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Spanning k-ended trees of 3-regular connected graphs

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Abstract

A vertex of degree one is called an *end-vertex* and the set of end-vertices of G is denoted by $End(G)$. For a positive integer k, a tree T be called k-ended tree if $|End(T)| \leq k$. In this paper, we obtain sufficient conditions for spanning k-trees of 3-regular connected graphs. We give a construction sequence of graphs satisfying the condition. At the end, we present a conjecture about spanning k-ended trees of 3-regular connected graphs.

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1. Introduction

Throughout this article we consider only finite undirected labeled graphs without loops or multiple edges. The vertex set and edge set of graph G is denoted by $V = V(G)$ and $E = E(G)$, respectively. For $u, v \in V$, an *edge* joining two vertices u and v is denoted by uv or vu. The *neighbourhood* $N_G(v)$ or $N(v)$ of vertex v is the set of all $u \in V$ which are adjacent to v. The *degree* of a vertex v, denoted by $\deg_G(u) = |N_G(v)|$.

The minimum degree of a graph G is denoted $\delta(G)$ and the maximum degree is denoted $\Delta(G)$. If all vertices of G have same degree k, then the graph G is called k*-regular*. The *distance* between vertices u and v, denoted by $d_G(u, v)$ or $d(u, v)$, is the length of a shortest path between u and v. A *Hamiltonian path* of a graph is a path passing through all vertices of the graph. A graph is

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Hamiltonian-connected if every two vertices are connected with a Hamiltonian path. In graph G, an *independent set* is a subset S of $V(G)$ such that no two vertices in S are adjacent. A *maximum independent set* is an independent set of largest possible size for a given graph G. This size is called the *independence number* of G, that denoted by $\alpha(G)$.

A vertex of degree one is called an *end-vertex*, and the set of end-vertices of G is denoted by $End(G)$. If T is a tree, an end-vertex of a T is usually called a leaf of T and the set of leaves of T is denoted by $leaf(T)$. A spanning tree is called *independence* if $End(G)$ is independent in G. For a positive integer k, a tree T is said to be a *k-ended tree* if $|End(T)| \leq k$. We define $\sigma_k(G) = \min\{d(v_1) + \ldots + d(v_k) \mid \{v_1, \ldots, v_k\}$ *is an independent set in G*$\}$. Clearly, $\sigma_1(G) = \delta(G)$.

By using $\sigma_2(G)$, Ore [4] obtain the following famous theorem on Hamiltonian path. Notice that a Hamiltonian path is spanning 2-ended tree. A Hamilton cycle can be interpreted as a spanning 1-ended tree. In particular, K_2 is hamiltonian and is a 1-ended tree.

Theorem 1.1. *[4] Let G be a connected graph, if $\sigma_2(G) \geq |G| - 1$, then G has Hamiltonian path.*

The following theorem of Las Vergnas Broersma and Tuinstra [1] gives a similar sufficient condition for a graph G to have a spanning k-ended tree.

Theorem 1.2. *[2] Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

Win [10] obtained a sufficient condition related to independent number for k-connected graph that confirms a conjecture of Las Vergnas Broersma and Tuinstra [1] gave a degree sum condition for a spanning k-ended tree.

Theorem 1.3. *[10] Let $k \geq 2$ and let G be a m-connected graph. If $\alpha(G) \leq m + k - 1$, then G has a spanning k-ended tree.*

A closure operation is useful in the study of existence of Hamiltonian cycles, Hamiltonian path and other spanning subgraphs in graph. It was first introduced by Bondy and Chavatal.

Theorem 1.4. *[1] Let G be a graph and let u and v be two nonadjacent vertices of G then,*
(1) *Suppose* $\deg_G(u) + \deg_G(v) \geq |G|$. *Then G has a Hamiltonian cycle if and only if $G + uv$ has a Hamiltonian cycle.*
(2) *Suppose* $\deg_G(u) + \deg_G(v) \geq |G| - 1$. *Then G has a Hamiltonian path if and only if $G + uv$ has a Hamiltonian path.*

After [1], many researchers have defined other closure concepts for various graph properties.

More on k-ended tree and spanning tree can be found in [6, 7, 8, 9]. In this paper, we obtain sufficient conditions for spanning k-ended trees of 3-regular connected graphs and with construction sequence of graphs like G_m, we will show this condition is sharp. At the end, we present a conjecture about spanning k-ended trees of 3-regular connected graphs.

2. Our results

Lemma 2.1. *Let T be a tree with n vertices such that $\Delta(T) \leq 3$. If $|leaf(T)| = k$ and p be the number of vertices of degree 3 in T, then $k = p + 2$.*

Proof. It is easy by the induction on p. □

Lemma 2.2. *Let G be a labelled graph and $k \geq 3$ be the smallest integer such that G has a spanning tree T with k leaves. Then, no two leaves of T are adjacent in G.*

Proof. Put $S = \{v_1, v_2, \ldots, v_k\}$ be the set of all leaves of T. By contradiction, suppose that v_1 and v_2 are adjacent vertices in G. If $T_1 = T + v_1v_2$, then T_1 contains a unique cycle as $C : v_1v_2c_1c_2 \ldots c_\ell v_1$ where $c_i \in G$ for $1 \leq i \leq \ell$. Since $k \geq 3$ then there exist vertex $v_s \in G$ such that it is not a vertex of C. Let P be the shortest path of vertex v_s to the cycle C such that its intersection with cycle C is c_j for $1 \leq j \leq \ell$.
Now, we omit the edge $c_{j-1}c_j$ of T_1, (If $j = 1$ put $c_{j-1} = v_2$). Let $T_2 = T_1 - c_{j-1}c_j$. Then T_2 is a spanning subtree of G such that $\deg_{T_2}(c_j) \geq 2$. The vertices of degree one in spanning subtree T_2 is equal to the set $\{v_3, v_4, \ldots, v_k\}$ either $\{v_3, v_4, \ldots, v_k, c_{j-1}\}$. That is a contradiction by minimality of k. □

Theorem 2.1. *Let G be a labeled 3-regular connected graph such that $|V(G)| = n \geq 6$. Then G has a spanning $\lfloor \frac{n+2}{4} \rfloor$-ended tree.*

Proof. For the graph T, we denote the vertices of degree 1 with the set A_1, the vertices of degree 2 with the set A_2 and the vertices of degree 3 with the set A_3.
If $v \in A_3$ then the two adjacent edges to v (those were in G but are not in T), each one connects v to a vertex of A_2 in G, because by Lemma 2.2 it can not connect v to a member of A_1. So, for each vertex in A_1 there exist two vertices in A_2 such that they are connected to v in G but not in T. Now, we have $2 \times |A_1| \leq |A_2|$. Let $|A_1| = k, |A_2| = s$ and $|A_3| = p$. By Lemma 2.1 we have $k = p + 2$ and since $2|A_1| \leq |A_2|$ then $2k \leq s$.
We have

$$n = p + s + k = k - 2 + s + k \geq k - 2 + 2k + k = 4k - 2,$$

Then $k \leq \lfloor \frac{n+2}{4} \rfloor$. □

3. Some concluding remarks

Now we construct the sequence G_m of 3-regular graphs, For $m = 1$, Consider the graph G_1 as Figure 1.

Clearly G_1 has spanning subtree like T that has 3 leaves and G has no spanning subtree with less than 3 leaves. Every part of G_1 like subgraph induced by vertices $\{1, 2, 3, 4, 5\}$ is called a branch, so G_1 has 3 branch. Let H be a branch of G_1 with vertices $\{1, 2, 3, 4, 5\}$ and set of edges $\{12, 15, 23, 24, 34, 35, 45\}$. Since the edge $\{01\}$ is a cut edge in G_1, So T must has a vertex with degree one in H. Also in every other branches of G_1, T must has a vertex with degree one. so G_1 is 3-ended tree and has no spanning tree with less than 3 leaves. Now, we counteract 3-regular graph

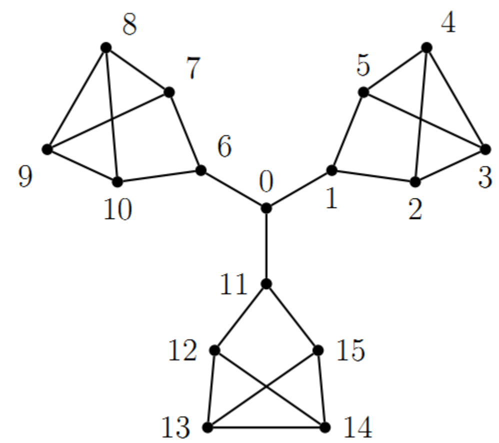

Figure 1. The 3-regular graph G_1 with 3 branch.

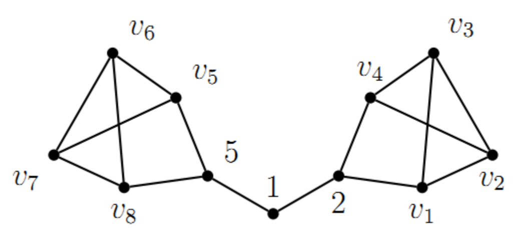

Figure 2. One part of G_2 constructed from G_1.

G_2, consider G_1 and for each branch of that like H defined as above, we removed two vertices $\{3, 4\}$ and add 8 new vertices $\{v_1, \ldots, v_8\}$ then we construct new 3-regular graph as Figure 2.

Clearly $|G_2| = 16 + 3 \times 6$ and minimum number leaves in every spanning subtree of G_2 is at least 2×3 and obviously G_2 has spanning subtree with 2×3 leaves.
Let the number of vertices of G_m is equal n and the number of branches of G_m is equal k, then we have the table 1.

m	n	k
G_1	16	3
G_2	$16 + 3 \times 6$	2×3
G_3	$16 + 3 \times 6 + 2 \times 3 \times 6$	$2 \times 2 \times 3$
$\ldots$	$\ldots$	$\ldots$
G_m	$16 + 3 \times 6 + \ldots + 2^{m-2} \times 3 \times 6$	$2^{m-1} \times 3$

Table 1. The number of vertices and branches of G_m for $m \in \mathbb{N}$.

It obvious for each $m \in \mathbb{N}$ if the number of vertices of G_m is equal n and the number of branches of G_m is equal k, then $\frac{n+2}{6} = k$, and so G_m is $\frac{n+2}{6}$-ended tree (such that $\frac{n+2}{6}$ is the minimum number for that G_m is $\frac{n+2}{6}$-ended tree).

Conjecture 1. *There exists $n \in \mathbb{N}$ such that each 3-regular graph with at least n vertices has a spanning $\lfloor \frac{n+2}{6} \rfloor$-ended tree.*

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