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The matching book embedding under some graph operations

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Abstract

The *matching book embedding* of a graph G is an embedding of G with the vertices on the spine, and each edge within a single page so that the edges on each page do not intersect and the degree of vertices on each page is at most one. The *matching book thickness* of G is the minimum number of pages in a matching book embedding of G, denoted by *mbt(G)*. In this paper, the exact matching book thickness of the corona product between a dispersible or nearly dispersible graph and a simple graph is determined. Additionally, the dispersibility of the edge product of a cycle and a simple graph is obtained. Finally the matching book thickness of the comb product of two dispersible graphs is obtained.

Keywords: matching book embedding, matching book thickness, corona product, comb product
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1. Introduction

The book embedding of a graph was first introduced by Bernhart and Kainen[1]. They defined an *n-book*, which is composed of a line L in 3-space (called *spine*) and *n* distinct half-planes (called *pages*), where L forms the common boundary of the *n* half-planes. An *n-book embedding* is an embedding of G such that each vertex of G is placed on the spine, and each edge is placed on

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at most one page, with no two edges on the same page intersecting. The *book thickness* of a graph G is the smallest *n* so that G has an *n-book embedding*, denoted by $bt(G)$.

Originally applied to VLSI routing algorithms[2], book embedding theory has been extensively utilized in TSV placement[3], network topology visualization[4], genome sequence compression[5], and stack number computation[6]. Although determining the book thickness of a graph is an NP-complete problem, the matching book thickness of graph provides a computable upper bound, thereby refining the page number estimation for book embedding.

A *matching book embedding* of a graph G is a book embedding where on each page the maximum degree of vertices is at most one. The *matching book thickness* of a graph G is the smallest k such that G has a k page matching book embedding, denoted by $mbt(G)$. A graph G is *dispersable* if $mbt(G) = \Delta(G)$, *nearly dispersable* if $mbt(G) = \Delta(G) + 1$. Matching book embeddings have been extensively studied for various families of graphs. In 1998, Overbay proved the complete bipartite graph $K_{n,n}$ $(n \geq 1)$, even cycle C_{2n} $(n \geq 2)$, binary n-cube $Q(n)$ $(n \geq 0)$ and tree are dispersible[7]. In [12], it was proved that $bt(H\Box G) \leq s + mbt(H)$, $bt(H \times G) \leq 2q \cdot mbt(H)$, and $bt(H \boxtimes G) \leq 2q \cdot mbt(H) + s + mbt(H)$, where H is a bipartite graph and G is a graph that admits a simultaneous s-stack q-queue layout. This finding establishes an upper bound for the book thickness of graph products by combining their matching book thickness, stack number, and queue number.

Significant progress has been made regarding the matching book thickness of the Cartesian product of two cycle. Kainen[13] proved that $mbt(C_p\Box C_q) = 4$, when p, q are both even, and $mbt(C_p\Box C_q) = 5$, when p is even, q is odd. In 2020, Shao, Liu, Li[14] established that for $n, q \geq 3$, $mbt(K_n\Box C_q) = \triangle(K_n\Box C_q) + 1$, which implies $mbt(C_3\Box C_q) = 5$. Joslin, Kainen, Overbay[15] further proved in 2021 that $mbt(C_5\Box C_q) = 5$. In 2023, Shao, Yu, Li[16] obtained that $mbt(C_p\Box C_q) = 5$ when both p and q are odd and $p \geq q \geq 7$, which completely solves the problem about the dispersibility of the Cartesian product of two cycles. Additionally, the matching book thickness of generalized Petersen graphs[17] and pseudo-Halin graphs[18] is obtained. References [19] and [20] respectively investigated the dispersibility of the circulant graphs $C(\mathbb{Z}_n, \{1, k\})$ and the Kronecker cover of the Cartesian product of complete graphs K_p and cycles C_q. Recently, Yuan, Geng, Yang, Guan[21] showed that $mbt(\theta_m\Box C_n) = 5$, when n is even, and $mbt(\theta_m\Box C_n) = 5$ or 6, when n is odd, where θ_m is a uni-chord cycle obtained by adding an edge connecting two non-consecutive vertices from a cycle C_m with $m \geq 4$.

In this paper, we obtain that matching book thickness of the corona product of a dispersible or nearly dispersible graph and a simple graph. Additionally, we get the dispersibility of the edge corona product between a cycle and an arbitrary graph. Finally, the matching book thickness of the comb product of two dispersible graphs is determined.

2. Preliminaries

In this section, we present some definitions and results which we needed in our work.

Definition 1. *[10] Let G and H be simple graphs. The corona product $G \circ H$ of G and H is constructed as follows: Choose a labeling of the vertices of G with labels $1, \ldots, m$. Take one copy of G and m disjoint copies of H, labeled $H_1, \ldots, H_m$, and connect each vertex of H_i to vertex i of G.*

It follows the definition of the corona product that the $G \circ H$ is not in general isomorphic to $H \circ G$. (See Fig.1 for the case $K_4 \circ C_4$ and $C_4 \circ K_4$).

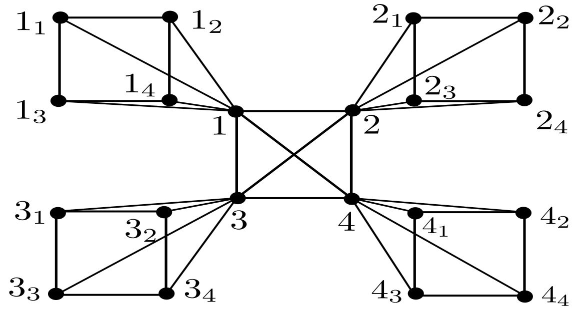

Fig.1 (Left)$K_4 \circ C_4$. (Right)$C_4 \circ K_4$.

Definition 2. *[9] Let G and H be simple graphs. The edge corona product $G \diamond H$ of G and H is constructed as follows: Choose a labeling of the edges of G with labels $1, \ldots, m$. Take one copy of G and m disjoint copies of H, labeled $H_1, \ldots, H_m$, and connect two endpoints of the edge i of G to all vertices of H_i.*

Definition 3. *[11] Let v be a vertex of H. The comb product between G and H, denoted by $G \rhd_v H$, is a graph obtained by taking one copy of G and $V(G)$ copies H and identify the i-th copy of H at the vertex v with the i-th vertex of G.*

By the definition of comb product, we say that $V(G \rhd_v H) = \{(g_i, h_j) \mid g_i \in V(G), h_j \in V(H)\}$ and two vertices (g_i, h_j) and (g_k, h_l) are adjacent if and only if

(a) $g_i = g_k$ and $h_j h_l \in E(H)$, or

(b) $g_i g_k \in E(G)$ and $h_j = h_l = v$.

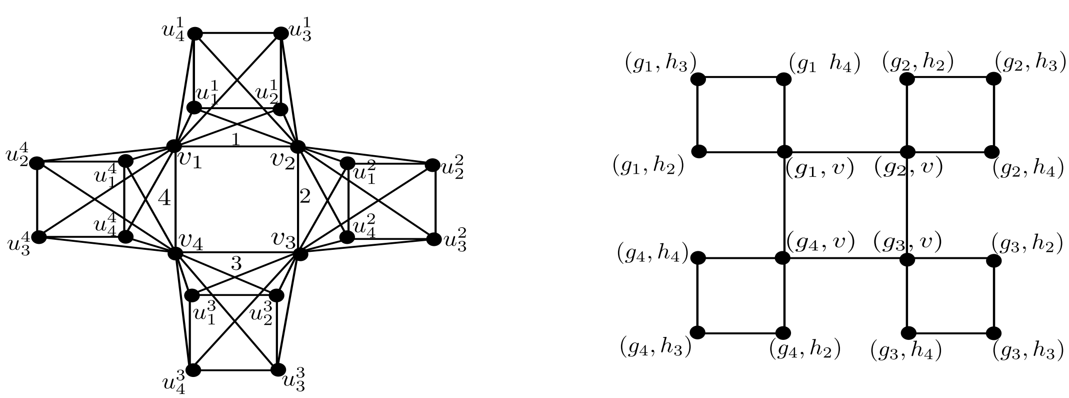

Fig.2 (Left)$C_4 \diamond C_4$. (Right)$C_4 \rhd_v C_4$.

Lemma 2.1. *[3] For any simple graph G, $\Delta(G) \leq \chi'(G) \leq mbt(G)$, where $\chi'(G)$ is the chromatic index of G. a*

Lemma 2.2. *[3] If H is the subgraph of G, then $mbt(H) \leq mbt(G)$.*

Lemma 2.3. *[1] Let G be a simple graph.*

(i) If G has an n-book matching embedding with printing cycle $v_1, v_2, \dots, v_p$, then G also has an n-book matching embedding with printing cycle $v_2, v_3, \dots, v_p, v_1$.

(ii) If G has an n-book matching embedding β with printing cycle $v_1, v_2, \dots, v_p$, then G also has an n-book matching embedding β^- with printing cycle $v_p, \dots, v_2, v_1$.

3. Results for the Corona Product

For a simple graph G with q vertices and a complete graph K_p, we assume that $V(G \circ K_p) = \{ v_i, u_j^i \mid i = 1, 2, \dots, q; j = 1, 2, \dots, p \}$ and $E(G \circ K_p) = E(G) \cup \{ (u_j^i, u_k^i) \mid i = 1, 2, \dots, q; j, k = 1, 2, \dots, p; j \neq k\} \cup \{ (v_i, u_j^i) \mid i = 1, 2, \dots, q, j = 1, 2, \dots, p \}$.

Next, we will compute the matching book thickness of the corona product between a dispersible graph and a simple graph.

Lemma 3.1. *Let G be an arbitrary dispersible graph with q vertices and $\triangle(G) = k$, and let K_p be a complete graph. Then, $G \circ K_p$ is dispersible.*

Proof. According to $\triangle(G \circ K_p) = p + k$, then $mbt(G \circ K_p) \geq \triangle(G \circ K_p)$ by Lemma 2.1. Because G is dispersible, without loss of generality, assume that $v_1, v_2, \dots, v_q$ is the vertex order of the dispersible book embedding of G. Then, we put the vertices of $G \circ K_p$ along the spine according to the following ordering of $v_1, u_1^1, u_2^1, \dots u_p^1, v_2, u_1^2, u_2^2, \dots u_p^2, \dots v_q, u_1^q, u_2^q, \dots u_p^q$. Obviously, the edges of G can be matching book embedded in k pages named page 1, page 2,…, page k, and one of these k pages can place the edges $\{ (u_j^i, u_{p-j+1}^i) \mid 1 \leq i \leq q, 1 \leq j \leq \lceil \frac{p}{2} \rceil \}$, which do not cause influence on the matching book embedding of G. The remaining edges of $G \circ K_p$ can be matching book embedded in additional p pages as follows.

Page $k+1$: edges $\{ (v_i, u_1^i) \mid 1 \leq i \leq q \}$, and edges $\{ (u_{1+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 1 \}$.

Page $k+2$: edges $\{ (v_i, u_2^i) \mid 1 \leq i \leq q \}$, and edges $\{ (u_{2+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 1 \}$.

Page $k+3$: edges $\{ (v_i, u_3^i) \mid 1 \leq i \leq q \}$, edges $\{ (u_{3+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 2 \}$, and edges $\{ (u_1^i, u_2^i) \mid 1 \leq i \leq q \}$.

Page $k+4$: edges $\{ (v_i, u_4^i) \mid 1 \leq i \leq q \}$, edges $\{ (u_{4+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 2 \}$, and edges $\{ (u_1^i, u_3^i) \mid 1 \leq i \leq q \}$.

Page $k+5$: edges $\{ (v_i, u_5^i) \mid 1 \leq i \leq q \}$, edges $\{ (u_{5+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 3 \}$, and edges $\{ (u_1^i, u_4^i), (u_2^i, u_3^i) \mid 1 \leq i \leq q \}$.

Page $k+6$: edges $\{ (v_i, u_6^i) \mid 1 \leq i \leq q \}$, edges $\{ (u_{6+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 3 \}$, and edges $\{ (u_1^i, u_5^i), (u_2^i, u_4^i) \mid 1 \leq i \leq q \}$.

…

Page $k+p-3$: edges $\{ (v_i, u_{p-3}^i) \mid 1 \leq i \leq q \}$, edges $\{ (u_j^i, u_{p-3-j}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 2 \}$. and edges $\{ (u_{p-2}^i, u_p^i) \mid 1 \leq i \leq q \}$.

Page $k+p-2$: edges $\{ (v_i, u_{p-2}^i) \mid 1 \leq i \leq q \}$, edges $\{ (u_j^i, u_{p-2-j}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 1 \}$ and edges $\{ (u_{p-1}^i, u_p^i) \mid 1 \leq i \leq q \}$.

Page $k+p-1$: edges $\{ (v_i, u^i_{p-1}) \mid 1 \le i \le q \}$, and edges $\{ (u^i_j, u^i_{p-1-j}) \mid 1 \le i \le q; 1 \le j \le \left[\frac{p}{2}\right] - 1 \}$.

Page $k+p$: edges $\{ (v_i, u^i_p) \mid 1 \le i \le q \}$, and edges $\{ (u^i_j, u^i_{p-j}) \mid 1 \le i \le q; 1 \le j \le \left[\frac{p}{2}\right] - 1 \}$.

It is clear that we can construct a matching book embedding of $G \circ K_p$ in a $(k+p)$-page book. Hence, $mbt\,(G \circ K_p) = \triangle(G_q \circ K_p) = k + p$. That is, $G \circ K_p$ is dispersible. (See Fig.3 for the case $C_4 \circ K_4$ and Fig.4 for the case $C_6 \circ K_5$).

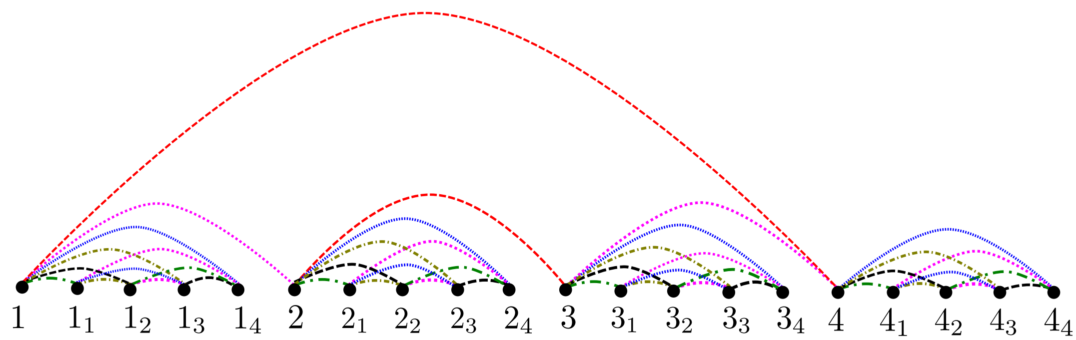

Fig.3 The matching book embedding of $C_4 \circ K_4$.

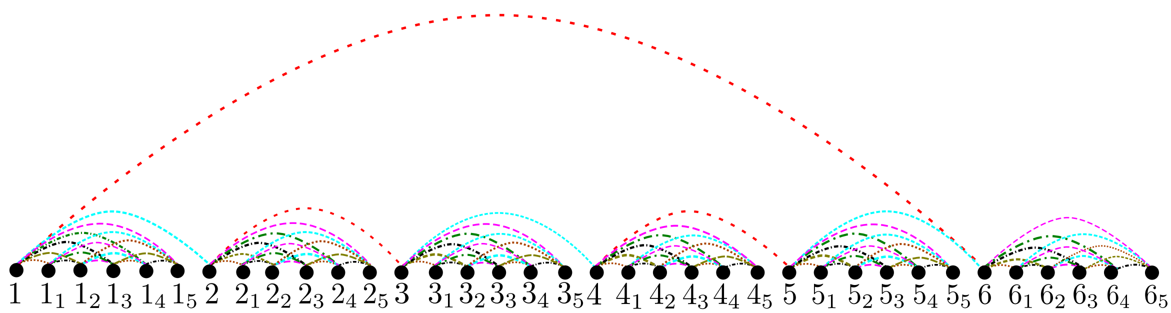

Fig.4 The matching book embedding of $C_6 \circ K_5$.

□

Theorem 3.1. *Let G be an arbitrary dispersible graph with q vertices, and let H be a simple graph with p vertices. Then $G \circ H$ is dispersible.*

Proof. According to $\triangle(G \circ H) = p + \triangle(G)$, then $mbt(G \circ H) \ge \triangle(G \circ H)$ by Lemma 2.1. Since $G \circ H$ is the subgraph of $G \circ K_p$, then $mbt(G \circ H) \le mbt(G \circ K_p)$ by Lemma 2.2. According to Lemma 3.1 and $\triangle(G \circ H) = \triangle(G \circ K_p) = p + \triangle(G)$, then $mbt(G \circ H) = p + \triangle(G)$.

Therefore, $G \circ H$ is dispersible. □

Next, we will show that the corona product between a nearly dispersible graph and a simple graph is dispersible.

Lemma 3.2. *Let G be an arbitrary nearly dispersible graph with q vertices $\triangle(G) = k$, and let K_p be a complete graph. Then $G \circ K_p$ is dispersible.*

Proof. Firstly, $mbt(G \circ K_p) \geq \triangle(G \circ K_p) = k+p$ by Lemma 2.1. Further, we consider that $G \circ K_p$ can be matching book embedded in $p+k$ pages. According to G is a nearly dispersible graph, without loss of generality, assume that $v_1, v_2, \ldots, v_q$ is the vertex order corresponding to the nearly dispersible book embedding of G. Then, put the vertices of $G \circ K_p$ along the spine according to the following ordering of $v_1, u_1^1, u_2^1, \ldots u_p^1, v_2, u_1^2, u_2^2, \ldots u_p^2, \ldots\ldots v_q, u_1^q, u_2^q, \ldots u_p^q$. Obviously, the edges of G can be matching book embedded in $k+1$ pages named page 1, page 2,..., page $k+1$. For $1 \leq i \leq q$, the edge (v_i, u_p^i) and edges $\{\, (u_j^i, u_{p-j}^i) \mid 1 \leq j \leq \lceil \frac{p}{2} \rceil - 1 \,\}$ can be placed on one of these $k+1$ pages where $d(v_i) = 0$ in the matching book embedding of G, and the edges $\{\, (u_j^i, u_{p-j+1}^i) \mid 1 \leq j \leq \lfloor \frac{p}{2} \rfloor \,\}$ can be added to one of these $k+1$ pages where $d(v_i) = 1$ in the matching book embedding of G. The remaining edges of $G \circ K_p$ can be embedded in additional $p-1$ pages as follows.

Page $k+2$: edges $\{\, (v_i, u_1^i) \mid 1 \leq i \leq q \,\}$, and edges $\{\, (u_{1+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 1 \,\}$.

Page $k+3$: edges $\{\, (v_i, u_2^i) \mid 1 \leq i \leq q \,\}$, and edges $\{\, (u_{2+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lfloor \frac{p}{2} \rfloor - 1 \,\}$.

Page $k+4$: edges $\{\, (v_i, u_3^i) \mid 1 \leq i \leq q \,\}$, edges $\{\, (u_{3+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 2 \,\}$, and edges $\{\, (u_1^i, u_2^i) \mid 1 \leq i \leq q \,\}$.

Page $k+5$: edges $\{\, (v_i, u_4^i) \mid 1 \leq i \leq q \,\}$, edges $\{\, (u_{4+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lfloor \frac{p}{2} \rfloor - 2 \,\}$, and edges $\{\, (u_1^i, u_3^i) \mid 1 \leq i \leq q \,\}$.

Page $k+6$: edges $\{\, (v_i, u_5^i) \mid 1 \leq i \leq q \,\}$, edges $\{\, (u_{5+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 3 \,\}$, and edges $\{\, (u_1^i, u_4^i), (u_2^i, u_3^i) \mid 1 \leq i \leq q \,\}$.

Page $k+7$: edges $\{\, (v_i, u_6^i) \mid 1 \leq i \leq q \,\}$, edges $\{\, (u_{6+j}^i, u_{p-j+1}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lfloor \frac{p}{2} \rfloor - 3 \,\}$, and edges $\{\, (u_1^i, u_5^i), (u_2^i, u_4^i) \mid 1 \leq i \leq q \,\}$.

...

Page $k+p-2$: edges $\{\, (v_i, u_{p-3}^i) \mid 1 \leq i \leq q \,\}$, edges $\{\, (u_j^i, u_{p-3-j}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lceil \frac{p}{2} \rceil - 2 \,\}$. and edges $\{\, (u_{p-2}^i, u_p^i) \mid 1 \leq i \leq q \,\}$.

Page $k+p-1$: edges $\{\, (v_i, u_{p-2}^i) \mid 1 \leq i \leq q \,\}$, edges $\{\, (u_j^i, u_{p-2-j}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lfloor \frac{p}{2} \rfloor - 1 \,\}$ and edges $\{\, (u_{p-1}^i, u_p^i) \mid 1 \leq i \leq q \,\}$.

Page $k+p$: edges $\{\, (v_i, u_{p-1}^i) \mid 1 \leq i \leq q \,\}$, and edges $\{\, (u_j^i, u_{p-1-j}^i) \mid 1 \leq i \leq q; 1 \leq j \leq \lfloor \frac{p}{2} \rfloor - 1 \,\}$.

Hence, $mbt\,(G \circ K_p) = \triangle(G_q \circ K_p) = k+p$. That is, $G \circ K_p$ is dispersible. (See Fig.5 for the case $K_3 \circ C_7$ and Fig.6 for the case $K_5 \circ C_4$).

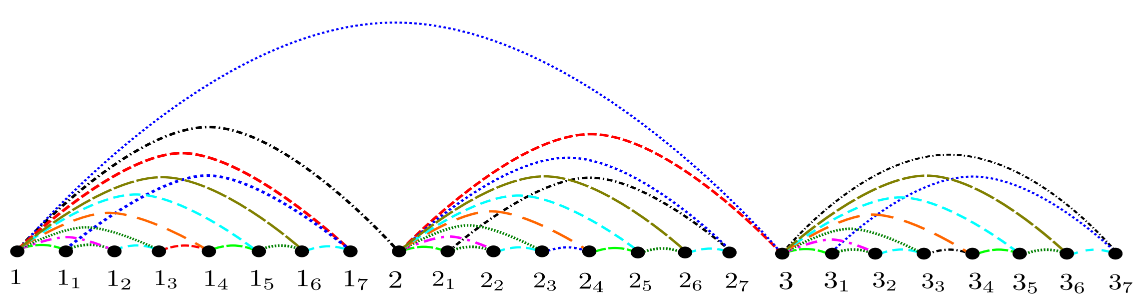

Fig.5 The matching book embedding of $K_3 \circ C_7$.

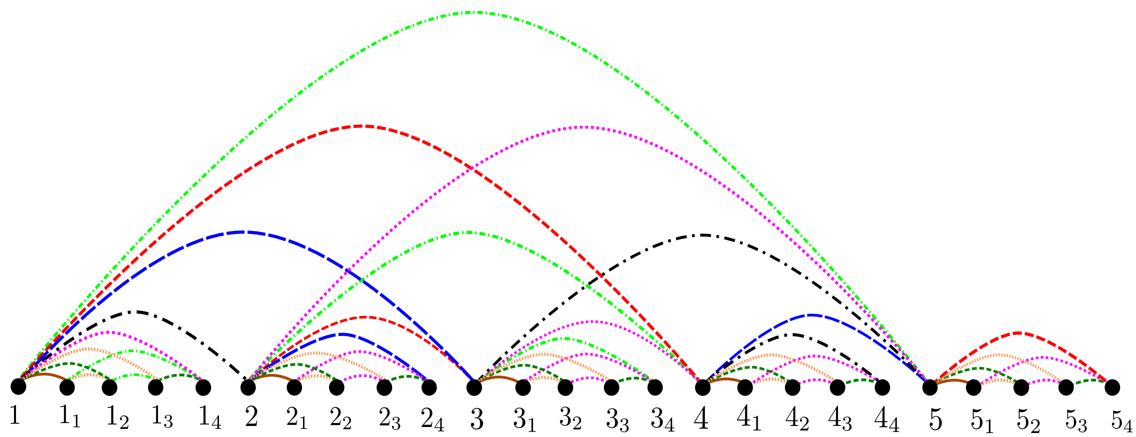

Fig.6 The matching book embedding of $K_5 \circ C_4$.

□

Theorem 3.2. *Let G be an arbitrary nearly dispersible graph with q vertices, and let H be a simple graph with p vertices. Then $G \circ H$ is dispersible,*

Proof. Firstly, since $\triangle(G \circ H) = p + \triangle(G)$, $mbt(G \circ H) \geq \triangle(G \circ H) = p + \triangle(G)$ by Lemma 2.1. In addition, Since $G \circ H$ is the subgraph of $G \circ K_p$, then $mbt(G \circ H) \leq mbt(G \circ K_p)$ by Lemma 2.2. According to $\triangle(G \circ H) = \triangle(G \circ K_p) = p + \triangle(G)$, we conclude that $G \circ H$ is dispersible by Lemma 3.2. □

Furthermore, the dispersibility of the edge corona product $C_q \diamond H$ is given as follows.

Theorem 3.3. *Let C_q be a cycle and let H be a simple graph with p vertices, then $C_q \diamond H$ is dispersible.*

Proof. Since $\triangle(C_q \diamond H) = 2p + 2$, then $mbt(C_q \diamond H) \geq 2p + 2$ by Lemma 2.1. Noting that $C_q \diamond H$ is the subgraph of $C_q \diamond K_p$ with $\triangle(C_q \diamond H) = \triangle(C_q \diamond K_p)$, then $mbt(C_q \diamond H) \leq mbt(C_q \diamond K_p)$ by Lemma 2.2. Thus, it is sufficient to consider the matching book embedding of $C_q \diamond K_p$.

Assume that $V(C_q \diamond K_p) = \{ v_i, u_j^i \mid i = 1, 2, \ldots, q; j = 1, 2, \ldots, p \}$ and $E(C_q \diamond K_p) = E(C_q) \cup \{ u_j^i, u_k^i \mid i = 1, 2, \ldots, q; j, k = 1, 2, \ldots, p; j \neq k\} \cup \{ (v_i, u_j^i) \mid i = 1, 2, \ldots, q, j = 1, 2, \ldots, p \} \cup \{ (v_i, u_j^{i-1}) \mid i = 2, \ldots, q, j = 1, 2, \ldots, p \} \cup \{ (v_1, u_j^q) \mid j = 1, 2, \ldots, p \}$. Arrange the vertices of $C_q \diamond K_p$ along the spine in the order $v_1, u_1^1, u_2^1, \ldots u_p^1, v_2, u_1^2, u_2^2, \ldots u_p^2, \ldots \ldots v_q, u_1^q, u_2^q, \ldots u_p^q$. Since $C_q \circ K_p$ induced by $E(C_q) \cup \{ u_j^i, u_k^i \mid i = 1, 2, \ldots, q; j, k = 1, 2, \ldots, p; j \neq k\} \cup \{ (v_i, u_j^i) \mid i = 1, 2, \ldots, q, j = 1, 2, \ldots, p \}$ is the subgraph of $C_q \diamond K_p$, then, the edges of $C_q \circ K_p$ can be matching book embedded in $p + 2$ pages by Theorem 3.2. For $1 \leq j \leq p$, $\{ (v_i, u_j^{i-1}), (v_1, u_j^q) \mid i = 2, \ldots, q \}$ can be assigned to additional distinctly p pages without crossing on a single page. $C_q \circ K_p$ can be matching book embedded in $2p + 2$ pages. Therefore, $C_q \circ H$ is dispersible. □

4. Results for the Comb Product

By the definition of the comb product, we get the reult as follows.

Lemma 4.1. *For two simple graphs G, H,*

$$mbt(G \rhd_v H) \geq \triangle(G \rhd_v H) = max\{\triangle(G) + d_H(v), \triangle(H)\}$$

.

Proof. Since $\triangle(G \rhd_v H) = max\{\triangle(G) + d_H(v), \triangle(H)\}$, by Lemma 2.1, $mbt(G \rhd_v H) \geq \triangle(G \rhd_v H) = max\{\triangle(G) + d_H(v), \triangle(H)\}$. □

Theorem 4.1. *Let G be an arbitrary dispersible graph with q vertices, and let H be an arbitrary dispersible graph with p vertices. Then, $G \rhd_v H$ is dispersible, where v is a vertex of H.*

Proof. According to Lemma 4.1, $mbt(G \rhd_v H) \geq \triangle(G \rhd_v H) = max\{\triangle(G) + d_H(v), \triangle(H)\}$. Besause G and H are all dispersible, without loss of generality, assume that $g_1, g_2, \ldots, g_q$ and $h_1, h_2, \ldots, h_p$ is the vertex order respectively corresponding to the dispersible book embedding of G and H. For $G \rhd_v H$, assume that $v = h_j$, then h_j, h_{j+1}, $\ldots$, h_p, h_1, h_2, $\ldots$, h_{j-2}, h_{j-1} is the vertex order corresponding to the dispersible book embedding of H by Lemma 2.3. Put the vertices of $G \rhd_v H$ on the spine in the ordering of (g_1, v), $(g_1, h_{j+1}), \ldots, (g_1, h_p)$, (g_1, h_1), (g_1, h_2), $\ldots$, (g_1, h_{j-1}), (g_2, v), (g_2, h_{j+1}), $\ldots$, (g_2, h_p), (g_2, h_1), (g_2, h_2), $\ldots$, (g_2, h_{j-1}), $\ldots$, $\ldots$, (g_q, v), (g_q, h_{j+1}), $\ldots$, (g_q, h_p), (g_q, h_1), (g_q, h_2), $\ldots$, (g_q, h_{j-1}), Next, $G \rhd_v H$ can be matching book embedding in $\triangle(G \rhd_v H)$ pages by considering two cases as follows.

Case 1. $\triangle(G) + d_H(v) > \triangle(H)$.

Since H is dispersible, it admits a matching book embedding in $\triangle(H)$ pages. Specificially, the edge set can be partitioned in to $\triangle(H)$ subsets $E_i (1 \leq i \leq \triangle(H))$, each assigned to a distinct page, such that for the vertex v,

(1). $d(v) = 1$ in $E_i (1 \leq i \leq d_H(v))$, and

(2). $d(v) = 0$ in $E_i (d_H(v) + 1 \leq i \leq \triangle(H))$.

Furthermore, let $E_{k,i} (1 \leq i \leq \triangle(H))$ denote all edges of the k-th copy of H in the graph $G \rhd_v H$.

Since G is dispersible, G can be matching book embedded in $\triangle(G)$ pages. For each $i(d_H(v) + 1 \leq i \leq \triangle(H))$, $\{E_{k,i} \mid 1 \leq k \leq |V(G)|\}$ can be matching book embedded in the $\triangle(H) - d_H(v)$ pages of the $\triangle(G)$ pages according to $\triangle(G) > \triangle(H) - d_H(v)$. For $1 \leq i \leq d_H(v)$, the remaining edges $\{E_{k,i} \mid 1 \leq k \leq |V(G)|\}$ can be placed in additional $d_H(v)$ pages.

Therefore, $G \rhd_v H$ can be matching book embedded in $\triangle(G) + d_H(v)$ pages.

Case 2. $\triangle(G) + d_H(v) \leq \triangle(H)$.

Since H is dispersible, $|V(G)|$ copies H of $G \rhd_v H$ can be matching book embedded in $\triangle(H)$ pages. Since $\triangle(G) \leq \triangle(H) - d_H(v)$, the edges of the graph G can be matching book embedded in $\triangle(G)$ pages selected from the $\triangle(H)$ pages where $d((g_i, v)) = 0$.

Therefore, $G \rhd_v H$ can be matching book embedded in $\triangle(H)$ pages. □

Using the similar method, we obtain that the dispersibility of the comb product between a dispersible graph and a nearly dispersible graph as follows.

Theorem 4.2. *Let G be an arbitrary nearly dispersible graph, and let H be an arbitrary dispersible graph. Then, $G \rhd_v H$ is dispersible, where v is a vertex of H and $\triangle(G) + d_H(v) \neq \triangle(H)$.*

Theorem 4.3. *Let G be an arbitrary dispersible graph, and let H be an arbitrary nearly dispersible graph. Then, $G \rhd_v H$ is dispersible, where v is a vertex of H and $\triangle(G) + d_H(v) \neq \triangle(H)$.*

5. Conclusions

In this paper, we study the matching book embedding of the corona product formed by a dispersible or nearly dispersible graph and a simple graph, and get the exact matching book thickness of the corona product of a dispersible or nearly dispersible graph and an arbitrary graph. Furthermore, the dispersibility of the edge corona product between a cycle and a simple graph is determined. Finally, the matching book thickness of the comb product of two dispersible graphs is obtained.

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