5^-) depicted in Figure 2. By plugging $(n, p) = (7, 2)$ in Theorem 4.1, the cubic equation transforms to $x^3 - 2x^2 - 13x + 22 = 0$ and its roots are -3.5033, 1.6151, 3.8882. Thus, we have

$$\mathrm{Spec}((K_7, F_5^-)) = \Big\{ -3.5033, -3, -1, 1^2, 1.6151, 3.8882 \Big\},$$

$$\mathrm{LSpec}((K_7, F_5^-)) = \Big\{ 2.1118, 4.3849, 5^2, 7, 9, 9.5033 \Big\}.$$

Similarly, in view of Theorem 4.2,

$$\mathrm{NSpec}((K_7, F_5^-)) = \Big\{ -3, 0, 1^2, 5, 7^2 \Big\}.$$

5. Conclusion

We have examined complete signed graphs Σ in which the set of negative edges induces either a complete tripartite graph or a friendship graph. Note that if $-\Sigma$ denotes the signed graph obtained by reversing the sign of every edge of Σ , then $A(-\Sigma) = -A(\Sigma)$ and $N(-\Sigma) = -N(\Sigma)$. Consequently, the corresponding results in the case where the positive edges (instead of the negative ones) induce any of the aforementioned graphs can be derived directly when dealing with the adjacency matrix or the net Laplacian matrix. In particular, the spectrum of $-\Sigma$ is obtained by negating each eigenvalue of Σ.

On the other hand, $L(-\Sigma) = -L(\Sigma)$ does not hold unless Σ is edgeless, since the main diagonal entries remain unchanged. Nevertheless, the analogous results for $-\Sigma$ can be derived in a similar manner, following the same arguments used for Σ.

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